Gauss's Divergence theorem :-

If $\vec{F}$ is a vector point function, finite and differentiable in a region R bounded by a closed surface S, then the surface integral of the normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ taken over V

$$\text{(i.e)} \iint_S \vec{F} \cdot \hat{n}\, dS = \iiint_V \nabla \cdot \vec{F}\, dv$$

where $\hat{n}$ is the unit vector in the positive

1. Verify the Gauss's divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x=0$, $x=1$, $y=1$, $z=0$, $z=1$

Soln:-

Gauss's divergence theorem is

$$\iint_S \vec{F} \cdot \hat{n} = \iiint_V \nabla \cdot \vec{F}\, dv$$

Given, $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$$\nabla \cdot \vec{F} = \left(\vec{i}\, \partial/\partial x + \vec{j}\, \partial/\partial x + \vec{k}\, \partial/\partial z\right) \cdot \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)$$

$$= \partial/\partial x\, (4xz) + \partial/\partial y\, (-y^2) + \partial/\partial z\, (yz)$$

$$= 4z - 2y + y$$

[$\because$ elementary value $dv = dx\,dy\,dz$]

$$\nabla \cdot \vec{F} = 4z - y$$

Now $\iiint_C \nabla \cdot \vec{F}\, dv = \int_0^1\int_0^1\int_0^1 (4z-y)\, dx\, dy\, dz$

$$= \int_0^1\int_0^1 \left[4zx - yx\right]_0^1 dy\, dz$$

$$= \int_0^1\int_0^1 (4z-y)\, dy\, dz$$

$$= \int_0^1 (4z - \tfrac{1}{2})\, dz.$$

$$= \left[4\frac{z^2}{2} - \tfrac{1}{2} z \right]_0^1$$

$$= \left[2z^2 - \tfrac{1}{2} z \right]_0^1$$

$$= 2 - \tfrac{1}{2} = \frac{4-1}{2}$$ (2)

$$\iiint_V \nabla \cdot \vec{F}\, dv = 3/2.$$

Now, $\iint_S \vec{F} \cdot \hat{n}\, dx = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

where S_1 = Face AEGD of the given cube. The unit normal vector to this surface is $\vec{i}$.

S_2 = Face OBFC of the given cube. the unit normal vector to this surface is $\vec{i}$

S_3 = Face EBFG of the given Cube. the unit normal vector to this surface is $-\vec{i}$

S_4 = Face OADC of the given Cube. the unit normal vector to this surface is $-\vec{j}$

S_5 = Face DGFC of the given Cube. The unit normal vector to this surface is $\vec{k}$

S_6 = Face DAEB of the given Cube. The unit normal vector to this surface is $-\vec{k}$

1. Evaluation of $\iint_{S_1} \vec{F} \cdot \hat{n}\, dx$

$$\iint_{S_1} \vec{F} \cdot \hat{n}\, ds = \iint_{AEGD} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \vec{i}\; dy\, dz.$$

∴ elemental area on this face is $dx = dy\, dz$]

$$= \iint_{AEGD} 4xz \, dy \, dz$$

[$\because$ $x = 1$ on this face]

$$= \int_0^1 \int_0^1 4z \, dy \, dz$$

$$= 4 \int_0^1 \left[\frac{z^2}{2} \right]_0^1 dy$$

$$= 4 \int_0^1 \tfrac{1}{2} \, dy$$

$$= 4 \left(\tfrac{1}{2}\right) [y]_0^1 = 2(1)$$

$$= 2 \quad \text{———} \; ①$$

2. Evaluation of $\iint_{S_2} \vec{F} \cdot \hat{n} \, ds$

$$\iint_{S_2} \vec{F} \cdot \hat{n} \, ds = \iint_{OBFC} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \, (-\vec{i}) \, dy \, dz$$

$$= \int_0^1 \int_0^1 -4xz \, dy \, dz$$

[$\because$ elemental area on this face is $ds = dy \, dz$]

$$= \int_0^1 \int_0^1 (0) \, dy \, dz$$

[$\because$ $x = 0$ in this face]

$$= 0 \quad \text{———} \; ②.$$

Evaluation of $\iint_{S_3} \vec{F} \cdot \hat{n} \, ds$

$$\iint_{S_3} \vec{F} \cdot \hat{n} \, ds = \iiint_{EBFG} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{j} \, dx \, dz.$$

[$\because$ Elemental area on this face is $ds = dx \, dz$]

$$= \int_0^1 \int_0^1 (-y^2) \, dx \, dz.$$

[$\because$ $y = 1$ on this face]

$$= \int_0^1 \int_0^1 (-1) \, dx \, dz$$

$$= \int_0^1 (-1) \, dz = [-z]_0^1$$

$$= -1 \quad \text{———} \; ③$$

Evaluation of $\iint_{S_4} \vec{F} \cdot \hat{n} \, ds.$

$$\iint_{S_4} \vec{F} \cdot \hat{n} \, ds = \iint_{OADC} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{j}) \, dx \, dz.$$

[$\because$ Elemental area on this face is $ds = dx \, dz$]

$$= \int_0^1 \int_0^1 (y^2)\, dx\, dz$$

$(\because y = 0$ on this face$)$

$$= \int_0^1 \int_0^1 0 \, dx\, dz$$

$$= 0$$

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Evaluation of $\iint_{S_5} \vec{F} \cdot \hat{n}\, ds$

$$\iint_{S_5} \vec{F} \cdot \hat{n}\, ds = \iint_{DGFC} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{k}\, dx\, dy$$

[$\because$ Elemental area on this face is $ds = dx\, dy$]

$$= \int_0^1 \int_0^1 (yz)\, dx\, dy$$

$(\because z = 1$ on this face$)$

$$= \int_0^1 \int_0^1 y\, dx\, dy$$

$$= \int_0^1 \left[y^2/2 \right]_0^1 dx$$

$$= \int_0^1 1/2 \, dx = 1/2 \, [x]_0^1 = 1/2 \quad \text{--- (5)}$$

Evaluation of $\iint_{S_6} \vec{F} \cdot \hat{n}\, ds$

$$\iint_{S_6} \vec{F} \cdot \hat{n}\, ds = \iint_{OAEB} (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot (-\vec{k})\, dx\, dy$$

$(\because$ Elemental area on this face is $ds = dx\,dy)$

$$= \int_0^1 \int_0^1 (-yz)\, dx\, dy$$

$$= \int_0^1 \int_0^1 0 \, dx\, dy$$

$$= 0 \quad \text{--- (6)}$$

(1) + (2) + (3) + (4) + (5) + (6) we get

$$\iint_{S_1} \vec{F} \cdot \hat{n}\, ds + \iint_{S_2} \vec{F} \cdot \hat{n}\, ds + \iint_{S_3} \vec{F} \cdot \hat{n}\, ds + \iint_{S_4} \vec{F} \cdot \hat{n}\, ds +$$

$$\iint_{S_5} \vec{F} \cdot \hat{n}\, ds + \iint_{S_6} \vec{F} \cdot \hat{n}\, ds = 2 + 0 - 1 + 0 - 1/2 + 0$$

$$= 2 - 1 + 1/2$$

$$= 1 + 1/2$$

$$\iint \vec{F} \cdot \hat{n}\, ds = 3/2$$

... divergence theorem is verified

1. Verify the Gauss's divergence theorem for $\vec{F} = 4xy\vec{i} + yz\vec{j} - xz\vec{k}$ over the cube bounded by $x=0, x=2, y=0, y=2, z=0, z=2$.

Soln:-

Gauss's divergence theorem is

(5)

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv.$$

$$\nabla\cdot\vec{F} = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot\left(4xy\vec{i} + yz\vec{j} - xz\vec{k}\right)$$

$$= \partial/\partial x\,(4xy) + \partial/\partial y\,(yz) - \partial/\partial z\,(xz)$$

$$\nabla\cdot\vec{F} = 4y + z - x.$$

Now, $\iiint_V \nabla\cdot\vec{F}\, dv = \int_0^2\int_0^2\int_0^2 (4y + z - x)\, dx\, dy\, dz.$

$$= \int_0^2\int_0^2 \left[4yx + zx - x^2/2\right]_0^2 dy\, dz.$$

$$= \int_0^2\int_0^2 (8y + 2z - 2)\, dy\, dz.$$

$$= \int_0^2 \left[\frac{8y^2}{2} + 2yz - 2y\right]_0^2 dz.$$

$$= \int_0^2 \left[4(2)^2 + 4z - 4\right] dz.$$

$$= \int_0^2 (16 + 4z - 4)\, dz.$$

$$= \int_0^2 (4z + 12)\, dz.$$

$$= \left[4z^2/2 + 12z\right]_0^2$$

$$= 2(2)^2 + 12(2) = 8 + 24$$

$$= 32$$

Now, $\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$ where,

S_1 = Face ~~EBFG~~ AEGD of the given Cube the unit normal vector to this Surface is $\vec{i}$

S_2 = Face OBFC of the given Cube the unit normal vector to this Surface is $-\vec{i}$

S_3 = Face EBFG of the given cube the unit normal vector to this surface is $\vec{j}$

S_4 = Face OADC of the given Cube the unit normal vector to this surface is $-\vec{j}$

S_5 = Face DGFC of the given Cube the unit normal vector to this surface is $\vec{k}$

~~S_6~~ = DAEB

S_6 = face OAEB of the given cube. the unit ~~f~~ normal vector to this surface is $-\vec{k}$

