3. Let V be the set of all polynomials in $F[x]$ of degree $\leq n$. Let $S = \{1, x, x^2, \ldots\ldots, x^n\}$. Then $L(S) = V$ and hence V is finite-dimensional.

4. $\mathbf{C}$ is a finite-dimensional vector space over $\mathbf{R}$, since $L(\{1, i\}) = \mathbf{C}$.

5. In $M_2(\mathbf{R})$ consider the set S consisting of the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix};$$

$$C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \text{ Then}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = aA + bB + cC + dD.$$

Hence $L(S) = M_2(\mathbf{R})$ so that $M_2(\mathbf{R})$ is finite-dimensional.

Note. All the vector spaces we have considered above are finite dimensional. However there are vector spaces which cannot be spanned by a finite number of vectors. For example, consider $\mathbf{R}[x]$. Let S be any finite subset of $\mathbf{R}[x]$. Let f be a polynomial of maximum degree in S. Let $deg\, f = n$. Then any element of $L(S)$ is a polynomial of degree $\leq n$ and hence $L(S) \neq \mathbf{R}[x]$ Thus $\mathbf{R}[x]$ is not finite-dimensional.

Throughout the rest of this chapter all the vector spaces we consider are finite dimensional.

Although we have defined what is meant by a finite dimensional space we have not yet defined what is meant by the *dimension* of a vector space. We now proceed to introduce the concepts necessary to define the dimension of a finite dimensional vector space.

Consider the vectors $e_1 = (1, 0, 0)$,

$e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ in $V_3(\mathbf{R})$.

Suppose that $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = \mathbf{0}$.

Then $(\alpha_1, 0, 0) + (0, \alpha_2, 0) + (0, 0, \alpha_3) = (0, 0, 0)$.

$$\therefore \quad (\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0).$$

$$\therefore \quad \alpha_1 = \alpha_2 = \alpha_3 = 0$$

(i.e) $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = \mathbf{0}$ iff $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Thus a linear combination of the vectors e_1, e_2 and e_3 will yield the zero vector iff all the coefficients are zero.

Definition. Let V be a vector space over a field F. A finite set of vectors $v_1, v_2, \ldots\ldots, v_n$ in V is said to be ***linearly independent*** if

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n = \mathbf{0}$$
$$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$$

If $v_1, v_2, \ldots\ldots, v_n$ are not linearly independent, then they are said to be ***linearly dependent***.

Note. If $v_1, v_2, \ldots\ldots, v_n$ are linearly dependent, then there exist scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ **not all zero,** such that $\alpha_1 v_1 + \ldots + \alpha_n v_n = \mathbf{0}$.

Examples

1. In $V_n(F)$, $\{e_1, e_2, \ldots\ldots, e_n\}$ is a linearly independent set of vectors, for,

$\alpha_1 e_1 + \alpha_2 e_2 + \ldots\ldots + \alpha_n e_n = \mathbf{0}$.

$$\Rightarrow \alpha_1(1, 0, \ldots\ldots, 0) + \alpha_2(0, 1, \ldots\ldots, 0) + \ldots\ldots + \alpha_n(0, 0, \ldots\ldots, 1) = (0, 0, \ldots\ldots 0)$$

$$\Rightarrow (\alpha_1 \alpha_2, \ldots\ldots, \alpha_n) = (0, 0, \ldots\ldots, 0)$$
$$\Rightarrow \alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = 0.$$

2. In $V_3(\mathbf{R})$ the vectors $(1, 2, 1)$, $(2, 1, 0)$ and $(1, -1, 2)$ are linearly independent. For, let

$\alpha_1(1, 2, 1) + \alpha_2(2, 1, 0)$
$+ \alpha_3(1, -1, 2) = (0, 0, 0)$.

$\therefore \quad (\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 + 2\alpha_3) = (0, 0, 0)$.

$$\therefore \quad \alpha_1 + 2\alpha_2 + \alpha_3 = 0 \ldots\ldots \quad (1)$$
$$2\alpha_1 + \alpha_2 - \alpha_3 = 0 \ldots\ldots \quad (2)$$
$$\alpha_1 + 2\alpha_3 = 0 \ldots\ldots \quad (3)$$

Solving equations (1), (2) and (3) we get $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

$\therefore$ The given vectors are linearly independent.

