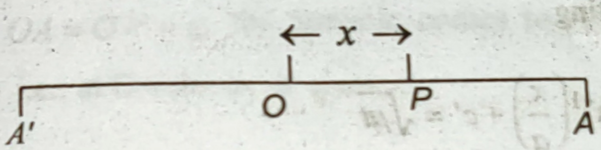
Chapter 4

Simple Harmonic Motion

Definition: A particle is said to execute simple harmonic motion if it moves in a straight line such that is acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point.



Let O be a fixed point on the line $A'OA$ and P be the position of the particle at time t where OP=x. So, that the acceleration of the particle in the sense OP is $\ddot{x}$.

$\therefore$ The equation of motion is

$$\ddot{x} = -\mu x \text{ where } \mu \text{ is a constant, taking } \ddot{x} = v\frac{dv}{dx}$$

$$\therefore\ v\frac{dv}{dx} = -\mu x \qquad \text{............ (1)}$$

Integrating *w.r.t.* x we get

$$\frac{v^2}{2} = \frac{-\mu x^2}{2} + c \quad \text{where } c \text{ is a constant}$$

If A be the extreme position of the particle *i.e.*, it is at rest at A *i.e.*, when

$x = a,\ v = 0$ where $OA = a$ we get $O = \dfrac{-\mu a^2}{2} + c \quad \therefore\ c = \dfrac{\mu a^2}{2}$

Hence $v^2 = \mu\left(a^2 - x^2\right)$

The distance $OA = a$ *i.e.*, the distance of the centre from one of the position of rest is called the amplitude.

The period is independent of the amplitude *i.e.*, whatever be the amplitude the period is the same.

The frequency is the number of complete oscillations in one second. If n be the frequency and T, the periodic time

$$n = \frac{1}{T} = \frac{\sqrt{\mu}}{2\pi}$$

Remark:

The solution of the equation $\ddot{x} = -\mu x$ namely

$$x = a\cos\sqrt{\mu t} \quad \text{is the simplest.}$$

This is because of the choice of initial conditions. In case $t = 0$ does not correspond to the instant when the particle is at the extreme position, then we have

$$\cos^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu t} + c'$$

Let $t = -t'$ when $x = a$ so that the origin of time is t' seconds after the particle is at the extreme position so that

$$O = -\sqrt{\mu t'} + c'$$

$$\text{Hence } \cos^{-1}\left(\frac{x}{a}\right) = \sqrt{\mu t} + \sqrt{\mu t'}$$

$$x = a\cos\left(\sqrt{\mu t} + \in\right)$$

where $\in = \sqrt{\mu t'}$ is called

the epoch of the S.H.M.

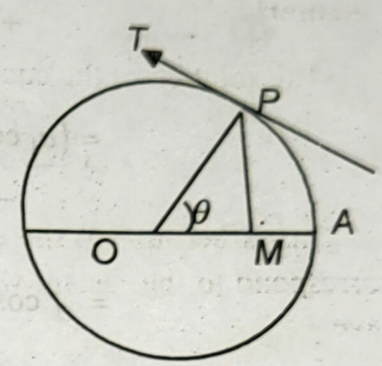
The phase of a S.H.M. is the time that has elapsed since the immediately previous instant when the particle is at the extreme position in the positive direction so that any time to phase is

$$t-(-t') = t + \frac{\epsilon}{\sqrt{\mu}}$$

$$= \frac{\sqrt{\mu}t + \epsilon}{\sqrt{\mu}}$$

The period remains the same and equal to $\frac{2\pi}{\sqrt{\mu}}$.

4.1 A geomentrical presentation of the S.H.M.



Let a particle P move on a circle with constant angular velocity w and let M be the foot of the perpendicular from P on any diameter OA. If a be the radius of the circle, the only acceleration of P towards O is $\omega^2 a$.

Let $\angle AOP = \theta$ and $OM = x$

the component of this accelartion along $OA = \omega^2 a \cos\theta$

$$= \omega^2 a \, \frac{x}{a} = \omega^2 x \text{ towards } O.$$

Hence the equation of motion of M is $\ddot{x} = -\omega^2 x$

This is clearly S.H.M.

Thus if a particle describes a circle with constant angular velocity, the foot of the perpendicular from it on any diameter executes a S.H.M.

4.2. Composition of two simple harmonic motions of the same period and in the same straight line.

Let $\frac{2\pi}{\sqrt{\mu}}$ be the common period. Let a_1, ϵ_1 and a_2, ϵ_2 be respectively the amplitudes and epochs of the two motions.

