Kinematics of a Point

Dynamics is that science which deals with the study of motion of a body. The part of Dynamics which confines to the motion without any reference to the cause of the motion is termed kinematics.

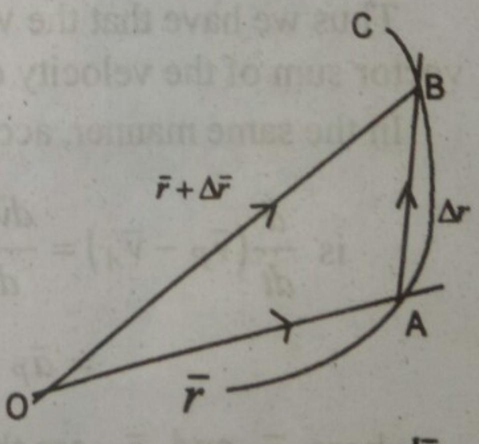
1.1 Displacement

If a particle at any particular instant is at a point P and at any subsequent instant comes to a point Q, then the length PQ is the displacement of the particle during that interval. Thus the displacement of a moving particle is the change of its position and is measured by the distance moved by it during a particular interval. Hence displacement is a vector.

1.2 Velocity

Let a point P, moving along a curre C be at A at time t, and at B at time $t+\Delta t$. Then $\overline{AB}$ is the displacement in the interval Δt. Then the ratio $\frac{\overline{AB}}{\Delta t}$ represents the displacement per unit of time. It is called the average velocity vector. We define the velocity vector of the moving point at time t as

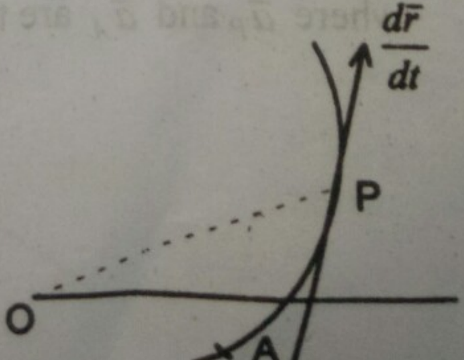
$$\bar{v} = \underset{\Delta t \to 0}{lt} \frac{\overline{AB}}{\Delta t} = \underset{\Delta t \to 0}{lt} \frac{d\bar{r}}{dt}$$



where $\Delta\bar{r} = \overline{AB} = \overline{OB} - \overline{OA}$

As $\Delta t \to 0$, the secant AB of the curve C tends to become the tangent and hence the velocity vector is always and only in the direction of the tangent to the path described by the moving point.

Remark :



if $AB = \Delta r = |\Delta\bar{r}|$

$$\therefore \left|\frac{d\bar{r}}{dt}\right| = \frac{dr}{dt}$$

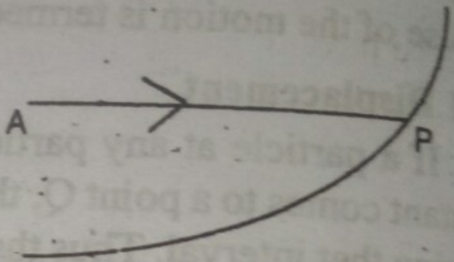
1.3 Acceleration

If $\bar{v}$ is the velocity vector at time t and $\bar{v}+\Delta\bar{v}$ at time $t+\Delta t$, then the ratio $\frac{\Delta\bar{v}}{\Delta t}$ represents the velocity change per unit of time and assuming $\lim_{\Delta t\to 0}\frac{\Delta\bar{v}}{\Delta t}$ exists, we define the acceleration $\bar{a}=\lim_{\Delta t\to 0}\frac{\Delta\bar{v}}{\Delta t}=\frac{d\bar{v}}{dt}=\frac{d^2\bar{r}}{dt^2}$

1.4 Relative velocity

Let A and P be two moving points then the position vector of P relative to A is $\overline{AP}$

$\therefore \frac{d}{dt}\overline{AP}$ is the velocity of P relative to A.



