Complex Analysis

UNIT IV

Taylor Series

Formula:

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \dots$$

5 Marks

1. Expand $\cos z$ into a Taylor's series about the point $z = \pi/2$ and determine the region of convergence.

Solution:

W.K.T.

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \dots$$

Let $f(z) = \cos z$.

$z_0 = \pi/2$.

$$\therefore \cos z = f(\pi/2) + \frac{f'(\pi/2)}{1!}(z-\pi/2) + \frac{f''(\pi/2)}{2!}(z-\pi/2)^2 + \dots \quad \to (1)$$

$f(z) = \cos z \Rightarrow f(\pi/2) = \cos \pi/2 = 0$

$f'(z) = -\sin z \Rightarrow f'(\pi/2) = -\sin \pi/2 = -1$

$f''(z) = -\cos z \Rightarrow f''(\pi/2) = -\cos \pi/2 = 0$

$f'''(z) = -(-\sin z)$

$= \sin z \Rightarrow f'''(\pi/2) = \sin \pi/2 = 1$

$\vdots \qquad \vdots$

By Substituting all the above values in ①, we get

$$\cos z = 0 + \frac{(-1)}{1!}(z-\pi/2) + \frac{0}{2!}(z-\pi/2)^2 + \frac{1}{3!}(z-\pi/2)^3 + \cdots$$

$$\cos z = -\frac{(z-\pi/2)}{1!} + \frac{(z-\pi/2)^3}{3!} - \frac{(z-\pi/2)^5}{5!} + \cdots$$

The expansion is valid throughout the complex plane.

2. Expand $f(z) = \sin z$ in a Taylor's series about $z = \pi/4$ and determine the region of convergence of this series.

Solution:

Here $f(z) = \sin z$, $z_0 = \pi/4$.

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots$$

$$\sin z = f(\pi/4) + \frac{f'(\pi/4)}{1!}(z-\pi/4) + \frac{f''(\pi/4)}{2!}(z-\pi/4)^2 + \cdots \quad \text{①}$$

$f(z) = \sin z \Rightarrow f(\pi/4) = \sin \pi/4 = 1/\sqrt{2}$.

$f'(z) = \cos z \Rightarrow f'(\pi/4) = \cos \pi/4 = 1/\sqrt{2}$.

$f''(z) = -\sin z \Rightarrow f''(\pi/4) = -\sin \pi/4 = -1/\sqrt{2}$.

$f'''(z) = -\cos z \Rightarrow f'''(\pi/4) = -\cos \pi/4 = -1/\sqrt{2}$.

By Substituting the above values in ①, we get

$$\sin z = 1/\sqrt{2} + \frac{1/\sqrt{2}}{1!}(z-\pi/4) - \frac{1/\sqrt{2}}{2!}(z-\pi/4)^2 - \frac{1/\sqrt{2}}{3!}(z-\pi/4)^3 + \cdots$$

$$\sin z = 1/\sqrt{2}\left[1 + \frac{(z-\pi/4)}{1!} - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right]$$

The expansion is valid in the entire complex plane

3. Expand ze^{2z} in a Taylor's Series about $z=-1$ and determine the region of convergence.

Solution:

Here $f(z) = ze^{2z}$.

$$f(z) = ze^{2z}$$

Multiply and $\div$ by e^2.

$$= ze^{2z} \cdot \frac{e^2}{e^2}$$

$$= z e^{2z} \cdot e^{2} \cdot e^{-2}$$

$$= z\, e^{2(z+1)} \cdot e^{-2}$$

$$= \frac{1}{e^2}\, z \cdot e^{2(z+1)}$$

$$= \frac{1}{e^2}\left[(z+1-1) \cdot e^{2(z+1)}\right] \quad \text{(Add and Sub 1)}$$

$$= \frac{1}{e^2}\left[(z+1)\, e^{2(z+1)} - e^{2(z+1)}\right]$$

$$= \frac{1}{e^2}\left[(z+1)\left\{1 + \frac{2(z+1)}{1!} + \frac{[2(z+1)]^2}{2!} + \cdots\right\} - \left\{1 + \frac{2(z+1)}{1!} + \frac{[2(z+1)]^2}{2!} + \cdots\right\}\right]$$