Let the displacements of the two motions be

$$x_1 = a_1 \cos\left(\sqrt{\mu}t + \epsilon_1\right) \qquad \text{............ (1)}$$

and $x_2 = a_2 \cos\left(\sqrt{\mu}t + \epsilon_2\right)$ .... (2)

The resultant displacement is

$$x = x_1 + x_2$$

$$= a_1 \cos\left(\sqrt{\mu}t + \epsilon_1\right) + a_2 \cos\left(\sqrt{\mu}t + \epsilon_2\right)$$

$$= a_1\left[\cos\sqrt{\mu}t \cos\epsilon_1 - \sin\sqrt{\mu}t \sin\epsilon_1\right] + a_2\left[\cos\sqrt{\mu}t \cos\epsilon_2 - \sin\sqrt{\mu}t \sin\epsilon_2\right]$$

$$= \left(a_1 \cos\epsilon_1 + a_2 \cos\epsilon_2\right)\cos\sqrt{\mu}t - \left(a_1 \sin\epsilon_1 + a_2 \sin\epsilon_2\right)\sin\sqrt{\mu}t$$

$$= A\cos\epsilon\cos\sqrt{\mu}t - A\sin\epsilon\sin\sqrt{\mu}t$$

$$= A\cos\left(\sqrt{\mu}t + \epsilon\right)$$

where

$$A\cos\epsilon = a_1\cos\epsilon_1 + a_2\cos\epsilon_2$$

$$A\sin\epsilon = a_1\sin\epsilon_1 + a_2\sin\epsilon_2$$

Squaring and adding

$$A^2 = \left(a_1\cos\epsilon_1 + a_2\cos\epsilon_2\right)^2 + \left(a_1\sin\epsilon_1 + a_2\sin\epsilon_2\right)^2$$

$$= a_1^2 + a_2^2 + 2a_1a_2\cos\left(\epsilon_1 - \epsilon_2\right) \quad \text{............ (3)}$$

Dividing

$$\tan\epsilon = \frac{a_1\sin\epsilon_1 + a_2\sin\epsilon_2}{a_1\cos\epsilon_1 + a_2\cos\epsilon_2} \quad \text{............ (4)}$$

Thus the composition is again a S.H.M. with the same period and its amplitu A and Epoch E given by equations (3) and (4).

4.3 Composition of two simple harmonic motions of the same period along two perpendicular lines.

Let the two perpendicular lines be taken as coordinate axes. Let $\frac{2\pi}{\sqrt{\mu}}$ be the common period.

Let for one of the motions the origin of time be at the extreme position so that the two displacements along the two perpendicular directions can be taken as

$$x = a\cos\sqrt{\mu}t \qquad \text{............ (1)}$$

$$y = b\cos\left(\sqrt{\mu}t + \in\right) \qquad \text{............ (2)}$$

a and b are the amplitudes

From (1)

$$\frac{x}{a} = \cos\sqrt{\mu}t$$

From (2) $\frac{y}{b} = \cos\sqrt{\mu}t \cos\in - \sin\sqrt{\mu}t \sin\in$

$$= \cos\in\frac{x}{a} - \sqrt{1-\frac{x^2}{a^2}}\sin\in$$

$$\frac{x}{a}\cos\in - \frac{y}{b} = \sqrt{1-\frac{x^2}{a^2}}\sin\in$$

Squaring and simplifying

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\in + \frac{y^2}{b^2} = \sin^2\in \text{ which is a conic.}$$

Note 1. If $\in = \frac{\pi}{2}$ and further a=b, the resultant motion will be a circular motion $x^2 + y^2 = a^2$.

2. If $\epsilon = 0$, the equation becomes $\frac{x^2}{a^2} + \frac{2xy}{ab} + \frac{y^2}{b^2} = 0$ which is a pair of concident lines.

3. If $\epsilon = \pi$, then again it represents a pair of concident lines given by $\frac{x^2}{a^2} + \frac{2xy}{ab} + \frac{y^2}{b^2} = 0$.

Solved Problems

Example 1

A particle is moving with S.H.M. and while making an excursion from one position of rest to the other, its distances from the middle point of its path at three consecutive seconds are observed to be x_1, x_2, x_3. Prove that the time of a complete oscillation is $2\pi \Big/ \cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)$.