Taking a fixed point O

$$=\frac{d(\overline{AP})}{dt}=\frac{d}{dt}(\overline{OP}-\overline{OA})$$

$$=\frac{d(\overline{OP})}{dt}-\frac{d(\overline{OA})}{dt}$$

$$=\bar{V}_P-\bar{V}_A$$

where $\bar{V}_P$ and $\bar{V}_A$ are the velocities of P and A.

Thus we have that the velocity of a point P relative to another point A is the vector sum of the velocity of P and the reversed velocity of A.

In the same manner, accelaration of a particle P relative to a moving point A is $\frac{d}{dt}(\bar{v}_P-\bar{v}_A)=\frac{d\bar{v}_P}{dt}-\frac{d\bar{v}_A}{dt}$

$$=\bar{a}_P-\bar{a}_A$$

where $\bar{a}_P$ and $\bar{a}_A$ are the accelerations of P and A respectively.

1.5 Velocity and Acceleration of a particle moving along a straight line

Let the particle be at a distance x from a fixed point O on the straight line at time t.

Then $v = \frac{dx}{dt} = \dot{x}$

$$a = \frac{d^2x}{dt^2} = \ddot{x}$$

Remark :

We have $\ddot{x} = \frac{d(\dot{x})}{dt} = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = \frac{dv}{dx}.v$

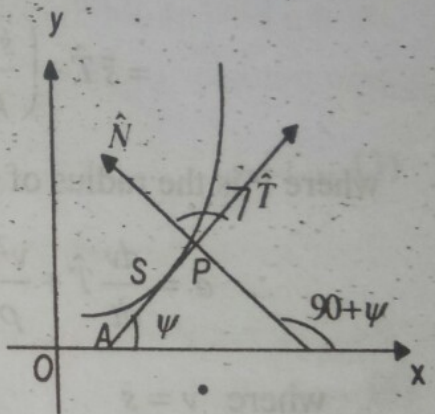
$$= v\,\frac{dv}{dx}$$

1.6 Velocity and acceleration in a coplanar motion

Book work: To find the components of the acceleration of a particle in the tangential and Normal directions.

Let P be the position of a particle on a curve. Let the arcual distance of P measured from a fixed point A be S.

Let $\hat{T}$ and $\hat{N}$ be the unit vectors along the tangent and normal at P. Let the tangent a P make an angle ψ with OX.



we know that $\bar{v} = v\hat{T} = \dot{s}\hat{T}$ .... (1)

$\hat{T}$ and $\hat{N}$ are not constant unit vectors since they change as P moves along the curve.

Now $\bar{T} = \cos\psi\, i + Sin\psi\, j$

$$\bar{N} = \cos(90+\psi)j + Sin(90+\psi)j$$

$$= -Sin\Psi i + Cos\Psi j$$

$$\frac{d\hat{T}}{dt} = -Sin\Psi\frac{d\Psi}{dt}i + Cos\Psi\frac{d\Psi}{dt}j$$

$$= \frac{d\Psi}{dt}(-Sin\psi i + Cos\psi j)$$

$$= \frac{d\psi}{dt}\hat{N} \qquad \text{.............. (2)}$$

From (1)

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d\left(\dot{s}\hat{T}\right)}{dt}$$

$$= \ddot{s}\hat{T} + \dot{s}\frac{d\hat{T}}{dt}$$

$$= \ddot{s}\hat{T} + \dot{s}\frac{d\psi}{dt}\hat{N} \text{ using (2)}$$

$$= \ddot{s}\hat{T} + \dot{s}\frac{d\psi}{ds}\frac{ds}{dt}\hat{N}$$

$$= \ddot{s}\hat{T} + \dot{s}\frac{1}{\rho}\dot{s}\hat{N}$$

$$= \ddot{s}\hat{T} + \left(\frac{\dot{s}^2}{\rho}\right)\hat{N}$$

where P is the radius of curvative of the curve at P.