(6)

Evaluation of $\iint_{S_1} \vec{F} \hat{n} \, ds$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, ds = \int\int_{AEGD} (4xy\vec{i} + yz\vec{j} - x^2\vec{k})\cdot\vec{i}\, dy\, dz$$

$$= \int_0^2 \int_0^2 4(2)\, y\, dy\, dz$$

$$= \int_0^2 \int_0^2 8y\, dy\, dz.$$

$$= 8 \int_0^2 \left[y^2/2 \right]_0^2 dz.$$

$$= 8/2 \int_0^2 4\, dz = 32/2 \int_0^2 dz.$$

$$= 16 [z]_0^2$$

$$= 16(2)$$

(1)

Evaluation of $\iint_{S_2} \vec{F}\cdot\hat{n}\, ds$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\, ds = \iint_{OBFC} (4xy\vec{i} + yz\vec{j} - xz\vec{k})\cdot(-\vec{i})\, dy\, dz$$

$$= \int_0^2\int_0^2 (-4xy)\, dy\, dz$$ [∵ Elemental area on this face is $ds = dy\,dz$]

$$= \int_0^2\int_0^2 0\, dy\, dz$$ [∵ $x=0$ on this face]

$$= 0 \quad \text{——} (2)$$

Evaluation of $\iint_{S_3} \vec{F}\cdot\hat{n}\, ds$.

$$\iint_{S_3} \vec{F}\cdot\hat{n}\, ds = \iint_{EBFG} (4xy\vec{i} + yz\vec{j} - xz\vec{k})\cdot\vec{j}\, dx\, dz$$

[∵ Elemental area on this face is $ds = dx\,dz$]

$$= \int_0^2\int_0^2 (yz)\, dx\, dz$$

$$= \int_0^2\int_0^2 2z\, dx\, dz$$ [∵ $y=2$ on this face]

$$= 2\int_0^2 \left[\frac{z^2}{2}\right]_0^2 dx$$

$$= \frac{2}{2}\int_0^2 4\, dx = 4[x]_0^2 = 4(2)$$

$$= 8 \quad \text{——} (3)$$

Evaluation of $\iint_{S_4} \vec{F}\cdot\hat{n}\, ds$.

$$\iint_{S_4} \vec{F}\cdot\hat{n}\, ds = \iint_{OADC} (4xy\vec{i} + yz\vec{j} - xz\vec{k})\cdot(-\vec{j})\, dx\, dz$$

[∵ elemental area on this face is $ds = dx\,dz$]

$$= \int_0^2\int_0^2 -yz\, dx\, dz$$

$$= \int_0^2\int_0^2 0\, dx\, dy$$ [∵ $y=0$ on this face]

$$= 0 \quad \text{——} (4)$$

Evaluation of $\iint_{S_5} \vec{F}\cdot\hat{n}\, ds$.

$$\iint_{S_5} \vec{F}\cdot\hat{n}\, ds = \iint_{DGFC} (4xy\vec{i} + yz\vec{j} - xz\vec{k})\cdot\vec{k}\, dx\, dy$$

[∵ Elemental area on this face is $ds = dx\,dy$]

$$= \int_0^2\int_0^2 (-xz)\, dx\, dy$$

$$= \int_0^2\int_0^2 (-2x)\, dx\, dy$$ [∵ $z=2$ on this face]

$$= -2\int_0^2 \left(\frac{x^2}{2}\right)_0^2 dy$$

$$= -\int_0^2 4\, dy = -4(y)_0^2 = -8 \quad \text{——} (5)$$

Evaluation of $\iint_{S_6} \vec{F}\cdot\hat{n}\, ds$.

$$\iint_{S_6} \vec{F}\cdot\hat{n}\, ds = \int_0^2\int_0^2 (4xy\vec{i} + yz\vec{j} - xz\vec{k})\cdot(-\vec{k})\, dx\, dy$$

$$= \int_0^2\int_0^2 (xy)\, dx\, dy.$$ [∴ Elemental area on this face is $ds = dx\, dy$]

$$= \int_0^2\int_0^2 0\, dx\, dy.$$ [∵ $z = 0$ on this face]

$= 0$ —— ⑥

① + ② + ③ + ④ + ⑤ + ⑥ we get.

$$\iint_{S_1} \vec{F}\cdot\hat{n}\, ds + \iint_{S_2} \vec{F}\cdot\hat{n}\, ds + \iint_{S_3} \vec{F}\cdot\hat{n}\, ds + \iint_{S_4} \vec{F}\cdot\hat{n}\, ds - \iint_{S_5} \vec{F}\cdot\hat{n}\, ds$$

$$+ \iint_{S_6} \vec{F}\cdot\hat{n}\, ds = 32 + 0 + 8 + 0 - 8 + 0$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

∴ Gauss's divergence theorem is verified.

2. Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ if $\vec{F} = (x^2 - yz)\vec{i} + 2x^2y\vec{j} + z\vec{k}$ & S is surface of cube bounded by $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$.

Soln:-

By Gauss's divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv.$$

$$\nabla\cdot\vec{F} = (\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z)\cdot((x^2 - yz)\vec{i} - 2x^2y\vec{j} + z\vec{k}).$$

$$= \partial/\partial x\,(x^2 - yz) + \partial/\partial y\,(-2yx^2) + \partial/\partial z\,(z)$$

$$\nabla\cdot\vec{F} = 2x - 2x^2 + 1$$

$$\iiint_V \nabla\cdot\vec{F}\, dv = \int_0^a\int_0^a\int_0^a (2x - 2x^2 + 1)\, dx\, dy\, dz.$$

$$= \int_0^a\int_0^a \left[\frac{2x^2}{2} - \frac{2x^3}{3} + x\right]_0^a dy\, dz$$

$$= \int_0^a\int_0^a (a^2 - 2/3\, a^3 + a)\, dy\, dz.$$

$$= \int_0^a \left[a^2y - \tfrac{2}{3}a^3y + ay\right]_0^a dz.$$

$$= \int_0^a (a^3 - \tfrac{2}{3}a^4 + a^2)\, dz.$$

$$= \left[a^3z - \tfrac{2}{3}a^4z + a^2z\right]_0^a$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = a^4 - \tfrac{2}{3}a^5 + a^3$$

(9)

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = a^4 - \tfrac{2}{3}a^5 + a^3.$$

Nov'19

1. Verify divergence theorem for $\vec{F} = 4x\vec{i} - 2y\vec{j} + z^2\vec{k}$ taken over the region bounded by $x^2+y^2=4$, $z=0$ & $z=3$

Soln :-

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

Gauss's divergence theorem is

$x^2+y^2=4$

$y^2=4-x^2$

$y=\pm\sqrt{4-x^2}$

$y=0$

$x^2=4$

$x=\pm 2$

Evaluation of $\iiint \nabla\cdot\vec{F}\,dv.$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \iiint \left[\partial/\partial x(4x) + \partial/\partial y(-2y^2) + \partial/\partial z(z^2)\right] dv$$

$$= \iiint_V (4-4y+2z)\,dv. \quad [\because dv = dx\,dy\,dz]$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4-4y+2z)\,dz\,dy\,dx.$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz - 2\tfrac{z^2}{2}\right]_0^3 dy\,dx.$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12-12y+9)\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [21-12y]\,dy\,dx$$

$$= \int_{-2}^{2}\left[21y - \tfrac{\cancel{12}^{6}y^2}{\cancel{2}}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx.$$

$$= \int_{-2}^{2}\left[21y - 6y^2\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left[21\sqrt{4-x^2} \cancel{dx} - 6(4-x^2) + 21\sqrt{4-x^2} + 6(4-x^2) \right] dx$$

$$= \int_{-2}^{2} 42\sqrt{4-x^2}\, dx$$

$\therefore f(x) = \sqrt{4-x^2}$ is an even fun

(i.e) $f(-x) = \sqrt{4-(-x)^2} = \sqrt{4-x^2} = f(x)$

$$= 42 \int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a)$

$$= 42 \cdot 2 \int_{0}^{2} \sqrt{4^2-x^2}\, dx$$

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$$= 84 \left[\frac{x^2}{2}\sqrt{4-x^2} + 4/2 \sin^{-1}(x/2) \right]_0^2$$

$$= 84 \left[2/2\sqrt{4-4} + 4/2 \sin^{-1}(2/2) \right] - \left[0 + 2\sin^{-1}(0) \right]$$

$$= 84 \left[2\sin^{-1}(1) + 2\cancel{6}\sin^{-1}(0) \right]$$

$$= 84 \left[2 \cdot \pi/2 \right]$$

$$= 84\pi.$$

Evaluation of $\iint_S \vec{F} \cdot \hat{n}\, ds$.