(Using the formula $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$)

$$= \frac{1}{e^2}\left[\left\{(z+1) + \frac{2(z+1)^2}{1!} + \frac{2^2(z+1)^3}{2!} + \cdots\right\} - \left\{1 + \frac{2(z+1)}{1!} + \frac{2^2(z+1)^2}{2!} + \frac{2^3(z+1)^3}{3!} + \cdots\right\}\right]$$

$$= \frac{1}{e^2}\left[(z+1) + \frac{2(z+1)^2}{1!} + \frac{2^2(z+1)^3}{2!} + \cdots - 1 - \frac{2(z+1)}{1!} - \frac{2^2(z+1)^2}{2!} - \frac{2^3(z+1)^3}{3!} - \cdots\right]$$

$$= \frac{1}{e^2}\left[-1 + \left(1-\frac{2}{1!}\right)(z+1) + \left(\frac{2}{1!} - \frac{2^2}{2!}\right)(z+1)^2 + \left(\frac{2^2}{2!} - \frac{2^3}{3!}\right)(z+1)^3 + \dots\right]$$

The expansion is valid throughout the complex plane.

10 Marks.

(1) Expand $f(z) = \frac{z-1}{z+1}$ as a Taylor's series (i) about the point $z=0$. (ii) about the point $z=1$.

Solution:

(i) $f(z) = \frac{z-1}{z+1}$

$= (z-1)(z+1)^{-1}$ $\boxed{(1+x)^{-1} = 1-x+x^2-x^3+\dots}$

$= (z-1)(1+z)^{-1}$

$= (z-1)(1-z+z^2-z^3+\dots)$ if $|z|<1$

$= z-z^2+z^3-z^4+\dots -1+z-z^2+z^3-\dots$

$= -1+2z-2z^2+2z^3+\dots$

(ii) $f(z) = \frac{z-1}{z+1}$

$= \frac{z-1}{z+1+1-1}$ (Add and sub 1 in the denominator)

$= \frac{z-1}{z+2-1}$

$= \frac{z-1}{2+z-1}$

$= \frac{z-1}{2\left(1+\frac{z-1}{2}\right)}$

$$= \frac{z-1}{2} \cdot \frac{1}{1+\frac{z-1}{2}}$$

$$\boxed{(1+x)^{-1} = 1-x+x^2-x^3+\dots}$$

$$= \frac{z-1}{2}\left[1+\frac{z-1}{2}\right]^{-1}$$

$$= \frac{z-1}{2}\left[1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3+\dots\right] \text{ if } \left|\frac{z-1}{2}\right|<1$$

$$= \frac{z-1}{2} - \frac{(z-1)^2}{2^2} + \frac{(z-1)^3}{2^3} - \frac{(z-1)^4}{2^4} + \dots$$

2. Find the Taylor's series to represent $\frac{z^2-1}{(z+2)(z+3)}$ in $|z|<2$.

Solution:

Given $\frac{z^2-1}{(z+2)(z+3)}$

Here the degree of the denominator is equal to the numerator. So we have to divide the numerator by the denominator.

So, $\frac{z^2-1}{(z+2)(z+3)}$ can be written as $\frac{z^2-1}{z^2+5z+6}$.

$$\begin{array}{r|l} & 1 \\ \hline z^2+5z+6 & \cancel{z^2}-1 \\ & \cancel{z^2}+5z+6 \\ \hline & -5z-7 \end{array}$$

$$\therefore \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{-5z-7}{(z+2)(z+3)}$$

$$= 1 - \left[\frac{5z+7}{(z+2)(z+3)}\right] \longrightarrow (A)$$

$$\frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3} \quad \text{(Partial fraction)} \longrightarrow (1)$$

$$\frac{5z+7}{\cancel{(z+2)}(z+3)} = \frac{A(z+3)+B(z+2)}{\cancel{(z+2)(z+3)}}$$

$$5z+7 = A(z+3)+B(z+2)$$

Put $z=-2$

$$5(-2)+7 = A(-2+3)+B(\cancel{-2+2})$$

$$-10+7 = A \Rightarrow \boxed{A=-3} \longrightarrow (2)$$

Put $z=-3$

$$5(-3)+7 = A(\cancel{-3+3})+B(-3+2)$$

$$-15+7 = -B$$

$$\cancel{-}8 = \cancel{-}B \Rightarrow \boxed{B=8} \longrightarrow (3)$$

Using (2) and (3) in (1), we get

$$\frac{5z+7}{(z+2)(z+3)} = \frac{-3}{z+2} + \frac{8}{z+3} \longrightarrow (B)$$

By Substituting (B) in (A), we get

$$\frac{z^2-1}{(z+2)(z+3)} = 1 - \left[\frac{-3}{z+2} + \frac{8}{z+3}\right]$$

$$= 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{2(1+z/2)} - \frac{8}{3(1+z/3)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\right]$$

