FUNCTIONAL ANALYSIS UNIT-IV(P16MA41)

II M.Sc. Mathematics

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Outline

General Preliminaries on Banach Algebra:

Banach Algebra:

Banach Algebra is a complex banach space which is also an algebra with identity 1 and in which the multiplicative structure is relative to the norm by

(i) $||xy|| \leq ||x|| \, ||y||$

(ii) $||1|| = 1$

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Theorem 1

Every element x for which $||x-1||<1$ is regular and the inverse of such element is given by
$x^{-1}=1+\sum_{n=1}^{\infty}(1-x)^n$

Proof:

Let $r=||x-1||$, So that $r<1$.
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For $n > m$,

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If G is an open set and therefore s is a closed set.

Proof:

Let x_0 be an element in G and let x be any element in A.

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Theorem 3

The mapping $x \to x^{-1}$ of G into G is continuous and is therefore a homeomorphism of G onto itself.

Proof:

- Let $f : G \to G$ be given by $f(x) = x^{-1} \forall x \in G$.

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The set of all topological divisor of zero Z is a subset of the set $S$of all singular element in A or Z is a subset of S.

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Implies $z \in G$ [G is the complement of S]

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Let A be a banach algebra and $x \in A$. Then spectrum of H is defined to be the following subsets of a complex plane.
$\sigma(x) = \{\lambda/(x - \lambda) \textit{ is singular}\}$

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$\sigma(x)$ is a subset of the closed disc $\{z/|z| \leq ||x||\}$

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Let λ be a complex such that $|\lambda| > ||x||$.

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If λ and μ are both in $\rho(x)$ then,

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$\sigma(x)$ is non-empty.

Proof:

Let f be a functional on A.
(*ie*) An element of the conjugate space A^* and define $f(\lambda)$ by

$$f(\lambda) = f(x(\lambda)) \tag{4}$$

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Spectral radius of x:

The number $r(x)$ defined by
$r(x) = \sup\{|\lambda| / \lambda \in \sigma(x)\}$ is called the spectral radius of x.
Note that $0 \leq r(x) \leq ||x||$.

Theorem 9

If A is a division algebra then it equals the set of all scalar multiples of the identity.

Proof:

- If $x \in A$ then $x = \lambda.1$ for some scalar λ
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If the norm in A satisfies the inequality $||x|| \geq k||x||||y||$ for some positive constant k. Then $A = C$.

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Theorem 12

If A^1 is a banach subalgebra of a banach algebra A then the spectra of an element x in A with respect to A and A^1 are related as follows

i $\sigma_{A^1}(x) \subseteq \sigma_A(x)$.

ii Each boundary point of $\sigma_A(x)$ is also a boundary point of $\sigma_{A^1}(x)$.

Proof:

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Lemma 1: The Formula for the Spectral Radius

$\sigma(x^n) = (\sigma(x))^n$

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- Let λ be a non-zero complex numbers.
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There fore,

$$x^n - \lambda.1 = (x - \lambda_1.1), (x - \lambda_2.1), ..., (x - \lambda_n.1) \quad (9)$$

- Let $\lambda \in \sigma(x^n)$. $\therefore$ $x^n - \lambda$ is singular.
- If all the factors on the $R.H.S$ of (9) are regular then the product is regular atleast $(x - \lambda_i.1)$ is singular for some i.

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$(ie) \quad \lambda \in (\sigma(x))^n$
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From (10) and (11) we get,
$\sigma(x^n) = (\sigma(x))^n$

Theorem 13

$r(x) = \lim ||x^n||^{\frac{1}{n}}$
Proof:
The formula for the spectral radius is
$r(x) = \sup\{|\lambda| / \lambda \in \sigma(x)\}$

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Now, $0 \leq r(x) \leq ||x||$ Implies $r(x^n) \leq ||x^n||$

$$r(x) \leq ||x^n||^{\frac{1}{n}} \forall n \tag{12}$$

To prove: If a is any real number such that $r(x) < a$ then $||x^n||^{\frac{1}{n}} \leq a$ for all but a finite number of n's

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Then, $x(\lambda) = (x - \lambda.1)^{-1} = \lambda^{-1}(\frac{x}{\lambda} - 1)^{-1}$

$$x(\lambda) = -\lambda^{-1}[1 + \sum_{n=1}^{\infty} \frac{x^n}{\lambda^n}] \quad (13)$$

If f is any functional on A, then (13) gives

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- $||x^n||^{\frac{1}{n}} \leq \alpha^n k^{\frac{1}{n}}$ for some positive constant k and every n.

$$||x^n||^{\frac{1}{n}} \leq a \tag{15}$$

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From (12) and (15), we have,

$r(x) = ||x^n||^{\frac{1}{n}} \leq a \forall n \geq m$ where a is any real number such that $r(x) < a$.

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$\therefore r(x) = \lim ||x^n||^{\frac{1}{n}}$

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Regular and Singular

- An element x in A is regular if there exist an element y such that $xy = yx = 1$.
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A maximal left ideal in A is defined to be a proper left ideal which is not properly contained in any other proper left ideal.

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If r is an element of R. Then $1 - r$ is left regular.

Proof

- Let $1 - r$ be left singular.
- So that $L = A(1 - r) = \{x - xr / x \in A\}$ is a proper left ideal which contain $1 - r$

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Since $r \in R$.

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If r is an element of A with the property that $1 - xr$ is regular for every x. Then r is in R.

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Let $m_1 + xr, m_2 + yr \in M + Ar$ where $m_1, m_2 \in M \text{ and } x, y \in A$.
Hence, $M + Ar$ is a linear subspace of A.