3. In $V_3(\mathbf{R})$ the vectors $(1, 4, -2), (-2, 1, 3)$ and $(-4, 11, 5)$ are linearly dependent. For, let

$$\alpha_1(1, 4, -2) + \alpha_2(-2, 1, 3) + \alpha_3(-4, 11, 5) = (0, 0, 0)$$

$$\therefore \quad \alpha_1 - 2\alpha_2 - 4\alpha_3 = 0 \ldots\ldots \quad (1)$$

$$4\alpha_1 + \alpha_2 + 11\alpha_3 = 0 \ldots\ldots \quad (2)$$

$$-2\alpha_1 + 3\alpha_2 + 5\alpha_3 = 0 \ldots\ldots \quad (3)$$

From (1) and (2),

$$\frac{\alpha_1}{-18} = \frac{\alpha_2}{-27} = \frac{\alpha_3}{9} = k \text{ (say)}$$

$\therefore \quad \alpha_1 = -18k, \alpha_2 = -27k, \alpha_3 = 9k.$

These values of α_1, α_2 and α_3, for any k satisfy (3) also.

Taking $k = 1$ we get

$\alpha_1 = -18, \alpha_2 = -27, \alpha_3 = 9$ as a non-trivial solution.

Hence the three vectors are linearly dependent.

4. Let V be a vector space over a field F. Then any subset S of V containing the zero vector is linearly dependent.

Proof. Let $S = \{\mathbf{0}, v_1, \ldots\ldots, v_n\}$

Clearly $\alpha\mathbf{0} + 0v_1 + 0v_2 + \ldots\ldots + 0v_n = \mathbf{0}$ where α is any element of F. Hence for any $\alpha \neq 0$, we get a non-trivial linear combination of vectors in S giving the zero vector. Hence S is linearly dependent.

Exercises

1. Determine whether the following sets of vector are linearly independent or linearly dependent in $V_3(\mathbf{R})$.
   (a) $\{(1, 0, 0), (0, 1, 0), (1, 1, 0)\}$.
   (b) $\{(1, 2, 3), (2, 3, 1)\}$.
   (c) $\{(1, 2, 3), (4, 1, 5), (-4, 6, 2)\}$.
   (d) $\{(0, 0, 0), (2, 5, 3), (-1, 0, 6)\}$.
   (e) $\{(1, 0, 0), (1, 1, 0), (1, 1, 1), (0, 1, 0)\}$.

2. Determine whether the following sets of vectors are linearly independent or not.
   (a) $\{(1, 1, 0, 0), (0, 0, 1, 1)(1, 0, 0, 4), (0, 0, 0, 2)\}$ in $V_4(\mathbf{R})$.
   (b) $\{(2i, 1, 0), (2, -i, 1), (0, 1, +i, -i)\}$ in $V_3(\mathbf{C})$.
   (c) $\{(\pi, 0, 0), (0, e, 0), (0, 0, \sqrt{5})\}$ in $V_3(\mathbf{R})$.
   (d) V = the set of all polynomials of degree $\leq n$ in $\mathbf{R}[x]$ and
   $$S = \{1, x, x^2, \ldots\ldots, x^n\}.$$
   (e) $\left\{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}\right\}$ in $M_2(\mathbf{R})$.

3. In $V_3(\mathbf{Z}_5)$ determine whether the following sets of vectors are linearly dependent.
   (a) $\{(1, 3, 2), (2, 1, 3)\}$
   (b) $\{(1, 1, 2), (2, 1, 0), (0, 4, 1)\}$.

4. In $V_2(\mathbf{R})$ prove that the vectors (a, b) and (c, d) are linearly dependent iff $ad - bc = 0$.

5. Let $\{v_1, v_2, v_3\}$ be a linearly independent set of vectors in $V_3(\mathbf{R})$.
   Show that
   (a) $\{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$ is linearly independent.
   (b) $\{2v_1 + v_2, v_1 + v_2, v_1 - v_3\}$ is linearly independent.

6. If the vectors $(0, 1, a)$, $(1, a, 1)$ and $(a, 1, 0)$ of $V_3(\mathbf{R})$ are linearly dependent then find the value of a.

Answers.

1. (b) is linearly independent.
2. (a), (b), (c), (d) and (e) are linearly independent.
3. (a) is linearly independent 6. $a = 0, \pm\sqrt{2}$.

Theorem 5.11. Any subset of a linearly independent set is linearly independent.

Proof. Let V be a vector space over a field F.

Let $S = \{v_1, v_2, \ldots\ldots, v_n\}$ be a linearly independent set.

Let S' be a subset of S. Without loss of generality we take $S' = \{v_1, v_2, \ldots\ldots, v_k\}$ where $k \leq n$.

Suppose S' is a linearly dependent set. Then there exist $\alpha_1, \alpha_2, \ldots\ldots, \alpha_k$ in F not all zero, such that

$\alpha_1 v_1 + \alpha_2 v_2 + \dots\dots\dots + \alpha_k v_k = \mathbf{0}.$

Hence $\alpha_1 v_1 + \alpha_2 v_2 + \dots\dots + \alpha_k v_k + 0v_{k+1} + \dots\dots + 0v_n = \mathbf{0}$ is a non- trivial linear combination giving the zero vector.

Here S is a linearly dependent set which is a contradiction.