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S_1} \vec{F} \cdot \hat{n}\, dx + \iint_{S_2} \vec{F} \cdot \hat{n}\, ds + \iint_{S_3} \vec{F} \cdot \hat{n}\, ds.$$

(i) On S_1, $\hat{n} = -\vec{k}$

[unit outward normal vector to the surface S_1 is $-\vec{k}$]

$\vec{F} = 4x\vec{i} - 2y^2\vec{j}$

[S_1, $z = 0$]

$\therefore \vec{F} \cdot \hat{n} = (4x\vec{i} - 2y^2\vec{j}) \cdot (-\vec{k})$

$= 0$

$\iint_{S_1} \vec{F} \cdot \hat{n}\, ds = 0$

(ii) On S_2, $\hat{n} = \vec{k}$

[unit outward normal vector to the surface S_2 is $\vec{k}$]

$\therefore \vec{F} = 4x\vec{i} - 2y^2\vec{j} + 9\vec{k}$

on this S_2, $z = 3$

$\therefore \vec{F} \cdot \hat{n} = (4x\vec{i} - 2y^2\vec{j} + 9\vec{k}) \cdot (\vec{k})$

$= 9$

$\therefore \iint_{S_2} \vec{F}\cdot\hat{n}\, ds = \iint_{S_2} 9\, ds_2$

[S_2 is a circle of radius 2]

$= 9\iint_{S_2} ds_2$

$= 9 S_2$

$= 9\,(\text{area of } S_2) = 9(\pi r^2)$

$= 9\,(\pi(2)^2)$

$= 9\pi \times 4$

$= 36\pi$

Another method of (ii)

$$\iint_{S_2} \vec{F}\cdot\hat{n}\, ds = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 9\, dy\, dx$$

$$= 9\int_{-2}^{2} [y]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\, dx$$

$$= 9\int_{-2}^{2} \left[\sqrt{4-x^2} + \sqrt{4-x^2}\right] dx$$

$$= 9\int_{-2}^{2} 2\sqrt{4-x^2}\, dx$$

$$= 9\times 2\int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$$= 18\int_{-2}^{2} \sqrt{4-x^2}\, dx = 18\times 2\int_{0}^{2} \sqrt{4-x^2}\, dx$$

$$= 36\left[\tfrac{x}{2}\sqrt{4-x^2} + \tfrac{4}{2}\sin^{-1}\tfrac{x}{2}\right] - \left[0 + \tfrac{4}{2}\sin^{-1}\tfrac{0}{2}\right]$$

$$= 36\left[2\sin^{-1}(1) - \sin 0\right]$$

$$= 36\left[2\cdot\tfrac{\pi}{2}\right]$$

$$= 36\pi$$

(iii) On S_3 $(x^2+y^2=4)$ the unit normal outward vector

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \left[\vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}\right]$$

$\nabla\phi = \vec{i}\,\partial/\partial x\,(x^2+y^2-4) + \vec{j}\,\partial/\partial y\,(x^2+y^2-4) + \vec{k}\,\partial/\partial z\,(x^2+y^2-4)$

$= \vec{i}\,2x + \vec{j}\,2y + \vec{k}\,(0)$

$= 2x\vec{i} + 2y\vec{j}$

$|\nabla\phi| = \sqrt{(2x)^2 + (2y)^2}$

$= \sqrt{4x^2 + (4y)}$

$= \sqrt{4(x^2+y^2)} = \sqrt{4(4)} = \sqrt{16}$ 12

$= 4$

$\hat{n} = \dfrac{2x\vec{i} + 2y\vec{j}}{4}$

$= \dfrac{2[x\vec{i} + y\vec{j}]}{4}$

$= \dfrac{x\vec{i} + y\vec{j}}{2}$

$\vec{F}\cdot\hat{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})\cdot\left(\dfrac{x\vec{i} + y\vec{j}}{2}\right)$

$= \dfrac{4x^2 - 2y^3}{2} = \dfrac{2(2x^2 - y^3)}{2} = 2x^2 - y^3$

$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds_3 = \iint_{S_3} (2x^2 - y^3)\,ds_3$

Here, S_3 is a Curved Surface Hence to find Elemental and ds_3 we Consider Co-ordinates .. Since is the Circle $x^2+y^2=4$, its polar Co-ordinates are,

$x = 2\cos\theta$, $y = 2\sin\theta$, $ds_3 = 2\,d\theta\,dz$

$= \int_{\theta=0}^{2\pi}\int_{z=0}^{3} [2(2\cos\theta)^2 - (2\sin\theta)^3]\,2\,dz\,d\theta$

$= \int_0^{2\pi}\int_0^{3} [4(4\cos^2\theta) - 2(8\sin^3\theta)]\,dz\,d\theta$

$= \int_0^{2\pi}\int_0^{3} 16\cos^2\theta - 16\sin^3\theta\,dz\,d\theta$

$= \int_0^{2\pi} \left[16\cos^2\theta\,z - 16\sin^3\theta\,z\right]_0^3\,d\theta$

$= \int^{2\pi} [16\cos^2\theta \ldots$

$$= \int_0^{2\pi} [16\cos^2\theta\, z - 16\sin^3\theta\, z]$$

$$= \int_0^{2\pi} [16\cos^2\theta\,(3) - 16\sin^3\theta\,(3)]\, d\theta$$

$$= \int_0^{2\pi} [48\cos^2\theta - 48\sin^3\theta]\, d\theta$$

$$\left[\because \int_0^{2\pi} \sin^3\theta\, d\theta = 0\right]$$

$$= 48\int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$\because \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= 48/2 \left(\theta + \frac{\sin 2\theta}{2}\right)_0^{2\pi}$$

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$$= 48/2 \left[2\pi - \frac{\sin 4\pi}{2} - 0 - 0\right]$$

$$= 48/2\,(2\pi) = 48\,[\pi] = 48\pi$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} = 0 + 36\pi + 48\pi$$

$$= 84\pi$$

2. Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by $x=0, x=2, y=0$ $y=1, z=0, z=1$

Soln:-

By divergence theorem we have.

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv.$$

Given $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$$\nabla\cdot\vec{F} = (\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z)\cdot(4xz\vec{i} - y^2\vec{j} + yz\vec{k})$$

$$= \partial/\partial x\,(4xz) + \partial/\partial y\,(-y^2) + \partial/\partial z\,(yz)$$

$$= 4z - 2y + y.$$

$$\nabla\cdot\vec{F} = 4z - y.$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V (4z - y)\, dz \qquad [\because dv = dx\,dy\,dz]$$

$$= \iiint_V (4z-y)\, dz\, dy\, dx.$$

$$= \int\int\int (4z-y)\, dz\, dy\, dx.$$

$$= \int_0^1 \int_0^1 \left(\frac{4xz^2}{2} - yz \right)_0^1 dy\, dx$$

$$= \int_0^1 \int_0^1 (2x^2 - yz)_0^1 \, dy\, dx$$

$$= \int_0^1 \int_0^1 (2 - y)\, dy\, dx$$

$$= \int_0^1 (2y - y^2/2)_0^1 \, dx = \int_0^1 (2 - 1/2)\, dx$$

$$= \int_0^1 3/2 \, dx = 3/2\,(x)_0^1 = 3/2\,(1)$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = 3/2$$

2\. Use divergence theorem to evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$

Soln:

By Gauss's divergence theorem

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Given $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$

$$\nabla \cdot \vec{F} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot (x^3\vec{i} + y^3\vec{j} + z^3\vec{k})$$

$$= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$$

$$= 3x^2 + 3y^2 + 3z^2$$

$$= 3(x^2 + y^2 + z^2)$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

$$= \iiint_V 3\,(x^2 + y^2 + z^2)\, dv$$

$$= 3 \iiint (x^2 + y^2 + z^2)\, dx\, dy\, dz$$

To evaluate this volume integral we have to change cartesian to spherical polar coordinates

$x^2 + y^2 + \ldots$

$dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = 3\int_0^a \int_0^\pi \int_0^{2\pi} r^2 (r^2 \sin\theta)\, d\phi\, d\theta\, dr$$

$$= 3\int_0^a \int_0^\pi r^4 \sin\theta \left[\phi\right]_0^{2\pi} d\theta\, dr$$

$$= 3\int_0^a \int_0^\pi r^4 \sin\theta\, (2\pi)\, d\theta\, dr$$

(15)

$$= 3\cdot 2\pi \int_0^a \int_0^\pi r^4 \sin\theta\, d\theta\, dr$$

$$= 6\pi \int_0^a r^4 (-\cos\theta)_0^\pi\, dr$$

$$= 6\pi \int_0^a r^4 \left[-\cos\pi + \cos 0\right] dr$$

$$= 6\pi \int_0^a 2r^4\, dr$$

$$= 12\pi \int_0^a r^4\, dr = 12\pi \left[\frac{r^5}{5}\right]_0^a$$

$$= \frac{12}{5}\pi (a^5)$$

3) Use divergence theorem to evaluate $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ and S is the surface bounded the region $x^2+y^2=4$, $z=0$ and $z=3$

Soln:-

By Gauss's divergence theorem.