$$- \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right]$$

$$= 1 + \frac{3}{2} - \frac{3}{2^2}\cdot z + \frac{3}{2\cdot 2^2}\cdot z^2 - \frac{3}{2\cdot 2^3}\cdot z^3 + \cdots$$

$$- \frac{8}{3} + \frac{8}{3^2}\cdot z - \frac{8}{3\cdot 3^2}\cdot z^2 + \frac{8}{3\cdot 3^3}\cdot z^3 - \cdots$$

$$= \left(1 + \frac{3}{2} - \frac{8}{3}\right) + \left(-\frac{3}{2^2} + \frac{8}{3^2}\right)z + \left(\frac{3}{2\cdot 2^2} - \frac{8}{3\cdot 3^2}\right)z^2 + \cdots$$

$$= \left(\frac{6+9-16}{6}\right) + \sum_{n=1}^{\infty} (-1)^{n+1}\left[\frac{8}{3^{n+1}} - \frac{3}{2^{n+1}}\right]z^n$$

$$= -\frac{1}{6} + \sum_{n=1}^{\infty} (-1)^{n+1}\left[\frac{8}{3^{n+1}} - \frac{3}{2^{n+1}}\right]z^n$$ and the expansion is valid in $|z| < 2$.

Laurent's Series

10 Marks

① Expand $\frac{-1}{(z-1)(z-2)}$ as a power series in z in the regions (i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$.

Solution:

Let $f(z) = \frac{-1}{(z-1)(z-2)}$

$$\frac{-1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \quad \text{[Partial fraction]} \to ①$$

$$\frac{-1}{(z-1)(z-2)} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$$-1 = A(z-2) + B(z-1)$$

Put $z = 2$

$$-1 = A(\cancel{2-2}) + B(2-1)$$

$$\boxed{-1 = B}$$

Put $z = 1$

$$-1 = A(1-2) + B(\cancel{1-1})$$

$$\cancel{-}1 = \cancel{-}A \Rightarrow \boxed{A = 1}$$

By Substituting the values of A and B in ①, we get

$$\frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

(i) The only points where $f(z)$ is not analytic are 1 and 2. Hence $f(z)$ is analytic in $|z|<1$.

Hence $f(z)$ can be expanded as a series in $|z|<1$.

$$\therefore f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{1}{-[1-z]} + \frac{1}{[2-z]}$$

$$= -(1-z)^{-1} + \frac{1}{2(1-z/2)}$$

$$= -(1-z)^{-1} + \frac{1}{2}(1-z/2)^{-1}$$

$$= -(1+z+z^2+z^3+\dots+z^n+\dots) + \frac{1}{2}\left(1+z/2+(z/2)^2+\dots+(z/2)^n+\dots\right)$$

$$= -(1+z+z^2+\dots+z^n+\dots) + \frac{1}{2}\left(1+z/2+\frac{z^2}{2^2}+\dots+\frac{z^n}{2^n}+\dots\right)$$

$$= \sum_{n=0}^{\infty}\left[-z^n + \frac{1}{2}\cdot\frac{z^n}{2^n}\right] = \sum_{n=0}^{\infty} z^n\left[-1+\frac{1}{2^{n+1}}\right]$$

$$\sum_{n=0}^{\infty}\left[\frac{1}{2^{n+1}} - 1\right]z^n$$

(ii) $f(z)$ is analytic in the annular region $1<|z|<2$ and hence can be expanded as a series in this region.