Hence S' is linearly independent.

Theorem 5.12. Any set containing a linearly dependent set is also linearly dependent.

Proof. Let V be a vector space. Let S be a linearly dependent set. Let $S' \supset S$.

If S' is linearly independent S is also linearly independent (by theorem 5.11) which is a contradiction. Hence S' is linearly dependent.

Theorem 5.13. Let $S = \{v_1, v_2, \dots\dots, v_n\}$ be a linearly independent set of vectors in a vector space V over a field F. Then every element of $L(S)$ can be uniquely written in the form

$\alpha_1 v_1 + \alpha_2 v_2 + \dots\dots + \alpha_n v_n$, where $\alpha_i \in F$.

Proof. By definition every elements of $L(S)$ is of the form

$\alpha_1 v_1 + \alpha_2 v_2 + \dots\dots + \alpha_n v_n.$

Now, let $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \dots + \alpha_n v_n.$

Hence $(\alpha_i - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \dots + (\alpha_n - \beta_n)v_n = \mathbf{0}.$

Since S is a linearly independent set, $\alpha_i - \beta_i = 0$ for all i.

$\therefore \quad \alpha_i = \beta_i$ for all i. Hence the theorem.

Theorem 5.14. $S = \{v_1, v_2, \dots\dots, v_n\}$ is a linearly dependent set of vectors in V iff there exists a vector $v_k \in S$ such that v_k is a linear combination of the preceding vectors $v_1, v_2, \dots\dots, v_{k-1}$.

Proof. Suppose $v_1, v_2, \dots\dots, v_n$ are linearly dependent.

Then there exist $\alpha_1, \alpha_2, \dots\dots, \alpha_n \in F$, not all zero, such that $\alpha_1 v_1 + \alpha_2 v_2 + \dots\dots + \alpha_n v_n = \mathbf{0}$.

Let k be the largest integer for which $\alpha_k \neq 0$.

Then $\alpha_1 v_1 + \dots\dots\dots + \alpha_k v_k = \mathbf{0}$.

$\therefore \quad \alpha_k v_k = -\alpha_1 v_1 - \alpha_2 v_2 \dots\dots\dots - \alpha_{k-1} v_{k-1}.$

$\therefore \quad v_k = (-\alpha_k^{-1}\alpha_1)v_1 + \dots\dots + (-\alpha_k^{-1}\alpha_{k-1})v_{k-1}.$

$\therefore \quad v_k$ is a linear combination of the preceding vectors.

Conversely, suppose there exists a vector v_k such that $v_k = \alpha_1 v_1 + \dots\dots + \alpha_{k-1} v_{k-1}$.

Hence $-\alpha_1 v_1 - \dots - \alpha_{k-1} v_{k-1} + v_k + 0v_{k+1} + \dots + 0v_n = \mathbf{0}.$

Since the coefficient of $v_k = 1$, we have

$S = \{v_1, \dots\dots\dots, v_n\}$ is linearly dependent.

Example.

In $V_3(\mathbf{R})$, let $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$

Here $(1, 1, 1) = (1, 0, 0) + (0, 1, 0) + (0, 0, 1)$.

Thus $(1, 1, 1)$ is a linear combination of the preceding vectors. Hence S is a linearly dependent set.

Theorem 5.15. Let V be a vector space over F. Let $S = \{v_1, v_2, \dots\dots, v_n\}$ and $L(S) = W$. Then there exists a linearly independent subset S' of S such that $L(S') = W$.

Proof. Let $S = \{v_1, v_2, \dots\dots, v_n\}$.

If S is linearly independent there is nothing to prove.

If not, let v_k be the first vector in S which is a linear combination of the preceding vectors.

Let $S_1 = \{v_1, v_2, \dots\dots, v_{k-1}, v_{k+1}, \dots\dots, v_n\}$.

(ie) S_1 is obtained by deleting the vector v_k from S.

We claim that $L(S_1) = L(S) = W$.

Since $S_1 \subseteq S$, $L(S_1) \subseteq L(S)$. (refer theorem 5.10).

Not, let $v \in L(S)$.

Then $v = \alpha_1 v_1 + \dots\dots + \alpha_k v_k + \dots + \alpha_n v_n$.

Now, v_k is a linear combination of the preceding vectors.

Let $v_k = \beta_1 v_1 + \dots\dots + \beta_{k-1} v_{k-1}$.

Hence $v = \alpha_1 v_1 + \dots\dots + \alpha_{k-1} v_{k-1} + \alpha_k(\beta_1 v_1 + \dots\dots + \beta_{k-1} v_{k-1}) + \alpha_{k+1} v_{k+1} + \dots\dots + \alpha_n v_n$.

