Vector Calculus & Fourier Series.

Unit - I 2 Marks

1. P.T Gradient of a Constant is zero
(or) Find $\nabla\phi$ if ϕ is a constant.

Ans:- Let $\phi(x,y,z)$ be Constant.

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial z} = 0.$$

$$\therefore \nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi(x,y,z)$$

$$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = \bar{i}(0) + \bar{j}(0) + \bar{k}(0)$$

$$\boxed{\nabla\phi = 0.}$$

2. Show that $\operatorname{div}(\bar{f}+\bar{g}) = \operatorname{div}\bar{f} + \operatorname{div}\bar{g}$.

Soln: $\operatorname{div}(\bar{f}+\bar{g}) = \nabla\cdot(\bar{f}+\bar{g})$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot(\bar{f}+\bar{g})$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\bar{f} + \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}+\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\cdot\bar{g}$$

$$= \nabla\cdot f + \nabla\cdot g$$

$$\therefore \operatorname{div}(\bar{f}+\bar{g}) = \operatorname{div}\bar{f} + \operatorname{div}\bar{g}.$$

3. If $\phi(x,y,z) = xy^2 + yz^3$ find $\nabla\phi$.

Ans:- $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$$= \bar{i}(y^2) + \bar{j}(2xy + z^3) + \bar{k}(3z^2y)$$

$$\nabla\phi = y^2\bar{i} + (2xy+z^3)\bar{j} + 3z^2y\,\bar{k}$$

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4) If $\bar{F} = x^2 z \bar{i} - 2y^3 z^2 \bar{j} + xy^2 z \bar{k}$, Find div $\bar{F}$ at $(-1, 1, 1)$.

Ans:- $\text{div } \bar{F} = \nabla \cdot \bar{F} = \frac{\partial}{\partial x}[F_1] + \frac{\partial}{\partial y}[F_2] + \frac{\partial}{\partial z}[F_3]$

$= \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3 z^2) + \frac{\partial}{\partial z}(xy^2 z)$

$= 2xz - 6y^2 z^2 + xy^2$

$(\text{div } \bar{F})_{\text{at }(-1,1,1)} = 2(-1)(1) - 6(1)^2(1)^2 + (-1)(1)^2$

$= -2 - 6 - 1$

$= -9.$

$\text{div } \bar{F} = -9$ at $(-1, 1, 1)$

5) If $\bar{f} = (ax + 3y + 4z)\bar{i} + (x - 3y + 3z)\bar{j} + (3x + 2y - z)\bar{k}$ is Solenoidal. Find a constant 'a'.

Ans:- Solenoidal : $\nabla \cdot \bar{f} = 0$.

$\Rightarrow \frac{\partial}{\partial x}(ax + 3y + 4z) + \frac{\partial}{\partial y}(x - 3y + 3z) + \frac{\partial}{\partial z}(3x + 2y - z) = 0$

$\Rightarrow a - 3 - 1 = 0$

$\boxed{a = 4}$ //

Unit - II

1. What is meant by Scalar potential of vector function?

Ans: $\bar{F}$ is Conservative if $\bar{F} = \nabla\phi$
Hence ϕ is called Scalar potentia

2. State physical interpolation of $\int \bar{F}.d\bar{r}$.

Sol: Line $\int_A^B \bar{F}.d\bar{r}$ is total work done by the force $\bar{F}$. A during a displacement from A to B.

3. Define Surface integral.
Let S be a Surface whose projection R_{xy} on xy plane is such that the points on S have a one to one on R_{xy}.

$$\int_S \bar{F}.d\bar{s} = \int_S \bar{F}.\hat{n}\,ds = \iint_{R_{xy}} \bar{F}.\hat{n}\,\frac{dx\,dy}{|\bar{n}.\bar{k}|}$$

$|||^{ly}$ $$\int_S \bar{F}.\hat{n}\,ds = \iint_{R_{yz}} \bar{F}.\hat{n}\,\frac{dy\,dz}{|\bar{n}.\bar{i}|}$$

$$\int_S \bar{F}.\hat{n}\,ds = \iint_{R_{xz}} \bar{F}.\hat{n}\,\frac{dx\,dz}{|\bar{n}.\bar{j}|}$$

4) If $\bar{f} = x^2\bar{i} - xy\bar{j}$ & C is straight line joining (0,0) & (1,1). Find $\int_C \bar{F}.d\bar{r}$.