Solution :

From $x = a\cos\sqrt{\mu}t$ we have

$$x_1 = a\cos\sqrt{\mu}t \qquad \text{............ (1)}$$

$$x_2 = a\cos\sqrt{\mu}(t+1) \qquad \text{............ (2)}$$

$$x_3 = a\cos\sqrt{\mu}(t+2) \qquad \text{............ (3)}$$

Adding (1) and (3)

$$x_1 + x_3 = a\left[\cos\sqrt{\mu}t + \cos\sqrt{\mu}(t+2)\right]$$

$$= 2a\cos\sqrt{\mu}(t+1)\cos\sqrt{\mu}$$

$$\frac{x_1 + x_3}{2} = x_2\cos\sqrt{\mu}$$

$$\therefore \cos\sqrt{\mu} = \frac{x_1 + x_3}{2x_2}$$

$$\sqrt{\mu} = \cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)$$

$$\text{Period } T = \frac{2\pi}{\sqrt{\mu}}$$

$$= \frac{2\pi}{\cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)}.$$

Example 2

A horizontal shelf moves vertically with S.H.M. of period 1 sec. What is the greatest amplitude that the shelf can have so that objects resting on it never leave it?

Solution :

$$\text{Given } \frac{2\pi}{\sqrt{\mu}} = 1$$

$$\therefore \ \mu = 4\pi^2$$

Weights rest on the shelf will be jerked off when the max acc. of SHM is greater than g and if it is not to be jerked off. max acc. of SHM must be g

$$\therefore \ \mu a = g$$

$$a = \frac{g}{\mu} = \frac{g}{4\pi^2}.$$

Example 3

A particle moves with S.H.M. in a straight line. In the first second after starting from rest, it travels a distance a metres and in the next second a distance b metres in the same direction. Prove that the amplitude of the motion is $\frac{2a^2}{3a - b}$.

Solution :

Since the particle starts from rest it is clear that at $t = 0$, it is at the extreme position. Hence displacement at any time t is

$$x = \alpha \cos \sqrt{\mu} t$$

where α is the amplitude

For $t = 1$ $\quad x = \alpha - a$

Similarly for $t = 2$

$$x = \alpha - (a + b)$$

$$\therefore \alpha - a = \alpha \cos \sqrt{\mu}$$

$$\alpha - (a + b) = \alpha \cos 2\sqrt{\mu}$$

$$= \alpha \left[2\cos^2 \sqrt{\mu} - 1\right]$$

$$= \alpha \left[2\left(\frac{\alpha - a}{\alpha}\right)^2 - 1\right]$$

$$\therefore \alpha - a - b = \frac{2}{\alpha}(\alpha - a)^2 - \alpha$$

$$\alpha^2 - \alpha a - \alpha b = 2\alpha^2 - 4\alpha a + 2a^2 - \alpha^2$$

$$3\alpha a - \alpha b = 2a^2$$

$$\alpha = \frac{2a^2}{3a - b}.$$

Example 4

A particle is performing S.H.M. of period T about a centre O, and it passes through a point P with velocity v in the direction OP. Show that the time which elapses before its return to P is $\frac{T}{\pi}\tan^{-1}\left\{\frac{vT}{2\pi OP}\right\}$.

Solution :

Let the origin of time concide with the origin of displacement.

$\therefore$ We have

$$x = a\sin\sqrt{\mu}t$$

$$T = \frac{2\pi}{\sqrt{\mu}} \Rightarrow \sqrt{\mu} = \frac{2\pi}{T}$$

$$\therefore\ x = a\sin\left(\frac{2\pi t}{T}\right)$$

$$\dot{x} = a\frac{2\pi}{T}\cos\left(\frac{2\pi t}{T}\right)$$

let $t = t_1$ when $x = OP$;

$\dot{x} = v$ and the particle is at P moving in the direction OP.

$$\therefore\ OP = a\sin\left(\frac{2\pi t_1}{T}\right)$$

$$v = a\frac{2\pi}{T}\cos\left(\frac{2\pi t_1}{T}\right)$$

$$\therefore\ \tan\left(\frac{2\pi t_1}{T}\right) = \frac{2\pi}{T}\frac{OP}{v}$$

$$\frac{2\pi t_1}{T} = \tan^{-1}\left(\frac{2\pi OP}{Tv}\right)$$

$$t_1 = \frac{T}{2\pi}\tan^{-1}\left\{\frac{2\pi OP}{vT}\right\}$$

Time for one-fourth oscillation is $\frac{T}{4}$

$\therefore$ Time taken to go from P to the extreme position is

$$= \frac{T}{4} - \frac{T}{2\pi}\tan^{-1}\left\{\frac{2\pi OP}{vT}\right\}$$

$$= \frac{T}{2\pi}\left[\frac{\pi}{2} - \tan^{-1}\left\{\frac{2\pi OP}{vT}\right\}\right]$$

$$= \frac{T}{2\pi}\cot^{-1}\left(\frac{2\pi OP}{vT}\right)$$

$$= \frac{T}{2\pi}\tan^{-1}\left(\frac{vT}{2\pi OP}\right)$$

$\therefore$ The required time

$$= 2 \times \frac{T}{2\pi}\tan^{-1}\left(\frac{vT}{2\pi OP}\right)$$

$$= \frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi OP}\right).$$

Example 5

If in a simple Harmonic motion u, v, w be the velocities at distances a, b, c from a fixed point on the straight line which is not the centre of force, show that the period T is given by the equation $\frac{4\pi^2}{T^2}(b-c)(c-a)(a-b) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$.