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{1}{z-1} - \frac{1}{-(2-z)}$$

$$= \frac{1}{z-1} + \frac{1}{2-z}$$

$$= \frac{1}{z(1-1/z)} + \frac{1}{2(1-z/2)}$$

$$= \frac{1}{z}(1-1/z)^{-1} + \frac{1}{2}(1-z/2)^{-1}$$

$$= \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\left(\frac{1}{z}\right)^n+\cdots\right] + \frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots+\left(\frac{z}{2}\right)^n+\cdots\right]$$

(since $1<|z|$, $|z|<2$

$\Rightarrow \frac{1}{|z|}<1$, $\frac{|z|}{2}<1$

$\Rightarrow \left|\frac{1}{z}\right|<1$, $\left|\frac{z}{2}\right|<1$)

$$= \left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots+\frac{1}{z^{n+1}}+\cdots\right] + \left[\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\cdots\frac{z^n}{2^{n+1}}+\cdots\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}.$$

(iii) $f(z)$ is analytic in $|z|>2$ $\Rightarrow \frac{|z|}{2}>1$

(ie) $\left|\frac{z}{2}\right|>1$

(ie) $\left|\frac{2}{z}\right|<1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{1}{z(1-1/z)} - \frac{1}{z(1-2/z)}$$

$$\frac{1}{z}(1-1/z)^{-1} - \frac{1}{z}(1-2/z)^{-1}$$

$$= \frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots+\left(\frac{1}{z}\right)^n+\cdots\right]$$

$$-\frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots+\left(\frac{2}{z}\right)^n+\cdots\right]$$

$$= \frac{1}{z}\left[\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots+\left(\frac{1}{z}\right)^n+\cdots\right)-\left(1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots+\left(\frac{2}{z}\right)^n+\cdots\right)\right]$$

$$= \frac{1}{z}\left[\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots+\frac{1}{z^n}+\cdots\right)-\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots\frac{2^n}{z^n}+\cdots\right)\right]$$

$$= \left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots+\frac{1}{z^{n+1}}+\cdots\right)-\left(\frac{1}{z}+\frac{2}{z^2}+\frac{2^2}{z^3}+\cdots+\frac{2^n}{z^{n+1}}+\cdots\right)$$

$$= \cancel{\frac{1}{z}}+\frac{1}{z^2}+\cdots+\frac{1}{z^{n+1}}+\cdots-\cancel{\frac{1}{z}}-\frac{2}{z^2}-\frac{2^2}{z^3}-\cdots-\frac{2^n}{z^{n+1}}-\cdots$$

$$= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\left[1-2^n\right].$$

2. Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent's series valid for (i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$ (iv) $|z-1|>1$ and (v) $0<|z-2|<1$.

Solution:

$$f(z) = \frac{z}{(z-1)(2-z)}$$

$$\frac{z}{(z-1)(2-z)} = \frac{A}{(z-1)} + \frac{B}{(2-z)} \longrightarrow (1)$$

$$\frac{z}{\cancel{(z-1)(2-z)}} = \frac{A(2-z)+B(z-1)}{\cancel{(z-1)(2-z)}}$$

$$z = A(2-z) + B(z-1)$$

Put $z = 2$

$$2 = A(2-\cancel{2}) + B(2-1) \Rightarrow \boxed{B=2}$$

Put $z = 1$

$1 = A(2-1) + B(1-1) \Rightarrow \boxed{A = 1}$

By substituting the values of A and B in ①, we get

$$f(z) = \frac{z}{(z-1)(2-z)} = \frac{1}{z-1} + \frac{2}{2-z}$$

(i) $|z| < 1$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$= \frac{1}{-(1-z)} + \frac{\cancel{2}}{\cancel{2}(1-z/2)}$$

$$= -(1-z)^{-1} + 1(1-z/2)^{-1}$$

$$= -(1+z+z^2+z^3+\cdots) + (1+(z/2)+(z/2)^2+(z/2)^3+\cdots)$$

$$= \cancel{1} - z - z^2 - z^3 - \cdots + \cancel{1} + z/2 + z^2/4 + z^3/8 + \cdots$$

$$= z(-1+1/2) + z^2(-1+1/4) + z^3(-1+1/8) + \cdots$$

$$= z(-1/2) + z^2(-3/4) + z^3(-7/8) + \cdots$$

$$= -z/2 - 3z^2/4 - 7z^3/8 - \cdots$$

(ii) $1 < |z| < 2$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$= \frac{1}{z(1-1/z)} + \frac{\cancel{2}}{\cancel{2}(1-z/2)}$$