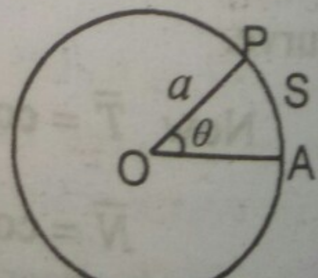
$$\therefore\ \bar{a} = \frac{dv}{dt}\hat{T} + \frac{v^2}{\rho}\hat{N}$$

where $v = \dot{s}$

Corrollary:

If the path of the particle is a circle of radius a and centre O and if $A\hat{O}P = \theta$, then $s = a\theta$, $v = \dot{s} = a\dot{\theta}$

$$\therefore\ \bar{v} = \dot{s}\hat{T} = a\dot{\theta}\hat{T}$$



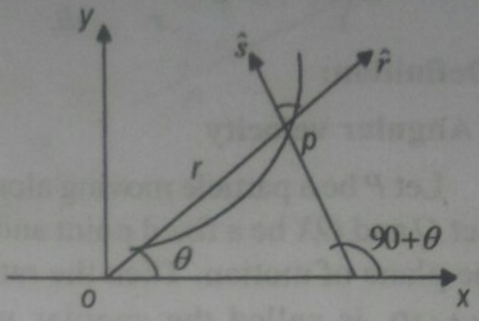
and $$\bar{a} = a\ddot{\theta}\hat{T} + \frac{\left(a\dot{\theta}\right)^2}{a}\hat{N}$$

$$= a\ddot{\theta}\hat{T} + a\dot{\theta}^2\hat{N}$$

1.7 Book Work:

To find the components of velocity and acceleration of a particle in the radial and transverse directions.

Let P be any point on a plane curve. Let O be the pole and OX be the initial line in the plane. Let (r, θ) be the polar coordinates of P.



Let $\bar{r}$ and $\hat{s}$ be the unit vectors along OP and perpendicular to OP. These unit vectors are clearly function of time. If i and j are the unit vectors along OX and OY we have

$$\hat{r} = \cos\theta i + \sin\theta j \quad \text{............ (1)}$$

$$\hat{s} = \cos(90+\theta)i + \sin(90+\theta)j$$

$$= -\sin\theta i + \cos\theta j \quad \text{............ (2)}$$

$$\frac{d\hat{r}}{dt} = -\sin\theta\,\dot{\theta}i + \cos\theta\,\dot{\theta}j$$

$$= \dot{\theta}\hat{s} \quad \text{............ (3)}$$

$$\frac{d\hat{s}}{dt} = -\cos\theta\,\dot{\theta}i - \sin\theta\,\dot{\theta}j$$

$$= -\dot{\theta}\hat{r} \quad \text{............ (4)}$$

$$\bar{r} = r\hat{r}$$

$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{d(r\hat{r})}{dt}$$

$$= \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \dot{r}\hat{r} + r\dot{\theta}\hat{s} \text{ using (3)}$$

$$\bar{a} = \frac{d\bar{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{s} + r\ddot{\theta}\hat{s} + r\dot{\theta}\frac{d\hat{s}}{dt}$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{s} + \dot{r}\dot{\theta}\hat{s} + r\ddot{\theta}\hat{s} + r\dot{\theta}(-\dot{\theta}\hat{r})$$

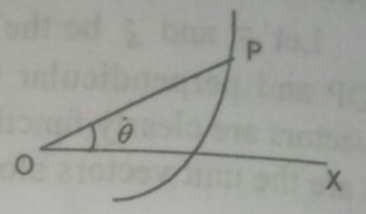
$$=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{r}+\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\hat{s}\right)$$

$$=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{r}+\frac{1}{r}\frac{d\left(r^2\dot{\theta}\right)\hat{s}}{dt}$$

Definition:

Angular velocity

Let *P* be a particle moving along plane curve. Let *O* and *OX* be a fixed point and a fixed line in the plane of motion. Then the rate of change of $\angle AOP$ is called the angular velocity of the particle about *O*.



i.e., if $\angle AOP=\theta$, then $\frac{d\theta}{dt}=\dot{\theta}$ is the angular velocity of the particle about O.

