Analytical Geometry 3D.

Unit - 1.

1. If l, m, n are direction cosines of a line, then Prove that $l^2+m^2+n^2=1$.

Proof:

Consider the line d which has the direction cosines l, m, n. Draw a line through O parallel to the line d. Take any point $P(x, y, z)$ on the line d.

Let $OP = r$. Then $r = \sqrt{x^2+y^2+z^2} \rightarrow (1)$.

Draw $PN \perp$ to OX.

From right $\triangle OMP$, $\cos\alpha = \frac{x}{r}$, $\cos\beta = \frac{y}{r}$.

similarly, $\cos\gamma = \frac{z}{r}$.

$$l^2+m^2+n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

$$l^2+m^2+n^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = \frac{x^2+y^2+z^2}{r^2} = \frac{r^2}{r^2}.$$

$$l^2+m^2+n^2 = 1 \quad (\text{Using } (1))$$

Hence the proof.

2. Find the distance between the points $(4, 3, -6)$ and $(-2, 1, -3)$.

Soln:

Given the two points $P(4, 3, -6)$, $Q(-2, 1, -3)$.

$$PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$$= \sqrt{(-2-4)^2+(1-3)^2+(-3+6)^2}$$

$$= \sqrt{36+4+9} = \sqrt{49}.$$

$$PQ = 7.$$

3. Write down the formula for the normal form of the equation of a plane.

The equation of a plane can be written as $lx+my+nz=p$.

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(2)

where l, m, n are the direction cosines of the normal to the plane, and p is the length of the perpendicular to from the origin to plane.

(4) Define : Direction Cosines.

Let α, β, γ be the angles made by a straight lines with the two directions of the co-ordinate axes these angles are called the direction angles and the cosines of these angles are called the direction cosines (d.c) of the line

(5) Define Direction ratios.

Any three numbers a, b, c which are proportional to the d.c of the line are called the direction ratio's or direction number of the line.

Hence $l = ak$, $m = bk$, $n = ck$, where $k \neq 0$.

Unit - (2).

(1). Define symmetric and non-symmetric form.

Symmetric form:

Let $A(x_1, y_1, z_1)$ be a given point on the line. Let l, m, n be the d.c of the line. Let $P(x, y, z)$ be any point on the line

$\therefore$ The direction ratio's of AP are

$x - x_1$, $y - y_1$, $z - z_1$.

Since the d.c are l, m, n we have

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z + z_1}{n}$$

Non- Symmetric form :-

We know that two planes in general intersect in line. Hence a line in space can be represented by two linear equations.

$\Pi_1 : a_1x + b_1y + c_1z + d_1 = 0$

$\Pi_2 : a_2x + b_2y + c_2z + d_2 = 0$.

(2). Define Skew lines:

Two straight lines in space which are not coplanar are called skew lines.

(3) when will say you that the two lines:

$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$.

are coplanar?

Soln:

The coplanar is $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$.

(4). Find the value of k, so that the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$ are perpendicular to each other.

Soln:

The direction ratios of the lines are $-3, 2k, 2$ and $3k, 1, -5$.

Since the lines are $\perp$ we have

$(-3)\,3k + (2k)1 + 2(-5) = 0$.

Hence $k = -10/7$.

(5). Write the condition for two lines are coplanar

Soln:

Let $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$.

(4)

are the two lines.

$$\therefore \text{The coplanar is } \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Unit - (3).

(1). Define Equation of a sphere.

A sphere is the locus of a point in space which moves such that it's distance from a fixed point is constant. The fixed point is called the centre of the sphere and the fixed distance is called the radius of the sphere.

Centre radius form: $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

General form: $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

(2). Define Tangent line:-

The straight line joining two points P and Q on a surface is called a chord of the surface. When Q moves along the surface and ultimately coincides with P the limiting position of PQ touches the surface at P and is called a tangent line of the surface.

(3). Define Tangent plane:

A sphere with centre C there are many tangent lines at a point P on it, all of them being $\perp$ to the radius CP. all these tangents lie on the plane through P $\perp$ to CP. This plane is called the tangent plane of the sphere at P.

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4. Write down the sphere equations, centre and radius?

The equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ represents a sphere with centre $(-u,-v,-w)$ and radius $\sqrt{u^2+v^2+w^2-d}$

5. Angle of intersection of two spheres.
The angle of intersection of two sphere at a common point is the angle between the tangent planes to them at that point.

Unit - 4.

1. Define Right circular cone:
A right circular cone is a surface generated by a line which passes through a fixed point and makes a constant angle with the fixed line through the fixed point.

2. Write down the conditions for the second order homogeneous equation. $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$, represents, (a) a cone, (b) a pair of planes.
$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ splits in two linear factors if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Hence (i) represents a cone if $\Delta \neq 0$ but (ii) a pair of planes if $\Delta = 0$.

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(3) Define :- Cone

If O is a fixed point and a variable point P moves along a fixed curve C, the surface generated by the straight line OP is a cone. O is the vertex and C a base curve and the line OP a generator of the surface.

(4) Find the general equation of the cone which touches the co-ordinate axes.

we now consider the more general equation $F(x, y, z) = 0 \rightarrow (1)$ points in the plane $z = k$, whose co-ordinates satisfy (1) have x and y co-ordinates such that $F(x, y, k) = 0$. Hence these points lie on a curve C_k in which the plane $z = k$ meets the surface $f(x, y, z) = 0$.

which (1) is the equation.

(5) Define : Tangent plane

If (x_1, y_1, z_1) lies on the cone $f(x_1, y_1, z_1) = 0$. The equation of $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ of the previous articles becomes,

$$f(l, m, n) d^2 + 2d\,[l(ax_1 + hy_1 + gz_1) + m(hx_1 + by_1 + fz_1) + n(gx_1 + fy_1 + cz_1)] = 0 \rightarrow (1)$$

If $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ is a tangent to the cone at (x_1, y_1, z_1) equation (1) has two zero roots

$$\therefore l(ax_1 + hy_1 + gz_1) + m(hx_1 + by_1 + fz_1) + n(gx_1 + fy_1 + cz_1) = 0.$$

Unit - (5).

(1) write the condition for the plane $lx+my+nz=p$. to touch the conicoid $ax^2+by^2+cz^2=1$.

Let the plane touch the conicoid at (x_1, y_1, z_1). The equation of the tangent plane at (x_1, y_1, z_1) is $axx_1+byy_1+czz_1=1 \rightarrow (1)$

This plane is also represented by the equation

$$lx+my+nz=p \rightarrow (2)$$

Since (x_1, y_1, z_1) lies on the conicoid

$ax_1^2+by_1^2+cz_1^2=1$

$$\therefore a\left(\frac{l}{ap}\right)^2+b\left(\frac{m}{bp}\right)^2+c\left(\frac{n}{cp}\right)^2=1.$$

(i.e) $$p^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}$$

(2). Give the general equation to a cone which touches the co-ordinate plane.

If the co-ordinate planes touch a cone. the $\perp$ to co-ordinate planes touch the reciprocal cone.

The equation of the cone passing through the axis is of the form $2fyz+2gzx+2hxy=0$.

The required cone is the reciprocal cone of this cone and its equation is

$f^2x^2+g^2y^2+h^2z^2-2ghyz-2hfzx-2fgxy=0$.

(3). Define : central quadric

If $P(x_1, y_1, z_1)$ lies on the surface,

$Ax^2+By^2+Cz^2=1. \rightarrow (i)$

$Q(-x_1, -y_1, -z_1)$ also lies on the surface, and O the origin is the mid-point of PQ. Hence all chords of (i) which pass through O are bisected at O. For this reason (1) is called a central quadric.

(5)

Q. write down the two tangent planes to a conicoid parallel to the plane $lx+my+nz=0$.

∴ Their equations are

$$lx+my+nz = \pm\left[\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right]^{1/2}.$$

(5) write down the condition that the cone has three mutually perpendicular generators.

The condition that the plane should cut the cone in ⊥ generators is that $\theta = 90^\circ$. In that case by (A) of the previous article

$(a+b+c)(u^2+v^2+w^2) = f(u, v, w)$.

The third generator is ⊥ to these two generators. Hence it is normal to the plane containing these ⊥ generators.

If the normal to the plane $ux+vy+wz=0$ lies on the cone we have $f(u, v, w)=0$.

$\therefore a+b+c=0.$