$$= 1/z(1-1/z)^{-1} + (1-z/2)^{-1}$$

$1 < |z| < 2 \Rightarrow 1 < |z|$ and $|z| < 2$

$\Rightarrow \frac{1}{|z|} < 1 \Rightarrow \left|\frac{1}{z}\right| < 1$; $\Rightarrow \frac{|z|}{2} < 1 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots\right] + \left[1 + (z/2) + (z/2)^2 + (z/2)^3 + \dots\right]$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots + 1 + z/2 + (z/2)^2 + (z/2)^3 + \dots$$

(iii) $|z| > 2$

Hence $2 < |z|$

$$\Rightarrow \frac{2}{|z|} < 1 \Rightarrow \left|\frac{2}{z}\right| < 1 \text{ and hence } \left|\frac{1}{z}\right| < 1$$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$= \frac{1}{z\left[1-1/z\right]} + \frac{2}{-z\left[1-2/z\right]}$$

$$= 1/z\,(1-1/z)^{-1} - 2/z\,(1-2/z)^{-1}$$

$$= 1/z\left[1 + 1/z + 1/z^2 + 1/z^3 + \dots\right] - 2/z\left[1 + 2/z + (2/z)^2 + (2/z)^3 + \dots\right]$$

$$= 1/z + 1/z^2 + 1/z^3 + 1/z^4 + \dots - 2/z - (2^2/z^2) - (2^3/z^3) - (2^4/z^4) - \dots$$

$$= +1/z(1-2) + 1/z^2(1-2^2) + 1/z^3(1-2^3) + 1/z^4(1-2^4) + \dots$$

$$= 1/z(-1) + 1/z^2(1-4) + 1/z^3(1-8) + 1/z^4(1-16) + \dots$$

$$= -1/z - 3/z^2 - 7/z^3 - 15/z^4 - \dots$$

(iv) $|z-1| > 1$

Hence $\frac{1}{|z-1|} < 1$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$= \frac{1}{z-1} + \frac{2}{-(z-2)}$$

$$= \frac{1}{z-1} - \frac{2}{z-2}$$

$$= \frac{1}{z-1} - \frac{2}{z-1-1}$$

$$= \frac{1}{z-1} - \frac{2}{(z-1)\left[1-\frac{1}{z-1}\right]}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\left(1-\frac{1}{z-1}\right)^{-1}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\left[1+\frac{1}{z-1}+\left(\frac{1}{z-1}\right)^2+\left(\frac{1}{z-1}\right)^3+\cdots\right]$$

$$= \frac{1}{z-1} - \frac{2}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \cdots$$

$$= \frac{1}{z-1}(1-2) - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \cdots$$

$$= -\frac{1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \cdots$$

(v) $0 < |z-2| < 1$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$= \frac{1}{z-1-1+1} + \frac{2}{2-z}$$

$$= \frac{1}{z-2+1} + \frac{2}{-(z-2)}$$

$$= \frac{1}{1+(z-2)} - \frac{2}{z-2}$$

$$= \left[1+(z-2)\right]^{-1} - \frac{2}{z-2}$$

$$= -\frac{2}{z-2} + (1+z-2)^{-1}$$

$$f(z) = \frac{-2}{z-2} + \left[1-(z-2)+(z-2)^2-(z-2)^3+\cdots\right]$$

$$\left(\because (1+x)^{-1} = 1-x+x^2-x^3+\cdots\right)$$

$$= \frac{-2}{z-2} + 1-(z-2)+(z-2)^2-(z-2)^3+\cdots$$

3. If $f(z) = \frac{z+4}{(z+3)(z-1)^2}$ find Laurent's series expansions in (i) $0<|z-1|<4$ and (ii) $|z-1|>4$.

Solution:

$$f(z) = \frac{z+4}{(z+3)(z-1)^2}$$

$$\frac{z+4}{(z+3)(z-1)^2} = \frac{A}{z+3} + \frac{B}{z-1} + \frac{C}{(z-1)^2} \longrightarrow (1)$$

$$\frac{z+4}{(z+3)\cancel{(z-1)^2}} = \frac{A(z-1)^2 + B(z+3)(z-1) + C(z+3)}{(z+3)\cancel{(z-1)^2}}$$

