Complex Analysis — 16BCCMM13

Unit - V

Evaluation of definite Integrals:

- Type 1. $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ & $\sin\theta$.

Put $z = e^{i\theta}$, $0 \le \theta \le 2\pi$,

z: Unit circle $|z| = 1$.

Also $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \dfrac{z+z^{-1}}{2}$ &

$\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \dfrac{z-z^{-1}}{2i}$.

$I = \int_c \phi(z)\,dz$, $\phi(z) = f\left(\dfrac{z+z^{-1}}{2}, \dfrac{z-z^{-1}}{2i}\right)$.

to evaluated by residue theorem.

Ex: 1) S.T $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2}$.

Soln Let $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$

Put $z = e^{i\theta}$

$dz = iz\,d\theta$. $\cos\theta = \dfrac{z+z^{-1}}{2}$

$\therefore I = \displaystyle\int_c \frac{dz/iz}{\left[5+3\left(\frac{z+z^{-1}}{2}\right)\right]}$

$= \displaystyle\int_c \frac{2\,dz}{iz\left[10+3z+3z^{-1}\right]} = \int_c \frac{2\,dz}{i\left[10z+3z^2+3\right]}$

$3z^2+10z+3=0$.

$= \dfrac{2}{i}\displaystyle\int_c \frac{dz}{(3z+1)(z+3)}$.

9, 10, $1/3$, $9/3$

$(3z+1)(z+3)=0$.

Let $f(z) = \dfrac{1}{3z^2+10z+3} = \dfrac{1}{(3z+1)(z+3)}$

$z = -\frac{1}{3}$ & $z = -3$ are simple poles of $f(z)$. The pole $-\frac{1}{3}$ lies inside on C.

$$\text{Res}\left\{f(z); -\tfrac{1}{3}\right\} = \lim_{z \to -\frac{1}{3}} \cancel{(z+\tfrac{1}{3})} \frac{1}{3\cancel{(z+\frac{1}{3})}(z+3)}$$

$$= \frac{1}{3(-\frac{1}{3}+3)}$$

$$= \frac{1}{8}$$

$$I = 2\pi \cancel{i}\left(\frac{2}{\cancel{i}} \cdot \frac{1}{8}\right)$$

$$= \pi/2$$

$$\therefore \boxed{\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2}}$$

- Type: II $\int_{-\infty}^{\infty} f(x)\, dx$ where $f(x) = \dfrac{g(x)}{h(x)}$

& $g(x)$, $h(x)$ are polynomials in x & degree of $h(x)$ exceeds that of $g(x)$ by at least two.

we take $f(z) = \dfrac{g(z)}{h(z)}$.

case(i): No pole of $f(z)$ lies on real axis

Curve C consisting interval $[-r, r]$ on real axis & semi circle $|z| = r$ lying in upper half plane

$$\int_C f(z)\,dz = \int_{-r}^{r} f(x)\,dx + \int_{C_1} f(z)\,dz$$

where C_1 is semi circle.

$\because$ deg $h(x)$ − deg $f(x) \geq 2$.

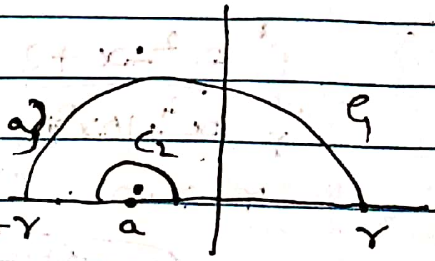
$\int_{C_1} f(z)\,dz \to 0$ as $r \to \infty$.

$$\int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

↳ Evaluate (Residue thm)

Case (ii) $f(z)$ has poles lying on Real axis.

$$\int_{C_2} f(z)\,dz = \pi i \operatorname{Res}(f(z), a)$$



Ex : 1, Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,dx$.

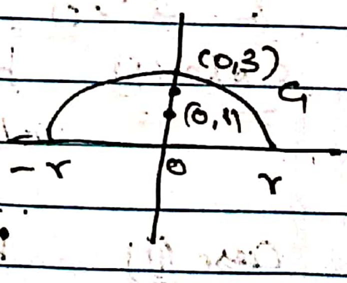
Sol. let $f(z) = \dfrac{z^2 - z + 2}{z^4 + 10z^2 + 9}$

$\therefore z^4 + 10z^2 + 9 = 0$

$(z^2 + 9)(z^2 + 1) = 0$

$\therefore z = \pm 3i, \pm i$ are simple poles of $f(z)$.