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla \vec{F}\, dv$$

$$\iiint_V \nabla\cdot\vec{F}\, dv$$

$$\nabla\cdot\vec{F} = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot\left(4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}\right)$$

$$= \partial/\partial x\,(4x) - \partial/\partial y\,(2y^2) + \partial/\partial z\,(z^2)$$

$$\nabla\cdot\vec{F} = 4 - 4y + 2z.$$

$$\iiint_V \nabla\cdot\vec{F}\, dv = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^3 (4-4y+2z)\, dz\, dy\, dx$$

$$= \int^{2} \int^{\sqrt{4-x^2}} \left[4z - 4yz - \frac{2z^2}{2}\right]_0^3 dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12-12y+9)\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21-12y)\, dy\, dx$$

$$= \int_{-2}^{2} \left[21y - \frac{12y^2}{2} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

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$$= \int_{-2}^{2} \left[21(\sqrt{4-x^2}) - 21(-\sqrt{4-x^2}) - 6(\sqrt{4-x^2})^2 + 6(-\sqrt{4-x^2})^2 \right] dx$$

$$= \int_{-2}^{2} \left[21(\sqrt{4-x^2}) + 21\sqrt{4-x^2} - 6\sqrt{4-x^2} + 6\sqrt{4-x^2} \right] dx$$

$$= \int_{-2}^{2} 42\sqrt{4-x^2}\, dx = 42 \int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$$= 42 \cdot 2 \int_{0}^{2} \sqrt{4-x^2}\, dx = 84 \int_{0}^{2} \sqrt{4-x^2}\, dx$$

$$= 84 \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2} \sin^{-1}(x/2) \right]_0^2$$

$$= 84 \left[2/2 \sqrt{4-4} + 2 \sin^{-1}(2/2) \right]$$

$$= 84 \left[2 \sin^{-1}(1) \right] = 84 \left[2 \cdot \pi/2 \right] = 84\pi.$$

4. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = (2x+3z)\vec{i} - (\cancel{2y}xz+y)\vec{j}$ $+ (y^2+2z)\vec{k}$ and S is the surface of the sphere. Sphere having centre at $(3,-1,2)$ and radius 3.

Soln:-

By Gauss's divergence theorem we have

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

where v is the volume of the given ~~the~~ sphere

given $\vec{F} = (2x+3z)\vec{i} - (xz-y)\vec{j} + (y^2+2z)\vec{k}$]

$$= \partial/\partial x\,(2x+3z) - \partial/\partial y\,(xz-y) + \partial/\partial z\,(y^2-2z)$$

$$-4-1 = 3$$

⑮

$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv = \iiint_V 3\, dv.$

$\therefore \iiint dv$ gives the volume of the sphere

$= 3V$.

Since radius of the given sphere is 3,

$V = \frac{4}{3}\pi r^3$

$V = \frac{4}{3}\pi (3)^3 = \frac{4}{3}\pi(27) = 36\pi$

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Hence $\iint_S \vec{F}\cdot\hat{n}\, ds = 3\times 36\pi = 108\pi$

AP. 14

5M 5) Evaluate $\iint_S \vec{r}\cdot\hat{n}\, ds$, where S is a closed surface

(or) P.T for a closed surface S, $\iint_S \vec{r}\cdot\hat{n}\, ds = 3V$

Soln:-

By Gauss's divergence theorem

$\iint_S \vec{r}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\hat{n}\, dv.$

$= \iiint_V \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot(x\vec{i}+y\vec{j}+z\vec{k})\, dv.$

$(\because \vec{r} = x\vec{i}+y\vec{j}+z\vec{k})$

$= \iiint_V (1+1+1)\, dv = \iiint_V 3\, dv$ $[\because \iiint_V dv = V$, volume of the closed surface$]$

$= 3\iiint_V dv = 3V$

$\iint_S \vec{r}\cdot\hat{n}\, ds = 3V.$

6. Using divergence theorem evaluate $\iint_S \vec{r}\cdot\hat{n}\, ds$ where S is the surface of the sphere $x^2+y^2+z^2=9$.

Soln:-

By the above Example.

we have $\iint_S \vec{r}\cdot\hat{n}\, ds = 3V$

$= 3\times\frac{4}{3}\pi r^3 = 4\pi r^3 = 4\pi(3)^3$

$(\because V$ is the volume of the closed surface$)$

$= 4\pi(27) = 108\pi$

7. P.T $\iint_S \phi \, \hat{n} \, ds = \iiint_V \nabla \phi \, dv$.

Soln:-

By Gauss's divergence theorem we have

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv.$$

Let $\vec{F} = \phi \vec{c}$ where $\vec{c}$ is a constant vector.

then, $\iint_S \phi \vec{c} \cdot \hat{n} \, ds = \iiint_V (\phi \vec{c}) \, dv.$

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$$\iint_S \vec{c} \cdot (\phi \cdot \hat{n}) \, ds = \iiint_V \vec{c} \cdot \nabla \phi \, dv.$$

$[\because \nabla(\phi c) = \nabla \phi \cdot \vec{c} = \vec{c} \cdot \nabla \phi$ and

$(\phi \vec{c}) \cdot \hat{n} = \vec{c} \cdot (\phi \hat{n})]$.

Taking $\vec{c}$ out side the integrals, we get

$$\vec{c} \cdot \iint_S \phi \cdot \hat{n} \, ds = \vec{c} \cdot \iiint_V \nabla \phi \, dv.$$

$$\iint_S \phi \cdot \hat{n} \, ds = \iiint_V \nabla \phi \, dv.$$

8. S.T $\iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds = 0$, where S is any closed surface.

Soln:

By divergence theorem we have.

$$\iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot (\text{curl } \vec{F}) \, dv.$$

where v is the volume of the closed surfaces.

Since $\nabla (\text{curl } \vec{F}) = 0$ we get.

$$\iiint_V \nabla \cdot (\text{curl } \vec{F}) \, dv = 0$$

$$\iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds = 0$$

9) Use divergence theorem evaluate $\iint_S \nabla r^2 \cdot \hat{n} \, ds$

Soln:-

By divergence theorem we have.

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv.$$

Here,

$\vec{F} = \nabla r^2$, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$.

$$\iint_S \nabla r^2 \cdot \hat{n}\, ds = \iiint_V \nabla \cdot (\nabla r^2)\, dv$$

$$= \iiint_V \nabla^2 r^2\, dv$$

$$= \iiint_V \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] r^2\, dv \quad (9)$$

Since $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$ and $r^2 = x^2+y^2+z^2$

$$= \iiint_V \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] (x^2+y^2+z^2)\, dv$$

$$= \iiint_V \left\{ \frac{\partial^2}{\partial x^2}(x^2+y^2+z^2) + \frac{\partial^2}{\partial y^2}(x^2+y^2+z^2) + \frac{\partial^2}{\partial z^2}(x^2+y^2+z^2) \right\} dv.$$

$$= \iiint_V (2+2+2)\, dv$$

[$\because$ where v is a volume of the closed surfaces]

$$= \iiint_V 6\, dv$$

$$= 6V$$

10) P.T $\displaystyle\iint_S \frac{\vec{r}\cdot\hat{n}}{r^2}\, ds = \iiint_V \frac{dv}{r^2}$

$\left[\vec{F} = \frac{\vec{r}}{r^2} \right]$

Soln:-

$$\cancel{\iint_S \frac{\vec{r}}{r^2} = \left[\vec{i}\, \partial/\partial x + \vec{j}\, \partial/\partial y + \vec{k}\, \partial/\partial z \right] \frac{\vec{r}}{r^2}}$$

$$\iint_S \frac{\vec{r}}{r^2}\, \hat{n}\, ds = \iiint_V \nabla \cdot \frac{\vec{r}}{r^2}\, dv \quad \text{— (1)}$$

[By divergence theorem]

Now, $\displaystyle \nabla \cdot \frac{\vec{r}}{r^2} = \left[\vec{i}\, \partial/\partial x + \vec{j}\, \partial/\partial y + \vec{k}\, \partial/\partial y \right] \cdot \frac{\vec{r}}{r^2}$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, $r^2 = x^2+y^2+z^2$

$$\nabla \cdot \frac{\vec{r}}{r^2} = \left[\vec{i}\, \partial/\partial x + \vec{j}\, \partial/\partial y + \vec{k}\, \partial/\partial z \right] \cdot \left[\frac{x\vec{i} + y\vec{j} + z\vec{k}}{x^2+y^2+z^2} \right]$$

$$= \partial/\partial x \left(\frac{x}{x^2+y^2+z^2} \right) + \partial/\partial y \left(\frac{y}{x^2+y^2+z^2} \right) + \partial/\partial z \left(\frac{z}{x^2+y^2+z^2} \right)$$

$$= \frac{(x^2+y^2+z^2)-2x^2}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2+z^2)-2y^2}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2+z^2)-2z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{-x^2+y^2+z^2+x^2-y^2+z^2+x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2} = \frac{1}{(x^2+y^2+z^2)}$$

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$$= \frac{1}{r^2} \quad - \iiint_V \nabla \cdot \frac{\vec{r}}{r^2}\, dv.$$

(1) $\Rightarrow \iiint_V 1/r^2 \, dv$ Hence the proof.

11) If $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$ a, b, c are constants show that $\iint_S \vec{F}\cdot\hat{n}\, ds = \frac{4\pi}{3}(a+b+c)$, where S is the surface of a unit sphere (radius = 1)

Sol:-

By Gauss's divergence theorem, we have

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv.$$

$$\nabla\cdot\vec{F} = (\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z)\cdot(ax\vec{i} + by\vec{j} + cz\vec{k})$$

$$= \partial/\partial x\,(ax) + \partial/\partial y\,(by) + \partial/\partial z\,(cz)$$

$$= a+b+c \quad = \iiint_V \nabla\cdot\vec{F}\, dv.$$

$$= (a+b+c)\iiint_V dv.$$

$$= \iiint_V (a+b+c)\, dv$$

$[\because V = 4/3\,\pi r^3 \quad \because r = 1]$

$$= (a+b+c)V$$

[$\because$ V is the volume of the unit sphere enclosed by S]

$$= (a+b+c)\, 4/3\, \pi r^3$$

$$= (a+b+c)\, 4/3\, \pi (1)^3$$

$$= 4/3\, \pi\, (a+b+c)$$

13) Evaluate using divergence theorem $\iint_S [xz^2\,dy\,dz + (x^2y - z^3)\,dz\,dx + (2xy + y^2z)\,dx\,dy]$ where S is the entire surface of the hemispherical region bounded by $x^2+y^2+z^2=a^2$ and $z=0$.