Sol: $\bar{f} = x^2\bar{i} - xy\bar{j}$, $\bar{r} = x\bar{i} + y\bar{j}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

$\bar{F}.d\bar{r} = x^2 dx - xy\,dy$

Straight line $y = mx + c$

Take $y = x \Rightarrow dy = dx$.

$\because 0 < x < 1,$

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 (x^2 dx - x^2 dx)$$
$$= 0$$

$$\therefore \boxed{\int_C \bar{F}\cdot d\bar{r} = 0} \text{ ——— } X.$$

5) If $\bar{u} = t\bar{i} + 2t\bar{j} + 3t\bar{k}$ & $\bar{v} = t^2\bar{i} + t^3\bar{j} + t^4\bar{k}$

Find $\int_0^1 \bar{u}.\bar{v}\, dt$.

$$\bar{u}.\bar{v} = (t\bar{i} + 2t\bar{j} + 3t\bar{k}) \cdot (t^2\bar{i} + t^3\bar{j} + t^4\bar{k})$$
$$= t^3 + 2t^4 + 3t^5.$$

$$\therefore \int_0^1 \bar{u}.\bar{v}\, dt = \int_0^1 [t^3 + 2t^4 + 3t^5]\, dt$$

$$= \left[\frac{t^4}{4} + \frac{2t^5}{5} + \frac{3t^6}{6}\right]_0^1$$

$$= \frac{1}{4} + \frac{2}{5} + \frac{\cancel{3}}{\cancel{6}_2}$$

$$= \frac{5 + 8 + 10}{20}$$

$$\int_0^1 \bar{u}.\bar{v}\, dt = 23/20 \,//\, \text{ ——— } x$$

——— x ———

Unit - III

1. Prove that a Closed Surface S, $\iint_S \bar{r}\cdot\bar{n}\, ds = 3V$. where V is volume enclosed by S.

Ans. $\iint_S \bar{r}\cdot\hat{n}\, ds = \iiint_V (\nabla\cdot\bar{r})\, dv$.

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

$\nabla\cdot\bar{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1$

$\nabla\cdot\bar{r} = 3$.

$\iint_S \bar{r}\cdot\hat{n}\, ds = \iiint_V 3\, dv = 3V$.

$$\boxed{\iint_S \bar{r}\cdot\hat{n}\, ds = 3V}$$

2. Define volume integral.

Ans. Define $f(P_r) = f(x_r, y_r, z_r)$ the sum $\sum_{r=1}^{n} f(P_r)\,\delta V_r$. Taking Limits when $n\to\infty$ such that Largest of volumes $\delta V_r \to 0$ it exists. Then $\iiint_V f(x,y,z)\, dv$ is called volume integral of $f(x,y,z)$ over V.

3. State Green's theorem.

If R is a closed region of the XOY plane bounded by a simple closed curve C & if M & N are continuous function of x & y having continuous derivatives in R, Then,

$$\oint_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

(where C – Transversed in anticlock direction). — x

4, State Gauss Divergence Theorem:

The Surface integral of normal component of a vector function $\bar{F}$ taken over a closed surface S enclosing a Volume V is equal to the volume integral of divergence of $\bar{F}$ taken throughout the volume V.

$$\iint_S \bar{F} \cdot ds = \iiint_V \operatorname{div} \bar{F}\, dv$$ — x

5, State Stoke's Theorem:-

If $\bar{F}$ is any Continuous differentiable vector function & S is a surface enclosed by a curve C. Then

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S \operatorname{curl} \bar{F} \cdot \hat{n}\, ds$$

where $\hat{n}$ - unit normal vector at any point of S.