Solution :

Let μ be the intensity of force and λ be the amplitude. The velocity at a distance x from the centre of force is $v^2 = \mu\left(\lambda^2 - x^2\right)$

Let d be the distance of the fixed point from the centre of force.

Hence we get

$$u^2 = \mu\left[\lambda^2 - (a+d)^2\right]$$
$$v^2 = \mu\left[\lambda^2 - (b+d)^2\right]$$
$$w^2 = \mu\left[\lambda^2 - (c+d)^2\right]$$

and $T = \dfrac{2\pi}{\sqrt{\mu}}$ (or) $\mu = \dfrac{4\pi^2}{T^2}$

Thus $$\frac{u^2}{\mu} + a^2 + 2ad + d^2 - \lambda^2 = 0$$

$$\frac{v^2}{\mu} + b^2 + 2bd + d^2 - \lambda^2 = 0$$

$$\frac{w^2}{\lambda} + c^2 + 2cd + d^2 - \lambda^2 = 0$$

Eliminating d and $d^2 - \lambda^2$,

we get $$\begin{vmatrix} \frac{u^2}{\mu} + a^2 & a & 1 \\ \frac{v^2}{\mu} + b^2 & b & 1 \\ \frac{w^2}{\mu} + c^2 & c & 1 \end{vmatrix} = 0$$

(or) $$\frac{1}{\mu}\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix} + \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = 0$$

$$i.e., \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \mu(b-c)(c-a)(a-b)$$

substituting for μ we get the answer.

Example 6

The displacement x of a particle moving along a straight line is given by $x = A\cos nt + B\sin nt$ where A, B, n are constants. Show that its motion S.H. If $A = 3,\ B = 4, n = 2$ find its period amplitude, maximum velocity and maximum acceleration.

Solution :

$$x = A\cos nt + B\sin nt \qquad \text{............ (1)}$$

$$\dot{x} = -An\sin nt + Bn\cos nt \qquad \text{............ (2)}$$

$$\ddot{x} = -An^2\cos nt - Bn^2\sin nt$$

$$= -n^2(A\cos nt + B\sin nt)$$

$$= -n^2 x \qquad \text{............ (3)}$$

$\therefore$ The motion is S.H.

$$\text{Period} = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$$

$$\frac{dx}{dt} = 0 \text{ gives}$$

$$-An\sin nt + Bn\cos nt = 0$$

$$\tan nt = \frac{B}{A} = \frac{4}{3}$$

$$\therefore \text{ Amplitude} = \text{Maximum } x$$

$$= 3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right)$$

$$= \frac{9}{5} + \frac{16}{5} = \frac{25}{5} = 5 \text{ units}$$

Maximum velocity = $na = 10$

Maximum acceleration = $n^2 a = 20$.

Example 7

A particle executing a S.H.M. with O as the mean position and a as the amplitude. When it is at a distance $\frac{a}{2}$ from O, its velocity is quadrupled by a blow. Show that its new amplitude is $\frac{7a}{2}$.

Solution :

Let v and $4v$ be the velocities immediately before and after the blow and a_1 the new amplitude.

From $v^2 = n^2\left(a^2 - x^2\right)$

$$v^2 = n^2\left(a^2 - \left(\frac{a}{2}\right)^2\right)$$

and $(4v)^2 = n^2\left(a_1^2 - \left(\frac{a}{2}\right)^2\right)$

$$\therefore 16\,n^2\left(a^2 - \frac{a^2}{4}\right) = n^2\left(a_1^2 - \frac{a^2}{4}\right)$$

$$a_1^2 = \frac{49}{4}a^2 \qquad \therefore\ a_1 = \frac{7a}{2}.$$

Example 8

Show that a particle executing a S.H.M. requires one-sixth of its period to move from the position of maximum displacement to one in which the displacement in one-half the amplitude.

Solution :

Let $t = 0,\ x = a$

and $t = t_1$ when $x = \dfrac{a}{2}$

Then we have

From "$x = a\cos nt$"

$$\frac{1}{2}a = a\cos nt_1$$

$$nt_1 = \frac{\pi}{3}$$

$$t_1 = \frac{\pi}{3n} = \frac{1}{6}\left(\frac{2\pi}{n}\right) = \frac{1}{6}(\text{period})$$

Hence the problem.