The unit of angular velocity is one radian per second

In the case of circular motion we have seen that the linear velocity *V* and angular velocity $\dot{\theta}$ about the centre are related as

$$\dot{\theta}=v/a$$

If the particle moved along a plane curve $\dot{\theta}$ can be obtained from the transverse component $r\dot{\theta}$ of the velocity by dividing it by *r*.

Thus we find that the angular velocity of a particle P about a point O is

$$\frac{\text{The Component of velocity of P perpendicular to OP}}{\text{OP}}$$

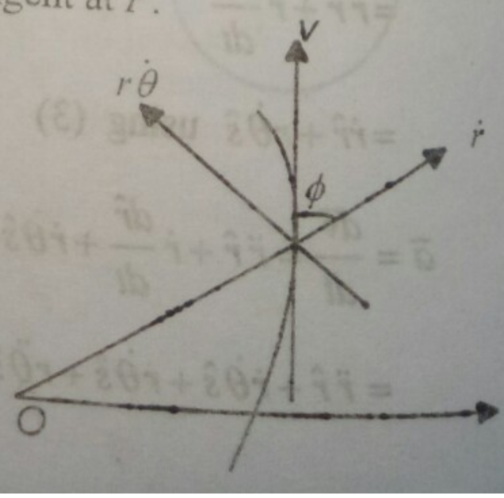
$$\therefore\ \dot{\theta}=\frac{v\sin\phi}{r}$$

where ϕ is the angle between *OP* and the tangent at *P*.

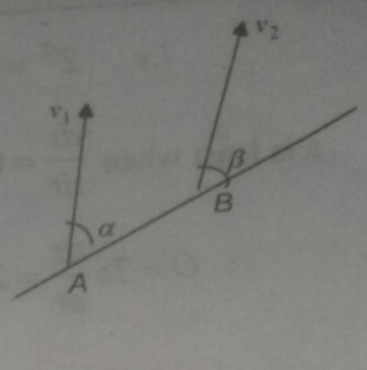
i.e., $r\dot{\theta}=v\sin\phi$

$$\dot{\theta}=\frac{v\sin\theta}{r}=\frac{rv\sin\phi}{r^2}$$

$$=\frac{\left|\bar{r}\times\bar{v}\right|}{r^2}$$



1.9 Relative angular velocity



Let A and B be two particles moving in a plane. If their velocities v_1 and v_2 make angles α and β with AB (See fig.), then the component in the direction perpendicular to AB of the velocity of B relative to A is $v_2 \sin\beta - v_1 \sin\alpha$.

$\therefore$ The angular velocity of B relative to A is

$$\frac{v_2 \sin\beta - v_1 \sin\alpha}{AB}$$

$$\dot{\theta} = \frac{\left|\overline{AB} \times (v_2 - v_1)\right|}{AB^2}$$

$$= \frac{\left|\overline{AB} \times (v_1 - v_2)\right|}{AB^2}$$

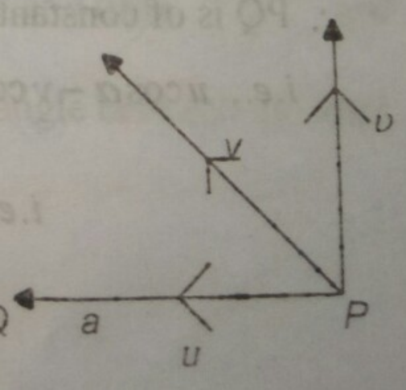
where $\overline{AB}$ is the P.V. of B relative to A.

Solved Problems

Example 1

The distance between two moving points at any time is a and their relative velocity is V, u and v being the components of V respectively along and perpendicular to the direction of a. Show that their distance when they are nearest to one another is $\frac{av}{V}$ and the time that elapses before they arrive at their nearest distance is $\frac{av}{V^2}$.