Soln:-

By cartesian form of Gauss's divergence theorem we have

$$= \iint_S \left[xz^2\,dy\,dz + (x^2y-z^3)\,dz\,dx + (2xy+y^2z)\,dx\,dy\right]$$

By Gauss's divergence theorem.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

(21)

$$\nabla\cdot\vec{F} = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot\left(xz^2\vec{i} + (x^2y-z^3)\vec{j} + (2xy+y^2z)\vec{k}\right)$$

$$= \partial/\partial x\,(xz^2) + \partial/\partial y\,(x^2y - z^3) + \partial/\partial z\,(2xy+y^2z)$$

$$= z^2+x^2+y^2$$ [v is the volume of sphere where radius is a]

$$= \iiint_V (z^2+x^2+y^2)\,dv$$ [$\because v = 4/3\,\pi r^3$, $r=a$]

$$= \iiint_V a^2\,dv.$$

$$= a^2\cdot 1/2\,v.$$

$$= a^2\cdot 1/2\cdot 4/3\,\pi a^3$$

$$= 2\pi/3\,a^5$$

Green's theorem :-

If $u, v, \partial u/\partial y, \partial v/\partial x$ are continuous and one valued functions in the region R enclosed by the curve c, then

$$\oint_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy.$$

Green's theorem (in vector notation)

Let c be a closed curve which covers the region R in anticlock direction in the xy plane, then

$$\int_C \vec{F}\cdot d\vec{r} = \int_R\int (\nabla\times\vec{F})\cdot\vec{k}\, ds$$

[The unit outward normal to the xy plane is $\vec{k}$]

By Taking $\vec{F} = u\vec{i} + v\vec{j}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$ 22

$ds = dx\,dy.$

we get Cartesian from of Greens theorem.

Note :-

Suppose $\nabla\times\vec{F}$ is zero, the walk done is independent of the path, in the Case the forse field is conservation.

1. versity Greeen's theorem in the plane for $\int_C (x^2 dx + xy\,dy)$ where C is the Curve in the xy plane given by $x=0,\ y=0,\ x=a,\ y=a\ (a>0)$

Soln:-

The Cartesion from of Green's theorem in the plane is

$$\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Here $u = x^2,\quad v = xy$

$$\frac{\partial u}{\partial y} = 0,\quad \frac{\partial v}{\partial x} = y.$$

Evaluation of $\int_C (u\,dx + v\,dy)$

To evaluate $\int_C (x^2 dx + xy\,dy)$, we shall take C is four different segments viz

i) along OA $(y=0)$, ii) along AB $= (x=a)$

ii) along BC$(y=a)$, iv) along CO$(x=0)$

iii) $\int_C (x^2 dx + xy\,dy) = \int_{BC} (x^2 dx + xy\,dy)$ $\quad y=0,\ dy=0$

$= \int (x^2 dx + dx)$ [along BC, x varies

$= \int_0^a x^2\, dx$

$= \left[\frac{x^3}{3} \right]_0^a$

$= 0/_3 - a^3/_3$

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$= -a^3/_3$

$\int_C (x^2\,dx + xy\,dy) = \int_{CO} (x^2\,dx + xy)\,dx$ $[\because x = 0 \quad dx = 0]$

$= \int_{CO} [0(0) + 0(y)\,dy]$

$= 0$

$\therefore \int (x^2\,dx + xy)\,dy = \int_{OA} (x^2\,dx + xy)\,dy + \int_{AB} (x^2\,dx + xy)\,dy$

$+ \int_{BC} (x^2\,dy + xy)\,dy + \int_{CO} (x^2\,dx + xy)\,dy.$

$= a^3/_3 + a^3/_2 - a^3/_3 + 0$

$= a^3/_2$ —— ⓐ

Evaluation of $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy.$

$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy = \iint_R (y - 0)\,dx\,dy$

$= \int_0^a \int_0^a y\,dx\,dy$

$= \int_0^a [yx]_0^a\,dy$

$= \int_0^a [ay]\,dy$

$= \left[a\, y^2/_2 \right]_0^a$

$= a \cdot a^2/_2$

$= a^3/_2$ ⟶ ⓑ

Q) Verify Green's theorem in the xy plane for $\int_C \{(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy\}$ where C is the boundary of the region given by $x=0,\ y=0$, $x+y=1$

Soln:-

Green's theorem is

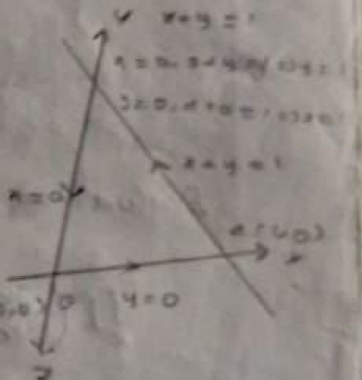
$$\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

Here $u = 3x^2 - 8y^2$ $\quad v = 4y - 6xy$

$$\frac{\partial u}{\partial y} = -16y \qquad \partial v/\partial x = -6y$$

Evaluation of $\int_C (u\,dx + v\,dy)$

To evaluate $\int_C u\,dx + v\,dy$, we shall take C in three different paths viz



i) along OA (y=0)

ii) along AB (x+y=1)

iii) along BO (x=0)

i) along OA (y=0)

$$= \int_C \{(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy\} \qquad [\because y=0,\ dy=0]$$

$$= \int_{OA} (3x^2 - 8(0))\,dx + (4(0) - 6x(0))0\}$$

$$= \int_{OA} 3x^2\,dx = \int_0^1 3x^2\,dx = 3\left(\frac{x^3}{3}\right)_0^1$$

$$= 3(1/3)$$

$$= 1$$

ii) along $AB\,(x+y=1) \Rightarrow \boxed{y=1-x}\ \boxed{dy=-dx}$

$$\int_C \left\{ (3x^2-8y^2)\,dx + (4y-6xy)\,dy \right\}$$

(28)

$$= \int_{AB} \left\{ (3x^2-8(1-x)^2)\,dx + 4(1-x)-6x(1-x)(-dx) \right\}$$

Along $x+y=1$, $y=1-x \Rightarrow dy=-dx$

$$= \int_{AB} \left\{ 3x^2-8(1+x^2-2(1)x)\,dx + (4-4x-6x+6x^2)(-dx) \right\}$$

$$= \int_{AB} \left\{ (3x^2-8-8x^2+16x-4+4x+6x-6x^2) \right\} dx$$

{$\because$ x varies from 1 to 0}

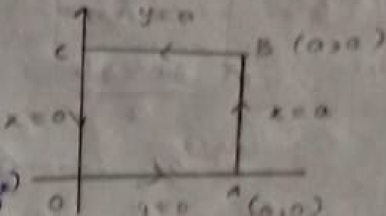
$$= \int_1^0 (-11x^2+26x-12)\,dx$$

$$= \left[-11\frac{x^3}{3} + 26\frac{x^2}{2} - 12x \right]_1^0 = -\left[\frac{-11}{3} + \frac{26}{2} - 12 \right]$$

$$= 11/3 - 13 + 12 = 11/3 - 1 = \frac{11-3}{3} = 8/3$$

ii) Along $BO\,(x=0)$

$$\int_C \left\{ (3x^2-8y^2)\,dx + (4y-6xy)\,dy \right\}$$



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i) along OA $y=0$

$$\int_C (x^2\,dx + xy\,dy) = \int_{OA} (x^2\,dx + x(0)\,dy)$$

$[y=0,\ dy=0]$

$$= \int_{OA} x^2\,dx$$

$$= \int_0^a x^2\,dx = \left[\frac{x^3}{3} \right]_0^a = a^3/3$$

ii) along AB $(x=a)$

$$\int_C (x^2 dx + xy\, dy) = \int_{AB} (x^2 dx + xy\, dy)$$

$$= \int_{AB} (a^2(0) + ay\, dy)$$

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$$= \int_{AB} ay\, dy.$$

$$= a \int_0 y\, dy$$ [Along AB, y varies from 0 to a]

$$= a \left[\frac{y^2}{2} \right]_0^a$$

$$= a\,(a^2/2)$$

$$= a^3/2$$

(ii)

$$= \int_{BO} 4y\, dy = 4 \int_1^0 y\, dy$$ [$\therefore x=0, dx=0$]

$$= 4 \left[\frac{y^2}{2} \right]_1^0 = 4(-1/2)$$ [Along BO, y varies from 1 to 0]

$$= -2$$

Hence $\int_C (u dx + v dy) = 1 + 8/3 - 2$

$$= \frac{3+8-6}{3}$$

$$= 5/3$$ —— (1)

Evaluation of $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy$

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy = \iint_R (-6y + 16y)\, dx\, dy$$

$$= \iint_R 10y\, dx\, dy \Rightarrow 10 \int_0^1 \int_0^{1-y} y\, dx\, dy$$

AP 11 3) 10 m

$= 10 \int_0^1 y(1-y)\,dy$

$= 10 \int_0^1 (y-y^2)\,dy = 10\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$

$= 10\left[\frac{1}{2} - \frac{1}{3}\right] = 10\left(\frac{3-2}{6}\right) = 10(1/6)$

$= 5/3$ —— ②

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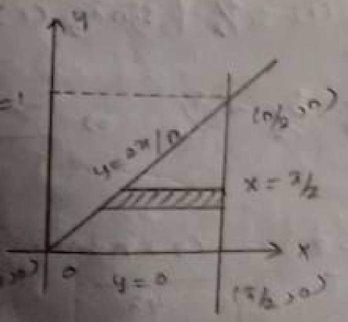
From ① and ② Green's theorem is verified.