$\therefore$ v can be expressed as a linear combination of the vectors of S_1 so that $v \in L(S_1)$. Hence $L(S) \subseteq L(S_1)$

Thus $L(S) = L(S_1) = W$.

Now, if S_1 is linearly independent, the proof is complete.

If not, we continue the above process of removing a vector from S_1, which is a linear combination of the preceeding vectors until we arrive at a linearly independent subset S' of S such that $L(S') = W$.

5.6. Basis and Dimension

Definition. A linearly independent subset S of a vector space V which spans the whole space V is called a *basis* of the vector space.

Theorem 5.16. Any finite-dimensional vector space V contains a finite number of linearly independent vectors which span V. (ie) A finite dimensional vector space has a basis consisting of a finite number of vectors.

Proof. Since V is finite dimensional there exists a finite subset S of V such that $L(S) = V$. By theorem 5.15 this set S contains a linearly independent subset $S' = \{v_1, v_2, \ldots\ldots, v_n\}$ such that

$L(S') = L(S) = V$.

Hence S' is a basis for V.

Theorem 5.17. Let V be a vector space over a field F. Then $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for V iff every element of V can be uniquely expressed as a linear combination of element of S.

Proof. Let S be a basis for V.

Then by definition S is linearly independent and $L(S) = V$. Hence by theorem 5.13 every element of V can be uniquely expressed as a linear combination of elements of S.

Conversely, suppose every element of V can be uniquely expressed as a linear combination of elements of S.

Clearly $L(S) = V$.

Now, let $\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n = \mathbf{0}$.

Also, $0v_1 + 0v_2 + \ldots + 0v_n = \mathbf{0}$.

Thus we have expressed $\mathbf{0}$ as a linear combination of vectors of S in two ways.

$\therefore$ By hypothesis $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = 0$.

Hence S is linearly independent. Hence S is a basis.

Examples

1. $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis for $V_3(\mathbf{R})$ for, $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$.

   $\therefore$ Any vector (a,b,c) of $V_3(\mathbf{R})$ has been expressed uniquely as a linear combination of the elements of S and hence S is a basis for $V_3(\mathbf{R})$

2. $S = \{e_1, e_2, \ldots\ldots, e_n\}$ is a basis for $V_n(F)$. This is known as the ***standard basis*** for $V_n(F)$.

3. $S = \{(1,0,0), (0,1,0), (1,1,1)\}$ is a basis for $V_3(\mathbf{R})$.

   Proof. We shall show that any element (a,b,c) of $V_3(\mathbf{R})$ can be uniquely expressed as a linear combination of the vectors of S.

   Let $(a,b,c) = \alpha(1,0,0) + \beta(0,1,0) + \gamma(1,1,1)$

   Then $\alpha + \gamma = a, \beta + \gamma = b, \gamma = c$.

   Hence $\alpha = a - c$ and $\beta = b - c$.

   Thus $(a,b,c) = (a-c)(1,0,0) + (b \quad [illegible])(0,1,0) + c(1,1,1)$.

   $\therefore$ S is a basis fo[illegible](R)

4. $S = \{1\}$ is a basis for the vector space $\mathbf{R}$ over $\mathbf{R}$.

5. $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_2(\mathbf{R})$, since any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be uniquely written as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

6. $\{1, i\}$ is a basis for the vector space $\mathbf{C}$ over $\mathbf{R}$.

7. Let V be the set of all polynomials of degree $\leq n$ in $\mathbf{R}[x]$. Then $\{1, x, x^2, \ldots\ldots, x^n\}$ is a basis for V.

8. $\{(1,0), (i,0), (0,1), (0,i)\}$ is a basis, for the vector space $\mathbf{C} \times \mathbf{C}$ over $\mathbf{R}$, for
$(a+ib, c+id) =$
$$a(1,0) + b(i,0) + c(0,1) + d(0,i).$$

9. $S = \{(1,0,0), (0,1,0), (1,1,1), (1,1,0)\}$ spans the vector space $V_3(\mathbf{R})$ but is not a basis.

Proof. Let $S' = \{(1,0,0), (0,1,0), (1,1,1)\}$.
Then $L(S') = V_3(\mathbf{R})$ (refer example 3).
Now, since $S \subseteq S'$, we get $L(S) = V_3(\mathbf{R})$.
Thus S spans $V_3(\mathbf{R})$.
But S is linearly dependent since
$$(1,1,0) = (1,0,0) + (0,1,0).$$
Hence S is not a basis.

10. $S = \{(1,0,0), (1,1,0)\}$ is linearly independent but not a basis of $V_3(\mathbf{R})$.

Proof. Let $\alpha(1,0,0) + \beta(1,1,0) = (0,0,0)$.
Then $\alpha + \beta = 0$ and $\beta = 0$.
$\therefore \quad \alpha = \beta = 0$. Hence S is linearly independent.
Also $L(S) = \{(a,b,0) / a, b \in \mathbf{R}\} \neq V_3(\mathbf{R})$.
$\therefore \quad S$ is not a basis.