Choose the contour C



$$\int_C f(z)\,dz = \int_{-r}^{r} f(x)\,dx + \int_{C_1} f(z)\,dz$$

The poles of $f(z)$ lying with in C are i, $3i$ & both are simple pole

$$\operatorname{Res}\{f(z); i\} = \frac{h(i)}{k'(i)}$$

$h(z) = z^2 - z + 2$

$k(z) = z^4 + 10z^2 + 9$

$k'(z) = 4z^3 + 20z$

$$= \frac{i^2 - i + 2}{4(i)^3 + 10(i)}$$

$$\text{Res}\{f(z); i\} = \frac{-1+2-i}{-4i+20i} = \frac{1-i}{16i}$$

$|||^{ly}$ $$\text{Re}\{f(z); 3i\} = \frac{-9-3i+2}{-108i+60i} = \frac{7+3i}{48i}$$

$\int_c dz = 2\pi i$ (sum of residues at poles)

$$= 2\pi i\left(\frac{1-i}{16i} + \frac{7+3i}{48i}\right)$$

$$= 2\pi i\left(\frac{10}{48i}\right) = \frac{5\pi}{12}$$

$$\int_{-r}^{r} \frac{x^2-x+2}{x^4+10x^2+9}\,dx + \int_{c_1} \frac{z^2-z+2}{z^4+10z^2+9}\,dz = \frac{5\pi}{12}$$

$r \to \infty$ $\int^{r'}$ over $c_1 \to 0$.

$$\boxed{\therefore \int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} = \frac{5\pi}{12}}$$

- Type - III $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos ax\,dx$ or $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \sin ax\,dx$, where $g(x)$ & $h(x)$ are real polynomials such that degree of $h(x)$ exceeds of $g(x)$ by at least one & $a > 0$.

<u>Case: (i)</u> $h(x)$ has no zeros on real axis

Take $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} e^{iax}\,dx$.

<u>Case (ii)</u> $h(x)$ has zeros of order one on the real axis

Ex. 1 P.T. $\int_0^\infty \frac{\cos ax}{(x^2+1)^2}\,dx = \frac{\pi}{4}(a+1)e^{-a}$, $a>0$.

Soln. Let $f(z) = \frac{e^{iaz}}{(z^2+1)^2} = \frac{e^{iaz}}{[(z-i)(z+i)]^2}$

$z^2+1=0$

$z=\pm i$,

The poles of $f(z)$ given by i & $-i$ are double poles.

The pole of $f(z)$ lies within C is i.

$\text{Res}\{f(z); i\} = \frac{1}{1!} g'(i)$

$g(z) = (z-i)^2 f(z) = \frac{e^{iaz}}{(z+i)^2}$

$g'(z) = \frac{(z+i)^2 ia e^{iaz} - e^{iaz}\, 2(z+i)}{(z+i)^4}$

$\text{Res}\{f(z); i\} = \frac{(2i)^2 ia e^{-a} - e^{-a}\, 2(2i)}{(2i)^4}$

$= \frac{-ie^{-a}(a+1)}{4}$

By Cauchy's Residue thm.

$\int_C f(z)\,dz = 2\pi i\left(\frac{-ie^{-a}(a+1)}{4}\right)$

$= \frac{\pi}{2} e^{-a}(a+1)$

$\therefore \int_{-r}^{r} f(x)\,dx + \int_{C_1} f(z)\,dz = \frac{\pi(a+1)e^{-a}}{2}$

As $r\to\infty$, $\int$ over $C_1 \to 0$.

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi(a+1)e^{-a}}{2}$

$\int_{-\infty}^{\infty} \frac{\cos ax\,dx}{(x^2+1)^2} = \frac{\pi}{2}(a+1)e^{-a}$

$\Rightarrow \boxed{\int_0^\infty \frac{\cos ax}{(x^2+1)^2}\,dx = \frac{\pi}{4}(a+1)e^{-a}}$

Head of the Department of Mathematics,
Mother Terasa College of Arts & Science,
Mettusalai, Illuppur, Pudukkottai Dt-622102.