AP'11 10m ③ Using Green's theorem, Evaluate $\int_C \{(y - \sin x)\,dx + \cos x\,dy\}$ where c is the triangle bounded by $y = 0$, $x = \pi/2$, $y = 2x/\pi$

Soln:-

Using Green's theorem we convert the line integral into double integral over the given region

Hence $\int_C \{(y-\sin x)\,dx + \cos x\,dy\} = \iint_R (-\sin x - 1)\,dx\,dy$



$y = 2x/\pi$

$\pi y = 2x$

$\frac{\pi y}{2} = x$

$y = 2x/\pi$

Put $x = \pi/2$

$y = \frac{2(\pi/2)}{\pi}$

$= \pi/\pi$

$\boxed{y = 1}$

where R is the triangle as shown in fig.

Note that point of intersection of the line $x = \pi/2$ and $y = 2x/\pi$ is $(\pi/2, 1)$

To cover the region we consider the horizontal strip PQ where x varies from $\pi y/2$ to $\pi/2$ and ... strip vertically from (i.e) $\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy$

Here

$u = y - \sin x \quad , \quad v = \cos x$

$\frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial x} = -\sin x$

$\therefore \iint_R (-\sin x - 1)\,dx\,dy = \int_0^1 \int_{\pi y/2}^{\pi/2} (-\sin x - 1)\,dx\,dy$

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$= \int_0^1 [\cos x - x]_{\pi y/2}^{\pi/2}\,dy$

$= \int_0^1 \left\{ [\cos(\pi/2) - \pi/2] - [\cos(\pi y/2) - \pi y/2] \right\} dy$

$= \int_0^1 (0 - \pi/2 - \cos \pi y/2 + \pi y/2)\,dy$

$= \left[-\pi/2\, y - \frac{\sin \pi y/2}{\pi/2} + \pi/2 \cdot y^2/2 \right]_0^1$

$= \left[-\pi/2 (1) - 2/\pi \left(\sin \pi/2 (1) + \pi/2 \cdot (1/2) \right) \right]$

$= -\pi/2 - 2/\pi + \pi/4 \Rightarrow \frac{-2\pi^2 - 8 + \pi^2}{4\pi}$

$= \frac{-\pi^2 - 8}{4\pi} = \frac{-(8 + \pi^2)}{4\pi}$

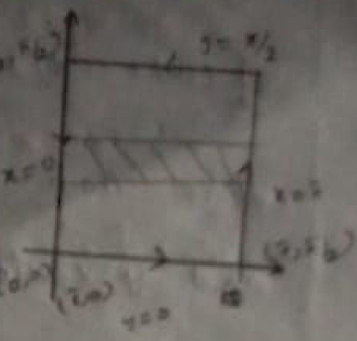
4. Evaluate by Green's theorem $\int_C e^{-x}(\sin y\,dx + \cos y\,dy)$ where c is the rectangle with vertices $(0,0), (\pi,0)$ $(\pi, \pi/2), (0, \pi/2)$

Soln:-

Given $u = e^{-x} \sin y \;,\; v = e^{-x} \cos y$

$\frac{\partial u}{\partial y} = e^{-x} \cos y \;,\; \frac{\partial v}{\partial x} = -e^{-x} \cos y$

From figure, x various from 0 to π. from figure, y various from 0 to $\pi/2$



$$\int_C (u\,dx + v\,dy) = \int_C (e^{-x}\sin y\,dx + e^{-x}\cos y)\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^{\pi} (-e^{-x}\cos y - e^{-x}\cos y)\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^{\pi} (-e^{-x}\cos y - e^{-x}\cos y)\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^{\pi} (-2e^{-x}\cos y)\,dx\,dy$$

$$= -2\int_0^{\pi/2} \left[-e^{-x}\cos y\right]_0^{\pi} dy$$

$$= -2\int_0^{\pi/2} \left[-e^{-\pi}\cos y + e^{0}\cos y\right] dy$$

$$= -2\left[-e^{-\pi}\sin y + \sin y\right]_0^{\pi/2}$$

$$= -2\left\{\left[-e^{-\pi}\sin(\pi/2) + \sin(\pi/2)\right] - \left[-e^{-\pi}\sin(0) + \sin 0\right]\right\}$$

$$= -2\left[-e^{-\pi}(1)+1\right] - \left[-e^{-\pi}(0)+0\right]$$

$$= -2\left[-(e^{-\pi}-1)\right] \quad \because \sin\pi/2 = 1$$

$$= 2(e^{-\pi}-1)$$

5) Evaluate $\int_C \{(xy+x^2)\,dx + (x^2+y^2)\,dy$ where C is the square formed by the lines $x=-1, x=1, y=-1, y=1$ using Green's theorem.

Soln:

Here $u = xy + x^2$, $v = x^2 + y^2$

$\frac{\partial v}{\partial x} = 2x$

By Green's theorem

$\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy$

Here from figure x variage from -1 to 1

and y variage from -1 to 1

$\int \{ (xy + x^2)\,dx + (x^2 + y^2)\,dy \}$

$= \int_{-1}^{1} \int_{-1}^{1} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy$

$= \int_{-1}^{1} \int_{-1}^{1} (2x - x)\,dx\,dy$

$f(x)$ is an odd func

$= \int_{-1}^{1} \int_{-1}^{1} x\,dx\,dy$ $\qquad \int_{-a}^{a} f(x)\,dx = 0$

$= \int_{-1}^{1} 0\,dx\,dy$

$= 0$

6 Using Green's theorem to evaluate $\int [(2x-y)\,dx + (x+y)\,dy]$ where C is the bounded of the circle $x^2 + y^2 = a^2$ in xy plane.

Soln :-

Here

$P = 2x - y \quad , \quad Q = x + y$

$\frac{\partial P}{\partial y} = -1 \quad , \quad \frac{\partial Q}{\partial x} = 1$

By Green's theorem, we have

$\oint (P\,dx + Q\,dy) = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx\,dy$

$\oint [(2x - y)\,dx + (x+y)\,dy] = \iint_R [1 - (-1)]\,dx\,dy$

$= 2$ (area of the region R)

$= 2\pi a^2$ (area of the circle πa^2)

~~Stoke's~~ Stoke's theorem :-

The line integral of the tangential component of a vector function $\vec{F}$ (finite & differentiable) around a simple closed curve C is equal to the surface integral of the normal component of curl $\vec{F}$ over any surface showing as its boundary.

Symbolically,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\, ds$$

$$\text{(i.e) } \int_C \vec{F}\cdot d\vec{r} = \iint_C (\nabla\times\vec{F})\cdot\hat{n}\, ds.$$

1. Verify Stoke's theorem for a vector field defined by $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the xy plane bounded by the lines $x=0,\ x=a,\ y=0,\ y=b$

Soln:-

Stokes theorem is

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\, ds$$

Evaluation of $\int_C \vec{F}\cdot d\vec{r}$

Given, $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$

$$\vec{r} = x\vec{i} + y\vec{j}$$

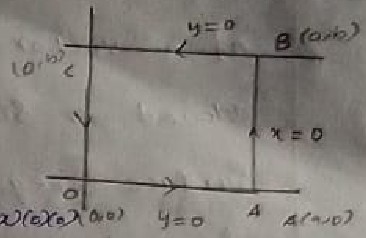
$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\vec{F}\cdot d\vec{r} = \left[(x^2-y^2)\vec{i} + 2xy\vec{j}\right]\cdot\left[dx\,\vec{i} + dy\,\vec{j}\right]$$

$$= (x^2-y^2)\,dx + 2xy\,dy.$$

Now $\int_C \vec{F}\cdot d\vec{r} = \int_C \left[(x^2-y^2)\,dx + 2xy\,dy\right]$

$$= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$



Along $OA\ (y=0) \Rightarrow dy=0$

$$\int_{OA} (x^2-y^2)\,dx + 2xy\,dy = \int_{OA} (x^2-0)\,dx + 2(x)(0)(0)$$

$$= \int_{OA} x^2\,dx \Rightarrow \int_0^a x^2\,dx$$ ($\because y=0, dy=0$ & along OA, x varies from 0 to a)

$$= \left(\frac{x^3}{3}\right)_0^a = a^3/3$$

along $AB\ (x=a)$

$$\int_{AB} (x^2-y^2)\,dx + 2xy\,dy = \int_{AB} (a^2-y^2)\,0 + 2ay\,dy$$

$$= \int_0^b 2ay\,dy.$$ ($\because x=a, dx=0$ and along AB, y varies from 0 to b)

$$= 2a\left(y^2/2\right)_0^b = ab^2$$

along $BC\ (y=b)$

$$\int_{BC} (x^2-y^2)\,dx + 2xy\,dy = \int_{BC} \left[(x^2-b^2)\,dx + 2xb(0)\right]$$

$$= \int_0^a (x^2-b^2)\,dx$$ [$\because y=b, dy=0$ & along BC, x varies from a to 0]

$$= \left[\frac{x^3}{3} - b^2x\right]_a^0 = -a^3/3 + ab^2$$

along $CO\ (x=0)$

$$\int_{CO} (x^2-y^2)\,dx + 2xy\,dy = \int_{CO} (0-y^2)\,0 + 2(0)y\,dy$$

$$= 0$$

Hence $\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$

$$= \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0$$

$$= 2ab^2 \quad \text{——— } (1)$$

Evaluation of $\iint_S \text{Curl } \vec{F}\cdot\hat{n}\, ds$.

