Odinary differential Equation, Partial differential equation, Laplace Transform and Vector Calculus.

Unit - 1

1. Define : Linear differential equation.

A linear equation is one in which the dependent variable y and it's derivatives of any order occur only in the first degree and are not multiplied together, their co-efficients being constants or functions of the independent variable x.

Consider the equation.

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_n y = X.$$

2. Solve $(D^2 - 5D + 4) y = 0$.

The auxiliary equation is $m^2 - 5m + 4 = 0$.

Solving $m_1 = 1$ and $m_2 = 4$.

The solution is $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$.

$y = C_1 e^{x} + C_2 e^{4x}$.

3. Solve $(D^3 - 3D^2 + 4) Y = 0$.

The auxiliary equation is $m^3 - 3m^2 + 4 = 0$.

$$\begin{array}{r|rrrr} & 1 & -3 & 0 & 4 \\ -1 & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$m_1 = -1$.

$m^2 - 4m + 4 = 0$.

$m_2 = 2$, and $m_3 = 2$.

Hence the solution is $y = C_1 e^{-x} + e^{2x}(C_2 + C_3)$.

(4). Solve : $y = (x-a)p - p^2$.

This is clairaut's equation ;

hence the solution is ($\because$ put $p = c$.)

$y = (x-a)c - c^2$.

(5). Solve : $(D^2 - 5D + 6)y = 0$.

The auxiliary equation $m^2 - 5m + 6 = 0$.

$m_1 = 3$. and $m_2 = 2$.

$\therefore$ This solution is.

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

$y = c_1 e^{3x} + c_2 e^{2x}$.

x x

Unit - (2).

(1). Eliminate the arbitrary function from $z = f(x^2+y^2)$.

Soln:- Differentiating partially w.r.to x and y.

$p = f'(x^2+y^2)\,2x$ and $q = f'(x^2+y^2)\,2y$.

Eliminating $f'(x^2+y^2)$ between the latter two equations $py = qx$.

(2). write down Lagrange's method of linear equation.

Consider the equations $u = a$ and $v = b$, where a and b are arbitrary constants.

$$\therefore \frac{dx}{u_y v_z - u_z v_y} = \frac{dy}{u_z v_x - u_x v_z} = \frac{dz}{u_x v_y - u_y v_x}$$

③. Solve. $(y+z)p + (z+x)q = x+y$.

<u>Soln</u>:. The Subsidiary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{d(\Sigma x)}{2\Sigma x}.$$

They are also equivalent to.

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dz-dx}{x-z} = \frac{\Sigma dx}{2\Sigma x}.$$

Taking the first two ratios and integrating

$$\frac{x-y}{y-z} = a.$$

Taking the first and last ratios and integrating

$$(x-y)^2 \Sigma x = b.$$

Hence the soln required is $\phi\left[\frac{x-y}{y-z}, (x-y)^2\Sigma x\right] = 0$.

where ϕ is arbitrary.

④ Solve. the equation $p+q = x+y$.

we can write the equation in the form $p-x = y-q$.

Let $p-x = a$. then $y-q = a$.

Hence $p = x+a$, $q = y-a$.

$\therefore dz = (x+a)dx + (y-a)dy$. , $z = \frac{(x+a)^2}{2} + \frac{(y-a)^2}{2} + b$.

⑤. write down the clairant's form.

This is of the form $z = px + qy + f(p,q)$.

The Solution of the equation is $z = ax + by + f(a,b)$.

for $p = a$. and $q = b$.

can easily verified to satisfy the given equation.

Unit - 3.

1. State final value theorem in Laplace transform

If $L\{f(t)\} = F(s)$. then $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

2. $L(t^2 + 2t)$. Find it.

Soln:

$L(t^2 + 2t) = L(t^2) + 2L(t)$.

$= \frac{2}{s^3} + \frac{2}{s^2}$.

3. Find $L[\cosh at]$.

Soln:

$f(t) = \cosh at$.

$L\{f(t)\} = L[\cosh at]$.

$= \frac{1}{s^2 - a^2}$.

4. Find $L(t^2 + 2t + 3)$.

Soln:

$L(t^2 + 2t + 3) = L(t^2) + 2L(t) + 3L(1)$.

$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}$.

5. Find $L(t^2 e^{-3t})$.

$L(t(f(t))) = -\frac{d}{ds} F(s)$; $L(t^2(f(t))) = \frac{d^2}{ds^2} F(s)$.

$\therefore L(t^2(e^{-3t})) = \frac{d^2}{ds^2} L(e^{-3t})$. $= \frac{d^2}{ds^2} \left(\frac{1}{s+3}\right)$.