Example 9

A point executes a S.H.M. such that in two of its positions on the same side of the mean position, its velocities are u, v and the corresponding accelerations are α and β. Show that the distance between these positions is $v^2 - u^2 / (\alpha + \beta)$.

Solution :

We have "$v^2 = n^2\left(a^2 - x^2\right)$"

When $x = d_1,\ v = u,\ \ddot{x} = \alpha$
$x = d_2,\ v = u,\ \ddot{x} = \beta$

$$\therefore\ u^2 = n^2\left(a^2 - d_1^2\right)$$

$$v^2 = n^2\left(a^2 - d_2^2\right)$$

$$v^2 - u^2 = n^2\left[d_1^2 - d_2^2\right]$$

$$= n^2(d_1 - d_2)(d_1 + d_2)$$

But $\alpha = n^2 d_1 \qquad \beta = n^2 d_2$

$$\therefore \; d_1 - d_2 = \frac{v^2 - u^2}{n^2 d_1 + n^2 d_2}$$

$$= \frac{v^2 - u^2}{\alpha + \beta}.$$

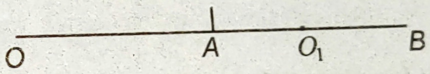
Example 10

A body moving in a straight line OAB with S.H.M. has zero velocity when at the points A and B whose distances from O are a and b respectively and has velocity v when half way between them. Show that the complete period is $\frac{\pi(b-a)}{v}$.

Solution :

Let a_1 be the amplitude of S.H.M. and O_1 be the centre of force.

O_1 is the midpoint of AB.



$$\therefore \; OO_1 = \frac{a+b}{2}$$

Amplitude $a_1 = AO_1$

$$= OO_1 - OA = \frac{a+b}{2} - a = \frac{b-a}{2}$$

We have $V^2 = \mu\left(a_1^2 - x^2\right)$

At the mid-point of AB, $V = v$ and $x = 0$

$$v^2 = \mu a_1^2$$

$$= \frac{\mu}{4}[b-a]^2$$

$$\therefore v = \frac{\sqrt{\mu}}{2}(b-a)$$

$$\text{Period} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\frac{2v}{b-a}}$$

$$= \frac{\pi(b-a)}{v}.$$

Example 11

Show that the resultant of two S.H.M's of the same period in the same straight line, the amplitudes of which are 3 metres and ametres respectively and the phase of the second a quarter of a period in advance of the first is another S.H.M. of the same period, whose amplitude is $3\sqrt{10}$ metres and whose phase is $\frac{1}{2\pi}\tan^{-1}(3)$ of a period in advance of the first.

Solution :

Let $T = \frac{2\pi}{\sqrt{\mu}}$ be the common period, and let us assume that the phase of the first be zero so that of the second is $\frac{\in}{\sqrt{\mu}} = \frac{T}{4} = \frac{2\pi}{4\sqrt{\mu}}$.

$$\therefore \in = \frac{\pi}{2}$$

Hence the two S.H.M's can be taken in the form

$$x_1 = 3\cos\sqrt{\mu}t$$

$$x_2 = 9\cos\left(\sqrt{\mu}t + \frac{\pi}{2}\right)$$

$$= -9\sin\sqrt{\mu}t$$

Since they are in the same straight line, the resultant motion is given by

$$X = x_1 + x_2$$

$$= 3\cos\sqrt{\mu}t - 9\sin\sqrt{\mu}t$$

This can be put in the form

$$X = A\cos\left(\sqrt{\mu}t + \epsilon\right)$$

where $A\cos\epsilon = 3$

$A\sin\epsilon = 9$

This is also S.H.M. of the same period with amplitude

$A = \sqrt{3^2 + 9^2} = 3\sqrt{10}$ metres and a phase in advance of the first

$$= \frac{\epsilon}{\sqrt{\mu}} = \frac{1}{\sqrt{\mu}}\tan^{-1}(3)$$

$$= \frac{T}{2\pi}\tan^{-1}(3)$$

$$= \frac{1}{2\pi}\tan^{-1}(3) \text{ of } T.$$

Exercise

1. The maximum velocity of a particle executing S.H.M is 1 metre/sec; and its period is $\frac{1}{5}$ of a second. Find the amplitude and maximum acceleration.

[Ans: $\frac{1}{10\pi}$ metre; 10π metre / sec^2]

2. Find the period and amplitude for each of the following S.H.M's described:

   (i) The maximum speed is 1 metre / sec and the maximum acceleration is 2 metre / sec^2.