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$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}\left[\partial/\partial x(2xy) - \partial/\partial y(x^2-y^2)\right]$$

$$= [2y-(-2y)]\vec{k} = (2y+2y)\vec{k}$$

$$= 4y\vec{k}$$

Here the surface S denotes the rectangle OABC and the unit outward normal vector is $\vec{k}$

(i.e) $\hat{n} = \vec{k}$ (xy plane)

$\therefore \text{Curl } \vec{F}\cdot\hat{n}\, ds = 4y\vec{k}\cdot\vec{k}\, ds$

$$\iint_S \text{curl } \vec{F}\cdot\hat{n}\, ds = \iint_R 4y\vec{k}\cdot\hat{n}\, ds.$$

$$= 4y\vec{k}\cdot\vec{k}\, dx\, dy.$$

$$= \iint_R 4y\vec{k}\cdot\hat{n}\, \frac{dx\, dy}{\vec{k}\cdot\hat{n}}$$

$$\iint_S \text{curl } \vec{F}\cdot\hat{n}\, ds = \iint_S 4y\, dx\, dy$$

where R is the projection of S to S to xy plane $= \iint_R 4y\, dx\, dy$

$$= \int_0^b \int_0^a 4y\, dx\, dy$$

$$= \int_0^b \left[4yx\right]_0^a dy$$

$$= \int_0^b 4ay\, dy = \left[\frac{4ay^2}{2} \right]_0^b$$

$$= 4ab^2/_2 = 2ab^2 \quad \text{— (B)}$$

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From (A) and (B) Stoke's from is verified.

AP'a 2. Verify Stoke's theorem for the Function

S.m $\vec{F} = x^2\vec{i} + xy\vec{j}$ integral around the square in the $z=0$ plane whose sides are along the lines $x=0$, $y=0$, $x=a$, $y=a$ (or) xy plane.

Soln:

Stoke's theorem is $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\hat{n}\, ds$

Evaluation of $\int_C \vec{F}\cdot d\vec{r}$

Given $\vec{F} = x^2\vec{i} + xy\vec{j}$

$\vec{r} = x\vec{i} + y\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j}$

$\vec{F}\cdot d\vec{r} = (x^2\vec{i} + xy\vec{j})\cdot(dx\vec{i} + dy\vec{j})$

$= x^2 dx + xy\, dy$

Now, $\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2 dx + xy\, dy)$

$= \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$

along OA ($y=0$)

$\int_{OA} \vec{F}\cdot d\vec{r} = \int_{OA} (x^2 dx + xy\, dy) = \int_{OA} x^2 dx + x(0)(0)$

$\because y=0,\ dy=0$ and

$= \int_0^a x^2 dx = \left(x^3/_3 \right)^a$ x varies from

$= a^3/3$

along AB ($x=a$)

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{AB} (x^2\,dx + xy\,dy)$$

$$= \int_{AB} a^2(0) + ay\,dy \qquad [\because x=a,\ dx=0\ \&\ y \text{ varies from } 0 \text{ to } a]$$

$$= \int_0^a ay\,dy = a\left(y^2/2\right)_0^a = a\cdot a^2/2$$

$$= a^3/2$$

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along BC ($y=a$)

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_{BC} x^2\,dx + xy\,dy = \int_{BC} x^2\,dx + x(a)(0)$$

$$= \int_a^0 x^2\,dx = \left[x^3/3\right]_a^0 \qquad [y=a,\ dy=0\ \&\ x \text{ varies from } a \text{ to } 0]$$

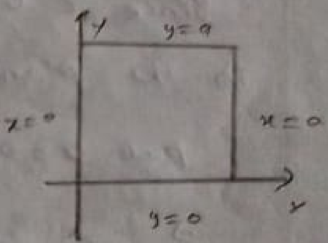
$$= -a^3/3$$

along CO ($x=0$)

$$\int_{CO} \vec{F}\cdot d\vec{r} = \int_{CO} (x^2\,dx + xy\,dy)$$

$$= \int_{CO} (0(0)) + (0)y\,dy$$

$$= 0$$



Hence $\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$

$$= a^3/3 + a^3/2 - a^3/3 + 0$$

$$= a^3/2 \quad \text{—— (A)}$$

Hence Stoke's theorem is verified from (A) and (B)

Nov'11 3. Verify Stoke's theorem for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$

10m where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is the boundary.

Soln:-

Stoke's theorem is $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds$

Evaluation of $\int_C \vec{F}\cdot d\vec{r}$

(36)

Given $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$ — (A)

$\vec{F}\cdot d\vec{r} = (y\vec{i} + z\vec{j} + x\vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$

$= y\,dx + z\,dy + x\,dz$ — (B)

Hence C is the boundary of the upper half of the given sphere which is clearly a circle $x^2+y^2=1$

Since C lines on the xy plane, we have in the plane $z=0$

put $z=0$ in (A) [in (B) put $z=0$ and $dz=0$]

$\vec{F} = y\vec{i} + x\vec{k}$, $d\vec{r} = dx\vec{i} + dy\vec{j}$

$\vec{F}\cdot d\vec{r} = (y\vec{i} + x\vec{k})\cdot(dx\vec{i} + dy\vec{j}) = y\,dx$

and the parametric representation of the circle

[$\because (x^2+y^2=a^2)$ $x = a\cos\theta$ $y = a\sin\theta$]

$x^2+y^2=1$ is $x=\cos\theta$; $y=\sin\theta$

$dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$

and θ varies from 0 to 2π

$\therefore \int \vec{F}\cdot d\vec{r} = \int_C y\,dx \Rightarrow \int_0^{2\pi} \sin\theta(-\sin\theta)\,d\theta$

$$= -\int_0^{2\pi} \sin^2\theta \, d\theta$$

$$= -\int_0^{2\pi} \frac{1-\cos 2\theta}{2} \, d\theta$$

$$= \left[\tfrac{1}{2}\theta - \frac{\sin 2\theta}{4} \right]_0^{2\pi} = -\left[\tfrac{1}{2}(2\pi) - \frac{\sin 4\pi}{4} \right]$$

$$= -\pi \quad \text{--- (2)}$$

Evaluation of $\iint \text{Curl } \vec{F} \cdot \hat{n} \, ds$ (37)

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y (x) - \partial/\partial z (z)\right] - \vec{j}\left[\partial/\partial x (x) - \partial/\partial z (y)\right]$$

$$+ \vec{k}\left[\partial/\partial x (z) - \partial/\partial y (y)\right]$$

$$= -\vec{i} - \vec{j} - \vec{k}$$

Note the surface S (upper half surface of the sphere) and the plane region S_1 of the xy plane bounded by $x^2+y^2=1$ constitute a closed surface S' (say). Also outward unit normal $\hat{n}$ is $-\vec{k}$ (i.e) $\hat{n} = -\vec{k}$

$$\text{Curl } \vec{F} \cdot \hat{n} = (-\vec{i} - \vec{j} - \vec{k}) \cdot (-\vec{k}) = 1$$

Now $\iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds = \iint_S 1 \, ds \Rightarrow \iint_S \frac{dx\,dy}{-1}$

$$\therefore ds = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|} = \frac{dx\,dy}{-1} \quad \text{Since } \hat{n} \cdot \vec{k} = -\vec{k}$$

$$\vec{k} = -1$$

where R is the region which is the projection of the surface is a circle whose radius is 1 and therefore is are is π. (38)

Hence $\iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds = \iint_R -dx\,dy = -\pi$ — (B)

Nov'14 From (A) and (B), Stoke's theorem is verified.

4) Using Stoke's theorem evaluate $\int_C [(2x-y)\,dx - yz^2\,dy - y^2 z\,dz]$ where C is the circle $x^2+y^2=1$ corresponding to the surface of the sphere of unit radius (or) verify Stoke's theorem for $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2 z\vec{k}$ where S is the upper half of the sphere $x^2+y^2+z^2=1$ and C is the boundary in the xy plane.

10m

Soln:-

By Stoke's theorem $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds$

Evaluation of $\int_C \vec{F}\cdot d\vec{r}$

$$\vec{F} \cdot d\vec{r} = \left((2x-y)\vec{i} - yz^2\vec{j} - y^2 z\vec{k}\right)\cdot\left(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}\right)$$

$$= (2x-y)\,dx - (yz^2)\,dy - y^2 z\,dz$$

$\vec{F}\cdot d\vec{r} = (2x-y)\,dx$ [along xy plane $z=0$, $dz=0$]

$x^2+y^2+z^2=1$

Hence $z=0 \Rightarrow x^2+y^2=1$

This is an equation of circle with unit radius.