$L(t^2(e^{-3t})) = \frac{2}{(s+3)^2}$

Unit - 4:

1. Find $L^{-1}\left[\frac{1}{(S-a)^2}\right]$

Soln::

$$L^{-1}\left[\frac{1}{(S-a)^2}\right] = e^{at} L^{-1}\left(\frac{1}{s^2}\right) = e^{+at}\, t$$

$$\therefore L^{-1}\left[\frac{1}{(S-a)^2}\right] = e^{at}.\, t.$$

2. Find $L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]$

Soln::

$$L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at.$$

3. Find $L^{-1}\left[\frac{1}{(s+2)^2+16}\right]$

Soln:-

$$L^{-1}\left[\frac{1}{(s+2)^2+16}\right] = e^{-2t} L^{-1}\left[\frac{1}{s^2+16}\right].$$

$$= e^{-2t} L^{-1}\left[\frac{1}{s^2+4^2}\right].$$

$$= e^{-2t} L^{-1}\left[\frac{4}{4(s^2+4^2)}\right].$$

$$= \frac{e^{-2t}}{4} L^{-1}\left[\frac{4}{(s^2+4^2)}\right].$$

$$= \frac{e^{-2t}}{4} \sin 4t.$$

4. Find $L^{-1}\left(\frac{s}{s^2+k^2}\right)$.

Soln:: W.K.T $L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)]$.

$$L^{-1}\left[s.\frac{1}{s^2+k^2}\right] = \frac{d}{dt} L^{-1}\left[\frac{1}{s^2+k^2}\right].$$

$$= \frac{d}{dt} L^{-1}\left[\frac{1}{k}\cdot\frac{k}{s^2+k^2}\right]$$

$$= \frac{1}{k}\frac{d}{dt} L^{-1}\left[\frac{k}{s^2+k^2}\right]$$

$$= \frac{1}{k} \frac{d}{dt} \sin kt.$$

$$= \frac{1}{k} k \cos kt.$$

$$= \cos kt.$$

$$L^{-1}\left[\frac{s}{s^2+k^2}\right] = \cos kt.$$

⑤. Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$.

Soln:

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2}\right].$$

$$= \frac{d}{dt}\left\{e^{-2t} L^{-1} \frac{1}{s^2}\right\}.$$

$$= \frac{d}{dt}(e^{-2t} t).$$

$$L^{-1}\left(\frac{s}{(s+2)^2}\right) = e^{-2t}(1-2t).$$

x x

Unit - ⑤.

①. Define Divergence.

Let $f = f_1 i + f_2 j + f_3 k$ be a vector valued function.

The divergence of f denoted by $\nabla \cdot f$ or div f.

defined by $\nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \sum i \cdot \frac{\partial f}{\partial x}$.

②. Define curl.

The curl of f denoted by $\nabla \times f$ or curl f is

defined by

curl $f = \sum i \times \frac{\partial f}{\partial x} = i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + j\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + k\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}.$$

(3). Define solenoidal and irrotational.

A vector f is called solenoidal if $\operatorname{div} f = 0$.

A vector f is called irrotational if $\operatorname{curl} f = 0$.

(4). Show that $\operatorname{curl}(r^n r) = 0$.

Soln:

$$r^n r = r^n x\, i + r^n y\, j + r^n z\, k.$$

$$\therefore \operatorname{curl}(r^n r) = \Sigma i \left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right].$$

$$= \Sigma i \left[z n r^{n-1} \frac{\partial r}{\partial y} - y n r^{n-1} \cdot \frac{\partial r}{\partial z}\right].$$

$$= n r^{n-1} \Sigma i \left[z(y/r) - y(z/r)\right] = 0.$$

(5). $\operatorname{curl} \operatorname{grad} \phi = \nabla \times (\nabla \phi) = 0$.

Proof:

$$\nabla \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k.$$

$$\therefore \nabla \times (\nabla \phi) = i\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) - j\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right)$$

$$+ k\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)$$

$$= 0.$$

1. Prove that $\nabla\times(\nabla\times f) = \nabla(\nabla\cdot f) - \nabla^2 f$ (or)

curl(curl f) = grad div f − $\nabla^2 f$.

Proof:

Let $f = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$.

$$\therefore \nabla\times f = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\vec{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\vec{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\vec{k}$$

$$\nabla\times(\nabla\times f) = \Sigma\left\{\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right\}\vec{i}$$

$$= \Sigma\left\{\left(\frac{\partial^2 f_2}{\partial y\partial x} + \frac{\partial^2 f_3}{\partial z\partial x}\right) - \left(\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right\}\vec{i}$$

$$= \Sigma\left\{\frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) - \left(\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right\}\vec{i}$$

$$= \Sigma\left\{\frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} + \frac{\partial f_1}{\partial x}\right) - \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right\}\vec{i}$$

$$= \Sigma\left\{\frac{\partial}{\partial x}(\nabla\cdot f) - (\nabla^2 f_1)\right\}\vec{i}$$

$$= \Sigma\left\{\frac{\partial}{\partial x}(\nabla\cdot f)\vec{i}\right\} - \Sigma(\nabla^2 f_1)\vec{i}$$

(Similarly

$$= \Sigma\left\{\frac{\partial}{\partial y}(\nabla\cdot f)\vec{j}\right\} - \Sigma(\nabla^2 f_2)\vec{j}$$

$$= \Sigma\left\{\frac{\partial}{\partial z}(\nabla\cdot f)\vec{k}\right\} - \Sigma(\nabla^2 f_3)\vec{k}$$)

$$\therefore \nabla\times(\nabla\times f) = \nabla(\nabla\cdot f) - \nabla^2 f.$$

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(2) Prove that $\operatorname{div}(r^n \vec{r}) = (n+3) r^n$; Deduce that $r^n \vec{r}$ is solenoidal iff $n = -3$.