Solution :



Let P and Q be the positions of the two moving points when the distance between them is a.

i.e., QP=a. At any subsequent time t, coordinates of P relative to Q are $x=a-ut$, $y=vt$

$\therefore$ The distance between P, Q is

$Z = \sqrt{x^2 + y^2}$

$$i.e., \quad Z^2 = (a-ut)^2 + v^2t^2$$

z is least when $\frac{dz}{dt} = 0$

$$\therefore \; O = 2z\frac{dz}{dt} = 2(a-ut)(-u) + v^2 2t$$

$$i.e., \quad t = \frac{au}{u^2+v^2} = \frac{au}{V^2}$$

$$z^2 = \left[a - \frac{au^2}{u^2+v^2}\right]^2 + \frac{a^2u^2}{(u^2+v^2)^2}v^2$$

$$= \frac{a^2v^2}{(a^2+v^2)} = \frac{a^2v^2}{V^2}$$

$$Z = \frac{av}{V}.$$

Example 2

The line joining two points P and Q is of constant length a and the velocities of P and Q are in directions which make angles α and β with PQ. Prove that the angular velocity of PQ is $\frac{u\sin(\alpha-\beta)}{a\cos\beta}$ where u is the velocity of P.

Solution :

Let v be the velocity of Q. Then angular velocity of PQ about Q is

$$\frac{u\sin\alpha - v\sin\beta}{PQ}.$$

$\therefore$ PQ is of constant length, the relative velocity along PQ is zero.

$$i.e., \; u\cos\alpha - v\cos\beta = 0$$

$$i.e., v = \frac{u\cos\alpha}{\cos\beta}$$

$\therefore$ Hence angular velocity of PQ is

$$= \frac{u\sin\alpha - \dfrac{u\cos\alpha}{\cos\beta}\sin\beta}{a}$$

$$= \frac{u\sin(\alpha-\beta)}{a\cos\beta}.$$

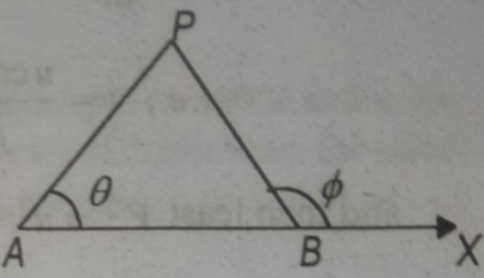
Example 3

If a point moves so that its angular velocities about two fixed points are the same, prove that it describes a circle.

Solution :

Let A and B be the fixed points and P a moving point.

Let AB be produced to X.



If $\angle XAP = \theta,\quad \angle XBP = \phi$ then it is given that

$$\frac{d\theta}{dt} = \frac{d\phi}{dt}$$

$$i.e., \quad \frac{d(\phi-\theta)}{dt} = 0$$

$$\therefore\ \phi - \theta = constant = c$$

$$i.e., \quad \angle APB = c$$

Hence P describes a circle.

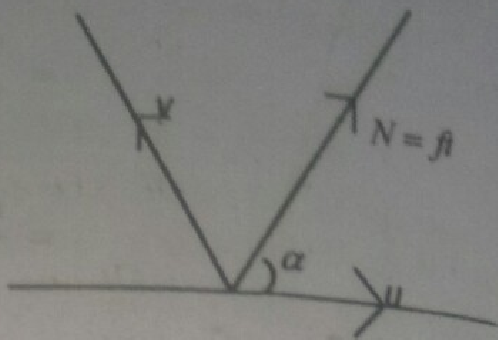
Example 4

Two particles start simultaneously from the same point and move along two straight lines one with uniform velocity u and the other from rest with uniform acceleration f. Show that their relative velocity is least after a time $\dfrac{u\cos\alpha}{f}$ and that the least relative velocity is $u\sin\alpha$ where α is the angle between the lines.

Solution :

The velocity of the second particle is $v = ft$.