Put $z=1$

$$1+4 = A(\cancel{1-1})^2 + B(1+3)\cancel{(1-1)} + C(1+3)$$

$$5 = 4C \Rightarrow \boxed{C = 5/4}$$

Put $z=-3$

$$-3+4 = A(-3-1)^2 + B\cancel{(-3+3)}(-3-1) + C\cancel{(-3+3)}$$

$$1 = (-4)^2 A \Rightarrow 1 = 16A \Rightarrow \boxed{A = 1/16}$$

Put $z=0$

$$0+4 = A(0-1)^2 + B(0+3)(0-1) + C(0+3)$$

$$= A(-1)^2 + B(-3) + C(3)$$

$$4 = A - 3B + 3C$$

$$= 1/16 - 3B + 3(5/4)$$

$$= 1/16 - 3B + 15/4$$

$$= \frac{1+60}{16} - 3B$$

$$A = \frac{61}{16} - 3B$$

$$3B = \frac{61}{16} - 4$$

$$3B = \frac{61-64}{16} = \frac{-3}{16}$$

$$\Rightarrow \boxed{B = -\frac{1}{16}}$$

By substituting the values of A, B and C in ① we get

$$f(z) = \frac{1/16}{z+3} - \frac{1/16}{z-1} + \frac{5/4}{(z-1)^2}$$

$$= \frac{1}{16(z+3)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

(i) $0 < |z-1| < 4$

$$\Rightarrow 0 < \frac{|z-1|}{4} < 1 \Rightarrow 0 < \left|\frac{z-1}{4}\right| < 1$$

$$f(z) = \frac{1}{16(z-1+4)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{16\left[4\left(1+\frac{z-1}{4}\right)\right]} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{64\left(1+\frac{z-1}{4}\right)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{64}\left(1+\frac{z-1}{4}\right)^{-1} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{64}\left[1 - \left(\frac{z-1}{4}\right) + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \cdots\right] - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{64} - \frac{1}{64}\left[\frac{z-1}{4} - \left(\frac{z-1}{4}\right)^2 + \cdots\right] - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{5}{4(z-1)^2} - \frac{1}{16(z-1)} + \frac{1}{64} - \frac{1}{64}\left[\frac{z-1}{4} - \left(\frac{z-1}{4}\right)^2 + \cdots\right]$$

(ii) $|z-1| > 4$

Hence $4 < |z-1| \Rightarrow \frac{4}{|z-1|} < 1 \Rightarrow \left|\frac{4}{z-1}\right| < 1$

$$f(z) = \frac{1}{16(z+3)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{16(z-1+4)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{16(z-1)\left[1+\frac{4}{z-1}\right]} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{16(z-1)}\left[1+\frac{4}{z-1}\right]^{-1} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{16(z-1)}\left[1-\left(\frac{4}{z-1}\right)+\left(\frac{4}{z-1}\right)^2-\left(\frac{4}{z-1}\right)^3+\cdots\right] - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \cancel{\frac{1}{16(z-1)}} - \frac{1}{\cancel{16}_4(z-1)}\cdot\frac{\cancel{4}}{z-1} + \frac{1}{\cancel{16}(z-1)}\cdot\frac{\cancel{16}}{(z-1)^2} - \frac{1}{\cancel{16}(z-1)}\cdot\frac{\cancel{64}^4}{(z-1)^3} + \cdots - \cancel{\frac{1}{16(z-1)}} + \frac{5}{4(z-1)^2}$$

$$= \frac{-1}{4(z-1)^2} + \frac{1}{(z-1)^3} - \frac{4}{(z-1)^4} + \cdots + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{4(z-1)^2}\left[-1+5\right] + \frac{1}{(z-1)^3} - \frac{4}{(z-1)^4} + \cdots$$

$$= \frac{1}{\cancel{4}(z-1)^2}(\cancel{4}) + \frac{1}{(z-1)^3} - \frac{4}{(z-1)^4} + \cdots$$

$$= \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \frac{4}{(z-1)^4} + \cdots$$

5 Marks

1. Find the Laurent's Series for $\frac{z}{(z+1)(z+2)}$ about $z=-2$

Solution:

Let $f(z) = \frac{z}{(z+1)(z+2)}$

$$\frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2} \longrightarrow (1)$$

$$z = A(z+2) + B(z+1)$$

Put $z = -2$

$$-2 = A(-2\cancel{+2}) + B(-2+1)$$

$$\cancel{+}2 = \cancel{+}B \Rightarrow \boxed{B=2}$$

Put $z = -1$

$$-1 = A(-1+2) + B(-1\cancel{+1})$$

$$\boxed{-1 = A}$$

By substituting the values of A and B in (1) we get

$$f(z) = \frac{z}{(z+1)(z+2)} = \frac{-1}{z+1} + \frac{2}{z+2}$$

$$= \frac{-1}{z+2+1-1} + \frac{2}{z+2}$$

$$= \frac{-1}{z+2-1} + \frac{2}{z+2}$$

$$= \frac{\cancel{-}1}{\cancel{-}[1-(z+2)]} + \frac{2}{z+2}$$

$$= [1-(z+2)]^{-1} + \frac{2}{z+2}$$

$$= 1 + (z+2) + (z+2)^2 + (z+2)^3 + \cdots + \frac{2}{z+2}$$

$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + (z+2)^3 + \cdots$$

(2.) Expand $\frac{1}{z^2-3z+2}$ in Laurent's series valid in the region $1<|z|<2$.

Solution:

Let $f(z) = \frac{1}{z^2-3z+2}$

$= \frac{1}{(z-1)(z-2)}$

$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)} \rightarrow (1)$

$1 = A(z-2) + B(z-1)$

Put $z=2$

$1 = A(\cancel{2-2}) + B(2-1) \Rightarrow \boxed{B=1}$

Put $z=1$

$1 = A(1-2) + B(\cancel{1-1}) \Rightarrow \boxed{A=-1}$

$\therefore (1) \Rightarrow f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$

$= \frac{1}{z-2} - \frac{1}{z-1}$

$f(z)$ is analytic in the region $1<|z|<2$;

$\Rightarrow 1<|z|, \quad |z|<2$

$\Rightarrow \frac{1}{|z|}<1, \quad \frac{|z|}{2}<1$

$\Rightarrow \left|\frac{1}{z}\right|<1, \quad \left|\frac{z}{2}\right|<1$

$\therefore f(z) = \frac{1}{-2(1-z/2)} - \frac{1}{z(1-1/z)}$

$= \frac{-1}{2}(1-z/2)^{-1} - 1/z\,(1-1/z)^{-1}$

$= -1/2\left[1+z/2+(z/2)^2+(z/2)^3+\dots\right] - 1/z\left[1+1/z+(1/z)^2+\dots\right]$

$= -1/2 \sum_{n=0}^{\infty} (z/2)^n - 1/z \sum_{n=0}^{\infty} (1/z)^n$

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$$= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{z^n}$$

$$= -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

3. Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z=1$ as a Laurent's series

Solution:

$$f(z) = \frac{e^{2z}}{(z-1)^3}$$

$$= \frac{e^{2z-2+2}}{(z-1)^3}$$ (Add and Sub 2 in the power of e)

$$= \frac{e^{2(z-1)+2}}{(z-1)^3}$$

$$= \frac{e^{2(z-1)}\cdot e^2}{(z-1)^3}$$

$$= \frac{e^2}{(z-1)^3}\cdot e^{2(z-1)}$$

$$= \frac{e^2}{(z-1)^3}\left[1 + \frac{2(z-1)}{1!} + \frac{[2(z-1)]^2}{2!} + \frac{[2(z-1)]^3}{3!} + \dots\right]$$

$$\left(\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots\right)$$

$$= \frac{e^2}{(z-1)^3}\left[1 + \frac{2(z-1)}{1!} + \frac{\cancel{4}^{\,2}(z-1)^2}{\cancel{2}} + \frac{\cancel{8}^{\,4}(z-1)^3}{\cancel{6}\,3} + \dots\right]$$

$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2(z-1)}{1!(z-1)^3} + \frac{2(z-1)^2}{(z-1)^3} + \frac{4(z-1)^3}{3(z-1)^3} + \frac{2^4(z-1)^4}{4!(z-1)^3} + \dots\right]$$

$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + \frac{2^4}{4!}(z-1) + \dots\right]$$