(or) $\nabla \cdot (r^n \vec{r}) = (n+3) r^n$.

Soln:

$r^n \vec{r} = r^n (x\vec{i} + y\vec{j} + z\vec{k})$.

$\therefore \operatorname{div}(r^n \vec{r}) = \frac{\partial}{\partial x}(x r^n) + \frac{\partial}{\partial y}(y r^n) + \frac{\partial}{\partial z}(z r^n)$.

$= r^n + x n r^{n-1} \frac{\partial r}{\partial x} + r^n + y n r^{n-1} \frac{\partial r}{\partial y} + r^n + z n r^{n-1} \frac{\partial r}{\partial z}$.

$= 3 r^n + n r^{n-1} \left(\frac{x^2 + y^2 + z^2}{r} \right)$

$\left(\text{Since}: \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r} \right)$.

$= 3 r^n + n r^{n-1} (x^2 + y^2 + z^2) r^{-1}$.

$= 3 r^n + n r^{n-2} (x^2 + y^2 + z^2)$.

$= 3 r^n + n r^{n-2} (r^2)$.

$= (3 + n) r^n$.

$\therefore$ Now, $r^n \vec{r}$ is solenoidal

iff $\operatorname{div} r^n \vec{r} = 0$, (i.e) iff $(3+n) r^n = 0$

iff $n = -3$.

③. Show that $\operatorname{curl}(r^n \vec{r}) = 0$.

soln:. $r^n \vec{r} = r^n x\hat{i} + r^n y\hat{j} + r^n z k$.

$$\operatorname{curl}(r^n \vec{r}) = \sum \hat{i}\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right]$$

$$= \sum \hat{i}\left[z n r^{n-1}\frac{\partial r}{\partial y} - y n r^{n-1}\frac{\partial r}{\partial z}\right].$$

$$= n r^{n-1} \sum \hat{i}\left[z\left(\frac{y}{r}\right) - y\left(\frac{z}{r}\right)\right].$$

(since $\frac{\partial r}{\partial x} = \frac{x}{r}$ lly).

$$= n r^{n-1} \sum \hat{i}\left[\frac{yz}{r} - \frac{yz}{r}\right]$$

$$= n r^{n-1} \sum \hat{i}\,(0).$$

$$= 0.$$

$\therefore \operatorname{curl}(r^n \vec{r}) = 0.$

④. Prove that $\operatorname{curl}(\vec{r}\times\vec{a}) = -2\vec{a}$, where $\vec{a}$ is constant vector.

Proof:. $\operatorname{curl}(\vec{r}\times\vec{a}) = \nabla \times (\vec{r}\times\vec{a})$.

$$= \sum\left[\hat{i}\times\frac{\partial}{\partial x}(\vec{r}\times\vec{a})\right].$$

$$= \sum\left[\hat{i}\times\left(\frac{\partial \vec{r}}{\partial x}\times\vec{a} + \vec{r}\times\frac{\partial \vec{a}}{\partial x}\right)\right].$$

$$= \sum\left[\hat{i}\times\left(\frac{\partial \vec{r}}{\partial x}\times\vec{a}\right)\right].$$

(since $\vec{a}$ is a constant vector $\frac{\partial \vec{a}}{\partial x} = 0$.)

$= \Sigma [\hat{i} \times (\hat{i} \times a)]$. $\left(\text{since } \frac{\partial r}{\partial x} = \hat{i}\right)$.

$= \Sigma [(\hat{i} \cdot a)\, i - (i \cdot i) \cdot a]$. $\quad (i \times i) = 1$.

$= \Sigma\, [(\hat{i} \cdot a)\, i - a]$.

$= [(\hat{i} \cdot a)\, \hat{i} - a] + [(\hat{j} \cdot a)\, \hat{j} - a] + [(k \cdot a)\, k - a]$.

$= (\hat{i} \cdot a)\, \hat{i} + (\hat{j} \cdot a)\, j + (k \cdot a)\, k - 3a$

$=$

$= a - 3a = -2a$.

(5). Prove that $\operatorname{div}(r \times a) = 0$ where a is constant vector.

Soln:. $\operatorname{div}(r \times a) = \nabla \cdot (r \times a)$

$= \Sigma \left[\hat{i} \cdot \frac{\partial}{\partial x} (r \times a)\right]$.

$= \Sigma \left[\hat{i} \cdot \left(\frac{\partial r}{\partial x} \times a + r \times \frac{\partial a}{\partial x}\right)\right]$.

$= \Sigma \left[\hat{i} \cdot \left(\frac{\partial r}{\partial x} \times a\right)\right]$. (Since a is constant vector $\frac{\partial a}{\partial x} = 0$).

$= \Sigma\, [\hat{i} \cdot (\hat{i} \times a)] = 0$.

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