If V be the relative velocity then



$$V^2 = u^2 + v^2 + 2uv\cos(180 - \alpha)$$

$$= u^2 + f^2t^2 - 2uft\cos\alpha$$

V is least when $\dfrac{dV}{dt} = 0$

$$\therefore\ 2V\frac{dV}{dt} = 2f^2t - 2uf\cos\alpha = 0$$

$$i.e.,\ t = \frac{u\cos\alpha}{f}$$

and then least $V^2 = u^2 + u^2\cos^2\alpha$

$$= -2u^2\cos^2\alpha$$

$$= u^2\sin^2\alpha$$

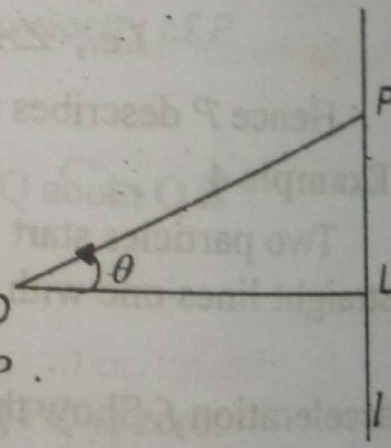
$$\therefore\ V = u\sin\alpha.$$

Example 5

Show that the angular velocity about a fixed point of a particle moving along a straight line varies in versely as the square of the distance from the point.

Solution :

Let P be the position of a particle at any time 't' on the line l. Let O be a fixed point about which the angular velocity is required. Draw OL perpendicular to l. Let $\angle LOP = \theta$. Let us assume that P starts from L with uniform velocity v.



Then $LP = vt$. From the right angled triangle ΔOLP.

$$\tan\theta = \frac{LP}{OL} = \frac{vt}{OL}$$

Differentiating w.r.t. t

$$\sec^2\theta\ \frac{d\theta}{dt} = \frac{v}{OL}$$

$$i.e.,\ \dot{\theta} = \frac{v}{OL}\cos^2\theta$$

$$= \frac{v}{OL}\left(\frac{OL}{OP}\right)^2$$

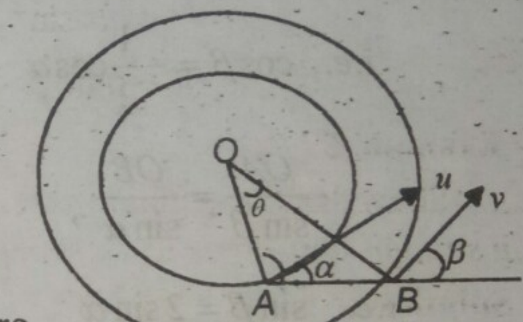
$$= \frac{v\,OL}{(OP)^2}$$

$$\alpha = \frac{1}{OP^2}.$$

Example 6

A and B describe concentric circles of radii a and b with speeds u and v, the motion being the same way round. If the angular velocity of either *w.r.t.* the other is zero, prove that the line joining them subtends at the centre an angle whose cosine is $\frac{au+bv}{av+bu}$.

Solution :



Let u and v make angles α and β with AB. $\angle AOB = \theta$.

Since the angular velocity of A w.r.t B is zero,

we have $v\sin\beta - u\sin\alpha = 0$ .... (1)

In ΔOAB, $\angle OAB = 90°+\alpha$

$\angle OBA = 90°-\beta$

By Projection formula

$$OA = OB\cos\theta - AB\cos(90°-\alpha)$$

$i.e.,\ a = b\cos\theta - AB\sin\alpha$ .... (2)

Similarly $b = a\cos\theta + AB\sin\beta$ .... (3)

Multiplying (2) by u and (3) by v and adding we get

$$au + bu = (bu + av)\cos\theta + AB(v\sin\beta - u\sin\alpha)$$

$$= (bu + av)\cos\theta \text{ using (1)}$$

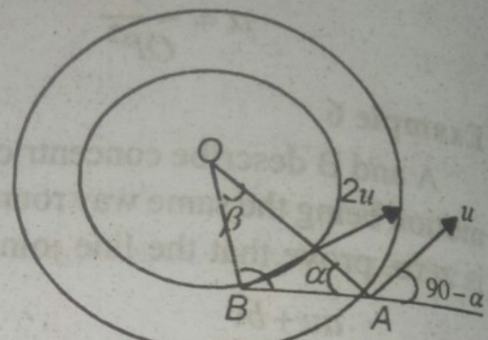
$$\therefore \cos\theta = \frac{au + bv}{av + bu}.$$

Example 7

Two points A and B move with speeds u and $2u$ in two concentric circles centre O and radius $2r$ and r respectively. If both of them move around the circles in the same sense and if $\alpha = \angle OAB$, when their relative motion is along AB, prove that $\cot\alpha = 2$.