Exercises

1. Show that the following three vectors from a basis for $v_3(\mathbf{R})$.
   (a) $(1,2,-3), (2,5,1), (-1,1,4)$
   (b) $(1,1,0), (0,1,1), (1,0,1)$
   (c) $(2,-3,1), (0,1,2), (1,1,2)$.
2. Show that the following sets of vectors do not form a basis for $V_3(\mathbf{R})$.
   (a) $\{(1,0,0), (1,1,0)\}$
   (b) $\{(1,2,1), (1,3,5), (-1,0,1), (1,-1,2)\}$
   (c) $\{(0,0,0), (1,0,0), (0,1,0), (0,0,$ ...
   (d) $\{(3,2,1), (3,1,5), (3,4,-7)\}$
   (e) $\{(1,2,3), (2,3,4), (3,4,5)\}$
3. Show that $(1, i, 0), (2i, 1, 1), (0, 1+i, 1-$ ... form a basis for $V_3(\mathbf{C})$.
4. Find a basis for the vector space consisting of all matrices of the form
   (a) $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ (b) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.
5. If $\{v_1, v_2, v_3\}$ is a basis for $V_3(\mathbf{R})$, show that $\{v_1+v_2, v_2+v_3, v_3+v_1\}$ is also a basis. Is this true in (a) $V_3(\mathbf{Z}_2)$ (b) $V_3(\mathbf{Z}_3)$?

Answers.

4. (a) $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$
   (b) $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

Theorem 5.18. Let V be a vector space over a field F. Let $S = \{v_1, v_2, \ldots, v_n\}$ span V. Let

$S = \{w_1, w_2, \ldots, w_m\}$ be a linearly independent set of vectors in V. Then $m \leq n$.

Proof. Since $L(S) = V$, every vector in V and in particular w_1, is a linear combination of $v_1, v_2, \ldots, v_n$.

Hence $S_1 = \{w_1, v_1, v_2, \ldots, v_n\}$ is a linearly dependent set of vectors. Hence there exists a vector $v_k \neq w_1$ in S_1 which is a linear combination of the preceding vectors.

Let $S_2 = \{w_1, v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$.

Clearly, $L(S_2) = V$.

Hence w_2 is a linear combination of the vectors in S_2.

Hence $S_3 = \{w_2, w_1, v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$ is linearly dependent. Hence there exists a vector in S_3 which is a linear combination of the preceding vectors. Since the w_i's are linearly independent, this vector cannot be w_2 or w_1 and hence must be some v_j where $j \neq k$ (say, with $j > k$). Deletion of v_j from the set S_3 gives the set

$S_4 = \{w_2, w_1, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots v_{j-1}, v_{j+1}, \ldots, v_n\}$ of n vectors spanning V.

In this process, at each step we insert one vector from $\{w_1, w_2, \ldots, w_m\}$ and delete one vector from $\{v_1, v_2, \ldots, v_n\}$.

If $m > n$ after repeating this process n times, we arrive at the set $\{w_n, w_{n-1}, \ldots, w_1\}$ which spans V.

Hence w_{n+1} is a linear combination of $w_1, w_2, \ldots, w_n$. Hence $\{w_1, w_2, \ldots, w_n, w_{n+1}, \ldots, w_m\}$ is linearly dependent which is a contradiction.

Hence $m \leq n$.

Theorem 5.19. Any two bases of a finite dimensional vector space V have the same number of elements.

Proof. Since V is finite dimensional, it has a basis say $S = \{v_1, v_2, \ldots, v_n\}$.

Let $S' = \{w_1, w_2, \ldots, w_m\}$ be any other basis for V.

Now, $L(S) = V$ and S' is a set of m linearly independent vectors. Hence by Theorem 5.18, $m \leq n$.

Also, since $L(S') = V$ and S is a set of n linearly independent vectors, $n \leq m$. Hence $m = n$.

Definition. Let V be a finite dimensional vector space over a field F. The number of elements in any basis of V is called the ***dimension*** of V and is denoted by $dim V$.

Examples

1. $dim V_n(\mathbf{R}) = n$, since $\{e_1, e_2, \ldots, e_n\}$ is a basis of $V_n(\mathbf{R})$.
2. $M_2(\mathbf{R})$ is a vector space of dimension 4 over $\mathbf{R}$ since $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_2(\mathbf{R})$.
3. $\mathbf{C}$ is a vector space of dimension 2 over $\mathbf{R}$ since $\{1, i\}$ is a basis for $\mathbf{C}$.
4. Let V be the set of all polynomials of degree $\leq n$ in $\mathbf{R}[x]$. V is a vector space over $\mathbf{R}$ having dimension $n + 1$, since $\{1, x, x^2, \ldots, x^n\}$ is a basis for V.