- Let $a \in A, m + xr \in M + Ar$
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Lemma 6

If $1 - xr$ is regular then $1 - rx$ is also regular.

Proof

- Assume that $1 - xr$ is regular with inverse $S = (1 - xr)^{-1}$
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Theorem 14

The radical R of A equals each of the four sets in $\cap MLI = \{r/1 - xr\}$ is regular for every x and $\cap MRI = \{r/1 - rx\}$ is regular for x is a proper two sided ideal.

Proof

- If $x \in R$ then $1 - xr$ is regular $\forall x$ [By lemma-4]
- Implies $\{\frac{r}{1 - xr}\}$ is regular $= \{r/x \in R\}$ [By lemma-5]

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$$R = \cap MLI \tag{19}$$

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A is said to be semi-simple if its radical equals the zero ideal $\{0\}$, that is if each non-zero element of A is outside of some maximal left ideal.

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The radical R of A is a proper closed two-sided ideal.

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A is said to be semi-simple if its radical equals the zero ideal $\{0\}$, that is if each non-zero element of A is outside of some maximal left ideal.

Theorem 15

The radical R of A is a proper closed two-sided ideal.

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If I is a proper closed two-sided ideal in A. Then the quotient algebra $\frac{A}{I}$ is Banach algebra.

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$$\alpha(x + I)(y + I) = \alpha(xy + I)$$

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Theorem 17

Let $\dfrac{A}{R}$ is a semi-simple banach algebra

Proof

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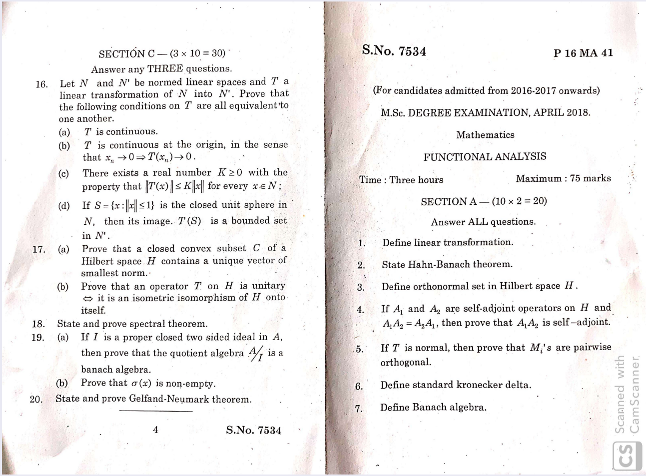
Old Question Paper



SECTION C — $(3 \times 10 = 30)$

Answer any THREE questions.

16. Let N and N' be normed linear spaces and T a linear transformation of N into N'. Prove that the following conditions on T are all equivalent to one another.
    (a) T is continuous.
    (b) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$.
    (c) There exists a real number $K \geq 0$ with the property that $\|T(x)\| \leq K\|x\|$ for every $x \in N$;
    (d) If $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in N, then its image. $T(S)$ is a bounded set in N'.
17. (a) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.
    (b) Prove that an operator T on H is unitary $\Leftrightarrow$ it is an isometric isomorphism of H onto itself.
18. State and prove spectral theorem.
19. (a) If I is a proper closed two sided ideal in A, then prove that the quotient algebra A/I is a banach algebra.
    (b) Prove that $\sigma(x)$ is non-empty.
20. State and prove Gelfand-Neumark theorem.

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S.No. 7534 P 16 MA 41

(For candidates admitted from 2016-2017 onwards)

M.Sc. DEGREE EXAMINATION, APRIL 2018.

Mathematics

FUNCTIONAL ANALYSIS

Time : Three hours Maximum : 75 marks

SECTION A — $(10 \times 2 = 20)$

Answer ALL questions.

1. Define linear transformation.
2. State Hahn-Banach theorem.
3. Define orthonormal set in Hilbert space H.
4. If A_1 and A_2 are self-adjoint operators on H and $A_1A_2 = A_2A_1$, then prove that A_1A_2 is self-adjoint.
5. If T is normal, then prove that M_i's are pairwise orthogonal.
6. Define standard kronecker delta.
7. Define Banach algebra.

Old Question Paper

8. Prove that Z (the set of all topological divisors of zero) is a subset of S.

9. Define multiplicative functional.

10. If $\|x^2\| = \|x\|^2$ for every x on arbitrary commutative Banach algebra A, then prove that $r(x) = \|x\|$ every x.

SECTION B — ($5 \times 5 = 25$)

Answer ALL questions, choosing either (a) or (b).

11. (a) If R is a commutative ring with identity, then prove that R is a field $\Leftrightarrow$ it has no non-trivial ideals.

Or

(b) If N is a normal linear space, then prove that the closed unit sphere S^* in N^* is a compact Hausdorff space in the weak* topology.

12. (a) If M is a closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^{\perp}$.

Or

(b) State and prove Bessel's inequality.

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13. (a) Prove that two matrices in A_n are similar $\Leftrightarrow$ they are the matrices of a single operator on H relative to (possibly) different bases.

Or

(b) Show that there exists a unique positive operator A on H such that $A^2 = T$.

14. (a) If $1 - xr$ is regular, then prove that $1 - rx$ is regular.

Or

(b) Prove that the boundary of S is a subset of Z.

15. (a) Prove that $M \to f_m$ is a one-to-one mapping of the set $\mathcal{M}$ of all maximal ideals in A into the set of all its multiplicative functions.

Or

(b) If A is self-adjoint, then prove that $\hat{A}$ is dense in $\mathcal{C}(\mathcal{M})$

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THANK YOU