Solution :

$OA = 2r$, $OB = r$. Let $\angle ABO = \beta$.



Since the relative motion is along AB, the relative velocity perpendicular to AB is zero.

$$u\sin(90-\alpha) - 2u\sin(\beta - 90) = 0$$

$$u\cos\alpha + 2u\cos\beta = 0$$

$$i.e., \cos\beta = -\frac{1}{2}\cos\alpha \quad \text{............ (1)}$$

$$\frac{OA}{\sin\beta} = \frac{OB}{\sin\alpha},$$

$$i.e., \sin\beta = 2\sin\alpha \quad \text{............ (2)}$$

squaring and adding (1) and (2) we get

$$1 = \frac{1}{4}\cos^2\alpha + 4\sin^2\alpha$$

$$i.e., 4\operatorname{cosec}^2\alpha = \cot^2\alpha + 16$$

$$4(1 + \cot^2\alpha) = \cot^2\alpha + 16$$

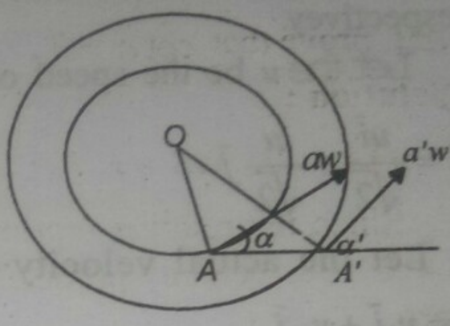
$$3\cot^2\alpha = 12$$

$$\cot^2\alpha = 4 \text{ or } \cot\alpha = 2.$$

Example 8

Two particles are describing concentric circles of radii a and a' with angular velocities w and w' respectively. Prove that the angular velocity of the line joining them when the length is r is $\dfrac{\left(r^2+a^2-a'^2\right)w+\left(r^2+a'^2-a^2\right)w'}{2r^2}$.

Solution :



Let O be the common centre and $A,\ A'$, the two paricles moving along the two circles with angular velocities w and w' respectively. Let $OA = a,\ OA' = a'$. Then the linear speeds of A and A' are $a\,w$ and $a'w'$ respectively and these make angles α and α' with AA'.

The relative angular velocity

of A' w.r.t $A = \dfrac{a'w'\sin\alpha' - aw\sin\alpha}{r}$

In $\Delta OAA'$ $\qquad \angle OAA' = 90+\alpha$

$$\angle OAA' = 90-\alpha'$$

Using cosine formula,

$$a^2 = a'^2 + r^2 - 2a'r\cos(90-\alpha')$$

$$= a'^2 + r^2 - 2a'r\sin\alpha'$$

$$\therefore a'\sin\alpha' = \frac{a'^2 + r^2 - a^2}{2r}$$

Similarly $a'^2 = a^2 + r^2 - 2ar\cos(90+\alpha)$

$$= a^2 + r^2 + 2ar\sin\alpha$$

$$a\sin\alpha = -\left(\frac{r^2 + a^2 - a'^2}{2r}\right)$$

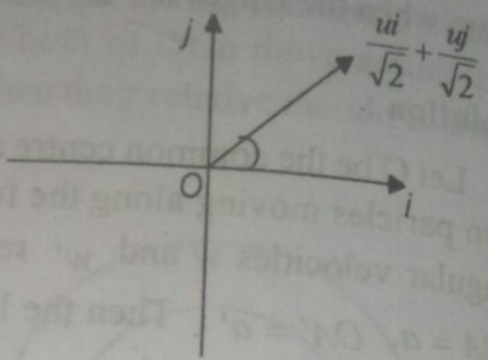
Substituting these values is (1) and Simplifying we get the result.