Theorem 5.20. Let V be a vector space of dimension n. Then

(i) any set of m vectors where $m > n$ is linearly dependent.

(ii) any set of m vectors where $m < n$ cannot span V.

Proof. (i) Let $S = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. Hence $L(S) = V$.

Let S' be any set consisting of m vectors where $m > n$. Suppose S' is linearly independent. Since S spans V by Theorem 5.18, $m \leq n$ which is a contradiction.

Hence S' is linearly dependent.

(ii) Let S' be a set consisting of m vectors where $m < n$. Suppose $L(S') = V$.

Now, $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and hence linearly independent. Hence by Theorem 5.18 $n \leq m$ which is a contradiction.

Hence S' cannot span V.

Theorem 5.21. Let V be a finite dimensional vector space over a field F. Any linearly independent set of vectors in V is part of a basis.

Proof. Let $S = \{v_1, v_2, \ldots\ldots, v_r\}$ be a linearly independent set of vectors.

If $L(S) = V$ then S itself is a basis.

If $L(S) \neq V$, choose an element $v_{r+1} \in V - L(S)$.

Now, consider $S_1 = \{v_1, v_2, \ldots\ldots, v_r, v_{r+1}\}$.

We shall prove that S_1 is linearly independent by showing that no vector in S_1 is a linear combination of the preceeding vectors. (refer theorem 5.14).

Since $\{v_1, v_2, \ldots\ldots\ldots, v_r\}$ is linearly independent, v_i where $1 \leq i \leq r$ is not a linear combination of the preceeding vectors.

Also $v_{r+1} \notin L(S)$ and hence v_{r+1} is not a linear combination of $v_1, v_2, \ldots\ldots\ldots, v_r$.

Hence S_1 is linearly independent.

If $L(S_1) = V$, then S_1 is a basis for V. If not we take an element $v_{r+2} \in V - L(S_1)$ and proceed as before. Since the dimension of V is finite, this process must stop at a certain stage giving the required basis containing S.

Theorem 5.22. Let V be a finite dimensional vector space over a field F. Let A be a subspace of V. Then there exists a subspace B of V such that $V = A \oplus B$.

Proof. Let $S = \{v_1, v_2, \ldots\ldots\ldots, v_r\}$ be a basis of A.

By theorem 5.21, we can find $w_1, w_2, \ldots\ldots\ldots,$ $w_s \in V$ such that such that $S' = \{v_1, v_2, \ldots\ldots, v_r, w_1, w_2, \ldots\ldots, w_s\}$ is a basis of V.

Now, let $B = L(\{w_1, w_2, \ldots\ldots\ldots, w_s\})$

We claim that $A \cap B = \{\mathbf{0}\}$ and $V = A + B$.

Now, let $v \in A \cap B$. Then $v \in A$ and $v \in B$.

Hence $v = \alpha_1 v_1 + \ldots\ldots \alpha_r v_r$

$= \beta_1 w_1 + \ldots\ldots + \beta_s w_s$

$\therefore \quad \alpha_1 v_1 + \ldots + \alpha_r v_r - \beta_1 w_1 - \ldots\ldots - \beta_s w_s = \mathbf{0}.$

Now, since S' is linearly-independent, $\alpha_i = 0 = \beta_j$ for all i and j.

Hence $v = \mathbf{0}$. Thus $A \cap B = \{\mathbf{0}\}$.

Now, let $v \in V$.

Then $v = (\alpha_1 v_1 + \ldots\ldots + \alpha_r v_r)$

$+(\beta_1 w_1 + \ldots\ldots + \beta_s w_s) \in A + B.$

Hence $A + B = V$ so that $V = A \oplus B$.

Exercises

1. Let V be a finite-dimensional vector space. Let A and B be subspaces of V such that $V = A \oplus B$. Then show that
$$dim\, V = dim\, A + dim\, B.$$
2. Construct 3 subspaces W_1, W_2, W_3 of a vector space V such that $V = W_1 \oplus W_2 = W_1 \oplus W_3$ but $W_2 \neq W_3$.
3. For each of the following subspaces A of $V_3(\mathbf{R})$ find another subspace B such that $A \oplus B = V_3(\mathbf{R})$
   [i] $A = L\{(1, 1, 0), (0, 1, 1)\}$.
   [ii] $A = L\{(1, 1, 1)\}$.
   [iii] $A = L(\{e_1, e_2, e_3\})$.

Definition. Let V be a vector space and

$S = \{v_1, v_2, \ldots\ldots\ldots, v_n\}$ be a set of independent vectors in V. Then S is called a ***maximal linearly independent set*** if for every $v \in V - S$, the set $\{v, v_1, v_2, \ldots\ldots\ldots, v_n\}$ is linearly dependent.