$\therefore$ The palametric eqn s are $x=\cos\theta$, $y=\sin\theta$, $dx=-\sin\theta\,d\theta$, $dy=\cos\theta\,dy$

from 0 to 2π

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (2\cos\theta - \sin\theta) - (-\sin\theta)\, d\theta$$

[if $x^2+y^2=a^2$,

$x = a\cos\theta,\ y = a\sin\theta$]

$$= -a \int_0^{2\pi} \sin\theta\cos\theta + \int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= -2/2 \int_0^{2\pi} 2\sin\theta\cos\theta\, d\theta + \int_0^{2\pi} \left(1/2 - \frac{\cos 2\theta}{2}\right) d\theta$$

$$= -\int_0^{2\pi} \sin 2\theta\, d\theta + 1/2 \int_0^{2\pi} (1-\cos 2\theta)\, d\theta$$

$$= -\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi} + 1/2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= -1/2\left[-\cos 4\pi + \cos 0\right] + 1/2\left[2\pi - \frac{\sin 4\pi}{2} - 0\right]$$

$$= 1/2\,[-1+1] + 1/2\,(2\pi)$$

$$= \pi$$

Evaluation of $\iint_S \text{curl}\, \vec{F}\cdot\hat{n}\, ds$

$$\text{Curl}\, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y\,(-y^2z) + \partial/\partial z\,(yz^2)\right] - \vec{j}\left[-\partial/\partial x\,(y^2-z) - \partial/\partial z\,(2x-y)\right]$$

$$+ \vec{k}\left[-\partial/\partial x\,(yz^2) - \partial/\partial y\,(2x-y)\right]$$

$$= \vec{i}\,(-2yz+2yz) - \vec{j}\,(0) + \vec{k}\,(0+1)$$

$$= \vec{k}$$

unit outward normal vector to the surface

is $\vec{k}$ and the elemental area

$ds = dx\,dy$ in xy plane.

(i.e) $\hat{n} = \vec{k}$

$$\iint_S \text{Curl } \vec{F} \cdot \hat{n} \, ds = \iint_R \vec{k} \cdot \vec{k} \, dx \, dy \Rightarrow \iint_R 1 \, dx \, dy$$

where R denotes the projection of the Surface on ~~the~~ xy plane which is clearly a Circle of unit radius.

$$\iint_S \text{Curl } \vec{F} \cdot \hat{n} \, ds = \pi \quad \text{--- (2)}$$

$\therefore$ Area of Circle $x^2+y^2=a^2$ is πa^2

Here a=1

$\therefore$ area $\pi(1)^2 = \pi$

From (1) & (2) Stoke's theorem is verified.

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5. Evaluate the integral $\int_c [(x+y)dx + (2x-z)dy + (y+z)dz]$ where c is the boundary of the triangle with vertices (2,0,0), (0,3,0) and (0,0,0) using Stoke's theorem.

Soln:-

Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_c \text{curl } \vec{F} \cdot \hat{n} \, ds$

where S is the surface of the triangle and $\hat{n}$ is the unit vector normal to the surface s.

$$\vec{F} \cdot d\vec{r} = (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz.$$

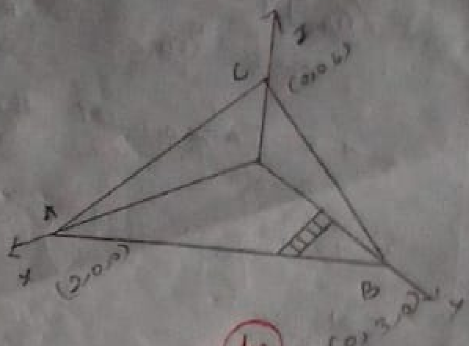
Now,
$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y\,(y+z) - \partial/\partial z\,(2x-z)\right] - \vec{j}\left[\partial/\partial x\,(y+z) - \partial/\partial z\,(x+y)\right] + \vec{k}\left[\partial/\partial x\,(2x-z) - \partial/\partial y\,(x+y)\right]$$

$$= (1+1)\vec{i} + (1)\vec{k}$$

Equation of the plane

ABC is,



$$x/2 + y/3 + z/6 = 1$$

$$\frac{3x+2y+z}{6} = 1$$

(i.e) $3x + 2y + z = 6$

$$\phi = 3x + 2y + z - 6$$

Now, $\nabla\phi = \left(\vec{i}\, \partial/\partial x + \vec{j}\, \partial/\partial y + \vec{k}\, \partial/\partial z\right) \cdot (3x + 2y + z - 6)$

$$= \partial/\partial x\,(3x)\,\vec{i} + \partial/\partial y\,(2y)\,\vec{j} + \partial/\partial z\,(z)\,\vec{k}$$

$$\nabla\phi = 3\vec{i} + 2\vec{j} + \vec{k}$$

unit normal vector to the surface ABC (or ϕ) is,

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$|\nabla\phi| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$$

$$\hat{n} = \frac{3\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{14}}$$

$$\text{Curl } \vec{F} \cdot \hat{n} = (2\vec{i} \cdot \vec{k}) \cdot \left(\frac{3\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{14}}\right)$$

$$= \frac{6+1}{\sqrt{14}} \Rightarrow 7/\sqrt{14}$$

Hence $\iint_S \text{Curl } \vec{F} \cdot \hat{n}\, ds = \iint_R 7/\sqrt{14}\, ds$

$$= \iint_R 7/\sqrt{14} \frac{dx\,dy}{|\hat{n} \cdot \vec{k}|} \Rightarrow 7/\sqrt{14} \iint_R \frac{dx\,dy}{|\hat{n} \cdot \vec{k}|}$$

$$\hat{n} \cdot \vec{k} = \frac{3\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{14}} \cdot \vec{k} - 1/\sqrt{14}$$

where R is the projection of the surface ABC on the xoy plane.

$= \frac{7}{\sqrt{14}} \iint_R \frac{dx\,dy}{1/\sqrt{14}} \Rightarrow 7 \iint_R dx\,dy$

$= 7 \times$ (Area of the triangle ABC)

$= 7 \times 3$ [Area of ABC $= \frac{1}{2} \times b \times h = \frac{1}{2} \times 3 \times 2$ $= 3$]

$= 21$

6) Evaluate by Stoke's theorem $\int_C (e^x\,dx + 2y\,dy - dz)$ where c is the curve $x^2 + y^2 = 4$, $z = 2$

Soln:

Stoke's theorem is $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds$

where $\hat{n}$ is the unit normal vector to the surface S.

$$\vec{F} \cdot d\vec{r} = e^x\,dx + 2y\,dy - dz$$

$$\vec{F} = e^x \vec{i} + 2y \vec{j} - \vec{k}$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y(-1) - \partial/\partial z(2y)\right] - \vec{j}\left[\partial/\partial x(-1) - \partial/\partial z(e^x)\right] + \vec{k}\left[\partial/\partial x(2y) - \partial/\partial y(e^x)\right]$$

$$= 0$$

Since Curl $\vec{F} = 0$, the R.H.S. of Stoke's theorem is zero, hence $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds$

$$= \iint_S 0 \cdot \hat{n}\, ds = 0$$

(i.e) $\int_C (e^x\,dx + 2y\,dy - dz) = 0$

7. Evaluate $\int_C (xy\,dx + xy^2\,dy)$ by Stoke's theorem where C is Square in the xy plane with vertices (1,0) (-1,0), (0,1) and (0,-1)

Soln:-

Stoke's theorem is $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds$

Given,

$$\vec{F}\cdot d\vec{r} = xy\,dx + xy^2\,dy$$

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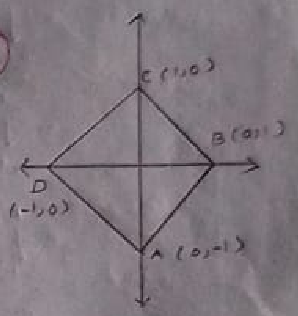

$$\vec{F} = xy\vec{i} + xy^2\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & xy^2 & 0 \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y\,(0) - \partial/\partial z\,(xy^2)\right] - \vec{j}\left[\partial/\partial x\,(0) - \partial/\partial z\,(xy)\right]$$

$$+ \vec{k}\left[\partial/\partial x\,(xy^2) - \partial/\partial y\,(xy)\right]$$

$$= \vec{k}\,(y^2 - x)$$

$$= (y^2 - x)\,\vec{k}$$

Here ABCD is a Square which lies on - xoy plane. Hence the unit normal vector to this Surface Stoke's is $\vec{k}$.

$$\therefore \hat{n} = \vec{k}$$

$$\text{Curl}\,\vec{F}\cdot\hat{n} = (y^2 - x)\,\vec{k}\cdot\vec{k} = y^2 - x$$

Also $ds = dx\,dy$ { ∵ The elemental area ds in the xy plane is dx dy }

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$= \iint (y^2 - x)\,dx\,dy$$

where R is the region in the XOY plane which is bounded by the square ABCD

$$= \int_{-1}^{1}\int_{-1}^{1} (y^2 - x)\, dx\, dy$$

$$= \int_{-1}^{1} \left[y^2 x - x^2/2 \right]_{-1}^{1} dy$$

$$= \int_{-1}^{1} \left\{ \left[y^2(1) - 1/2 \right] - \left[-y^2 - 1/2 \right] \right\} dy$$

$$= \int_{-1}^{1} (y^2 - 1/2 + y^2 + 1/2)\, dy$$

$$= \int_{-1}^{1} 2y^2\, dy$$

$$= \left[2\, y^3/3 \right]_{-1}^{1}$$

$$= 2\left[1/3 + 1/3 \right]$$

$$= 2\left[2/3 \right]$$

$$= 4/3$$

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