Example 9

A person travelling towards North-East finds that the wind appears to come from North, but on doubling his speed it seems to come from a direction inclined at an angle $\cot^{-1} 2$ on the East of North. Find the true velocity of the wind.

Solution :

Let $\bar{i}$ and $\bar{j}$ represent unit vectors in the Eastern and in the Northern directions respectivey.



Let the u be the speed of the man. Then

$\bar{u}=\frac{u\bar{i}}{\sqrt{2}}+\frac{u}{\sqrt{2}}\bar{j}$.

Let the actual velocity of the wind be $\bar{v}=v_1\bar{i}+v_2\bar{j}$.

So that the relative velocity is $\bar{v}-\bar{u}$

$$=\left(v_1-\frac{u}{\sqrt{2}}\right)\bar{i}+\left(v_2-\frac{u}{\sqrt{2}}\right)\bar{j}$$

This is given to be a vector from North. Hence $\bar{i}$ component is zero.

$$\therefore\ v_1-\frac{u}{\sqrt{2}}=0 \qquad \text{............ (1)}$$

In the second case (the speed of the man has been doubled) we have

$$\bar{u}=\frac{2u}{\sqrt{2}}\bar{i}+\frac{2u}{\sqrt{2}}\bar{j}$$

Relative velocity $=\bar{v}-\bar{u}$

$$=\left(v_1-\frac{2u}{\sqrt{2}}\right)\bar{i}+\left(v_2-\frac{2u}{\sqrt{2}}\right)\bar{j}$$

$$=\left(\frac{u}{\sqrt{2}}-\frac{2u}{\sqrt{2}}\right)\bar{i}+\left(v_2-\frac{2u}{\sqrt{2}}\right)\bar{j} \quad \text{(using (1))}$$

$$=\frac{-u}{\sqrt{2}}\bar{i}+\left(v_2-\frac{2u}{\sqrt{2}}\right)\bar{j}.$$

By problem, this makes an angle $\theta = \cot^{-1} 2$ with the north ($\bar{j}$ direction).

$$\therefore \tan\theta = \frac{-\frac{u}{\sqrt{2}}}{v_2 - \frac{2u}{\sqrt{2}}} = -\frac{1}{2} \qquad (\because \cot\theta = 2)$$

Solving, $v_2 = 0$

$\therefore$ Actual velocity of the wind is

$$v = v_1\bar{i} + v_2\bar{j} = \frac{u}{\sqrt{2}}\bar{i}$$

This is in the eastern direction.

Exercise

1. To a man walking due east at the rate of $4m/Sec$. the wind appears to come from north. To a cyclist riding at 12m / Sec. north east, the wind appears to come N 15° E. Find the velocity of wind.
2. A person travelling eastwards finds that the wind seems to blow from north; on doubling his speed it seems to come from north west. Show that if he trebled his speed, the wind would appear to blow from a direction making an angle θ to the north of east given by $\tan\theta = \frac{1}{2}$.
3. Two cars A and B are moving due north and due east at 40 kms per hour and 30 kms per hour respectively. At noon B is west of A at a distance of 20 kms. When are the cars nearest to each other and what is the distance between them at that time?
4. A ship B observes another ship A, 20 kms due west of B. A is steaming north at 40 kmph. After $\frac{6}{25}$ th of an hour, the ships are nearest to each other at a distance 16 kms. Find the velocity of B.
5. Two straight lines OX and OY are inclined to each other at an acute angle α. One car moves along XO with speed u while the second car moves along OY with speed v. If initially the first car is at a distance 'd' from O while the second is at O, show that the cars are at their least distance after a time $\frac{d(u+v\cos\alpha)}{u^2+v^2+2uv\cos\alpha}$ and ratio of their distances from O is

$v + u\cos\alpha$ /