Definition. Let $S = \{v_1, v_2, \ldots\ldots\ldots, v_n\}$ be a set of vectors in V and let $L(S) = V$. Then S is called a ***minimal generating set*** if for any $v_i \in S$,

$L(S - \{v_i\}) \neq V.$

Theorem 5.23. Let V be a vector space over a field F. Let $S = \{v_1, v_2, \ldots\ldots, v_n\} \subseteq V$. Then the following are equivalent.

(i) S is a basis for V.

(ii) S is a maximal linearly independent set.

(iii) S is a minimal generating set.

Proof. **(i) ⇒ (ii)** Let $S = \{v_1, v_2, \ldots\ldots, v_n\}$ be a basis for V. Then by theorem 5.20 any $n + 1$ vectors in V are linearly dependent and hence S is a maximal linearly independent set.

(ii) ⇒ (i) Let $S = \{v_1, v_2, \ldots\ldots\ldots, v_n\}$ be a maximal linearly independent set. Now to prove that S is a basis for V we shall show that $L(S) = V$.

Obviously $L(S) \subseteq V$.

Now, let $v \in V$.

If $v \in S$, then $v \in L(S)$. (since $S \subseteq L(S)$)

If $v \notin S$, $S' = \{v_1, v_2, \ldots\ldots, v_n, v\}$ is a linearly dependent set (since S is a maximal linearly independent set)

$\therefore$ There exists a vector in S' which is a linear combination of the preceeding vectors.

Since $v_1, v_2, \ldots, v_n$ are linearly independent, this vector must be v. Thus v is a linear combination of $v_1, v_2, \ldots\ldots, v_n$. Therefore $v \in L(S)$.

Hence $V \subseteq L(S)$. Thus $V = L(S)$.

(i) ⇒ (iii) Let $S = \{v_1, v_2, \ldots\ldots\ldots, v_n\}$ be a basis. Then $L(S) = V$.

If S is not minimal, there exists $v_i \in S$ such that $L(S - \{v_i\}) = V$.

Since S is linearly independent, $S - \{v_i\}$ is also linearly independent. Thus $S - \{v_i\}$ is a basis consisting of $n-1$ elements which is a contradiction.

Hence S is a minimal generating set.

(iii) $\Rightarrow$ (i) Let $S = \{v_1, v_2, \ldots\ldots, v_n\}$ be a minimal generating set. To prove that S is a basis, we have to show that S is linearly independent.

If S is linearly dependent, there exists a vector v_k which is a linear combination of the preceeding vectors.

Clearly $L(S - \{v_k\}) = V$ contradicting the minimality of S.

Thus S is linearly independent and since

$L(S) = V$, S is a basis for V.

Theorem 5.24. Any vector space of dimension n over a field F is isomorphic to $V_n(F)$.

Proof. Let V be a vector space of dimension n. Let $\{v_1, v_2, \ldots\ldots\ldots, v_n\}$ be a basis for V.

Then we know that if $v \in V$, v can be written uniquely as $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n$, where $\alpha_i \in F$.

Now, consider the map $f : V \to V_n(F)$ given by

$$f(\alpha_1 v_1 + \ldots + \alpha_n v_n) = (\alpha_1, \alpha_2, \ldots\ldots, \alpha_n).$$

Clearly f is 1-1 and onto.

Let $v, w \in V$.

Then $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ and

$w = \beta_1 v_1 + \ldots\ldots + \beta_n v_n$.

$$\begin{aligned} f(v+w) &= f[(\alpha_1 + \beta_1)v_1 + \ldots + (\alpha_n + \beta_n)v_n] \\ &= ((\alpha_1 + \beta_1), (\alpha_2 + \beta_2), \ldots, (\alpha_n + \beta_n)) \\ &= (\alpha_1, \alpha_2, \ldots\ldots, \alpha_n) + (\beta_1, \beta_2, \ldots\ldots, \beta_n) \\ &= f(v) + f(w). \end{aligned}$$

$$\begin{aligned} \text{Also } f(\alpha v) &= f(\alpha\alpha_1 v_1 + \ldots + \alpha\alpha_n v_n) \\ &= (\alpha\alpha_1, \alpha\alpha_2, \ldots\ldots, \alpha\alpha_n) \\ &= \alpha(\alpha_1, \alpha_2, \ldots\ldots, \alpha_n) \\ &= \alpha f(v). \end{aligned}$$

Hence f is an isomorphism of V to $V_n(F)$.

Corollary. *Any two vector spaces of the same dimension over a field F are isomorphic*, for, if the vector spaces are of dimension n, each is isomorphic to $V_n(F)$ and hence they are isomorphic.

Theorem 5.25. Let V and W be vector spaces over a field F. Let $T : V \to W$ be an isomorphism. Then T maps a **basis of V** onto a basis of W.