— x

— x —

Unit - IV Fourier Series

1. Find Fourier Constant a_0 for function $f(x) = k$ in $(0, 2\pi)$.

Soln.
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$
$$= \frac{1}{\pi}\int_0^{2\pi} k\,dx = \frac{k}{\pi}[x]_0^{2\pi} = \frac{k}{\pi}[2\pi - 0]$$
$$\boxed{a_0 = 2k}$$

2. Write any two Conditions for fourier Series converges:

Dirichlet Conditions

1. $f(x)$ is well defined & Single valued except possibly at a finite no of points.
2. $f(x)$ has only a finite no. of finite discontinuous & no infinite discontinuity
3. $f(x)$ has only a finite no. of maxima & minima.

3. Define Fourier Series:-
$$f(x) = a_0 + \sum_{n=1}^{n}[a_n \cos nx + b_n \sin nx]$$

where $a_0, a_1, \ldots a_n$ & $b_1, b_2 \ldots b_n$ are constant Co-efficient of the Series.

4. Find fourier Coefficient a_0. fun $f(x) = \frac{1}{2}(\pi - x)$, $0 < x < 2\pi$.

Soln.
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$
$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\,dx = \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$
$$= \frac{1}{2\pi}\left[2\pi^2 - \frac{4\pi^2}{2}\right] = \frac{1}{2\pi}\left[2\pi^2 - 2\pi^2\right]$$
$$\boxed{a_0 = 0}$$

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5. Define odd & even function:

Even function:

If $f(-x) = f(x)$. Then function of x is called even function.

Ex: $\cos x$, $\cos^2 x$, x^2, $x^2 - \cos x$ ... etc

Odd function:-

If $f(-x) = -f(x)$ is said to be odd fun of x.

Ex: $\sin x$, x^3, $x^3 + \sin x$ ... etc, — o

— x —

Unit - V

1. Define half range Sine Series in the interval $(0, \pi)$.

Ans: Let $f(x)$ be the fourier Series of an interval $(0,\pi)$. If the fun, $f(x)$ is odd. Then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$.

2. Define half range Cosine Series in interval $(0,\pi)$

Ans A function $f(x)$ on an interval $(0,\pi)$ if $f(x)$ is even-periodicity. Then the Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$,

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx . dx.$$

Range $(0, l)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right).$$

$$a_0 = \frac{2}{l}\int_0^{l} f(x) dx, \quad a_n = \frac{2}{l}\int_0^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx.$$

3. Find Sine Series $f(x) = x$ in $0 < x < \pi$.

Sol: Sine Series $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$\int u\,dv = uv - u'v_1 + \dots$

$u = x$ $\quad dv = \sin nx$

$u' = 1$ $\quad v = -\frac{\cos nx}{n}$

$u'' = 0$ $\quad v_1 = -\frac{\sin nx}{n^2}$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx.$$

$$= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n} + 0 + \frac{(0-0)}{n^2}\right]$$

$$\boxed{b_n = \frac{2}{n}(-1)^{n+1}}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

$$= 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$$

4. Evaluate $\int_0^{\pi} \sin mx \cdot \sin nx \, dx$ if $m = n$.

Sol: If $m = n$ $\therefore \int_0^{\pi} \sin^2 mx \, dx = \frac{1}{m}\left[\frac{x}{2} - \frac{1}{4}\sin 2mx\right]_0^{\pi}$

$$= \frac{1}{m}\left[\frac{\pi}{2} - \frac{1}{4}\sin 2m\pi - 0\right]$$

$$\boxed{\int_0^{\pi} \sin mx \sin nx \, dx = \frac{\pi}{2m}}$$

5) If $f(x) = x^3$, $-\pi < x < \pi$. Find a_0

Sol: $f(-x) = (-x)^3 = -x^3 = -f(x)$.

$\Rightarrow f(-x) = -f(x)$

$\therefore f(x)$ is odd function.

Then $a_0 = a_n = 0$.

$$\boxed{a_0 = 0}$$

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