Proof. Let $\{v_1, v_2, \ldots\ldots, v_n\}$ be a basis for V.

We shall prove that $T(v_1), T(v_2), \ldots\ldots, T(v_n)$ are linearly independent and that they span W.

Now, $\alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots\ldots + \alpha_n T(v_n) = \mathbf{0}$.

$\Rightarrow T(\alpha_1 v_1) + T(\alpha_2 v_2) + \ldots\ldots + T(\alpha_n v_n) = \mathbf{0}$.

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n) = \mathbf{0}$.

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$ (since T is 1-1)

$\Rightarrow \alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = 0$

(since $v_1, v_2, \ldots\ldots, v_n$ are linearly independent).

$\therefore$ $T(v_1), T(v_2), \ldots\ldots, T(v_n)$ are linearly independent.

Now, let $w \in W$. Then since T is onto, there exists a vector $v \in v$ such that $T(v) = w$

Let $v = \alpha_1 v_1 + \ldots\ldots + \alpha_n v_n$.

$$\begin{aligned} \text{Then } w &= T(v) \\ &= T(\alpha_1 v_1 + \ldots\ldots + \alpha_n v_n). \\ &= \alpha_1 T(v_1) + \ldots\ldots + \alpha_n T(v_n). \end{aligned}$$

Thus w is a linear combination of the vectors

$$T(v_1), \ldots\ldots\ldots, T(v_n).$$

$\therefore$ $T(v_1), \ldots\ldots, T(v_n)$ span W and hence is a basis for W.

Corollary. Two finite dimensional vector spaces V and W over a field F are isomorphic iff they have the same dimension.

Theorem 5.26. Let V and W be finite dimensional vector spaces over a field F. Let $\{v_1, v_2, \ldots\ldots, v_n\}$ be a basis for V and let $w_1, w_2, \ldots\ldots, w_n$ be any n vectors in W (not necessarily distinct) Then there exists a unique linear transformation $T : V \to W$ such that $T(v_i) = w_i$, $i = 1, 2, \ldots\ldots n$.

Proof. Let $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n \in V$.

We define $T(v) = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n$.

Now, let $x, y \in V$.

Let $x = \alpha_1 v_1 + \ldots + \alpha_n v_n$ and

$y = \beta_1 v_1 + \ldots + \beta_n v_n$.

$\therefore \quad x + y = (\alpha_1 + \beta_1)v_1 + \ldots + (\alpha_n + \beta_n)v_n$.

$$\therefore \quad T(x + y) = (\alpha_1 + \beta_1)w_1 + \ldots + (\alpha_n + \beta_n)w_n = (\alpha_1 w_1 + \ldots + \alpha_n w_n) + (\beta_1 w_1 + \ldots + \beta_n w_n) = T(x) + T(y).$$

Similarly $T(\alpha x) = \alpha T(x)$.

Hence T is a linear transformation.

Also $v_1 = 1v_1 + 0v_2 + \ldots + 0v_n$.

Hence $T(v_1) = 1w_1 + 0w_2 + 0w_n = w_1$.

Similarly $T(v_i) = w_i$ for all $i = 1, 2, \ldots\ldots, n$

Now, to prove the uniqueness, let $T' : V \to W$ be any other linear transformation such that $T'(v_i) = w_i$.

$$\text{Let } v = \alpha_1 v_1 + \ldots + \alpha_n v_n \in V$$

$$T'(v) = \alpha_1 T'(v_1) + \ldots + \alpha_n T'(v_n) = \alpha_1 w_1 + \ldots + \alpha_n w_n = T(v).$$

Hence $T = T'$.

Remark. The above theorem shows that a linear transformation is completely determined by its values on the elements of a basis.

Theorem 5.27. Let V be a finite dimensional vector space over a field F. Let W be a subspace of V. Then

(i) $\dim W \leq \dim V$.

(ii) $\dim \frac{V}{W} = \dim V - \dim W$.

Proof.

(i) Let $S = \{w_1, w_2, \ldots\ldots, w_m\}$ be a basis for W. Since W is a subspace of V, S is a part of a basis for V.
Hence $\dim W \leq \dim V$.

(ii) Let $\dim V = n$ and $\dim W = m$.
Let $S = \{w_1, w_2, \ldots, w_m\}$ be a basis for W. Clearly S is a linearly independent set of vectors in V.
Hence S is a part of a basis in V. Let $\{w_1, w_2, \ldots w_m, v_1, v_2, \ldots, v_r\}$ be a basis for V. Then $m + r = n$.

Now, we claim $S' = \{W + v_1, W + v_2, \ldots, W + v_r\}$ is a basis for $\frac{V}{W}$.

$$\alpha_1(W + v_1) + \alpha_2(W + v_2) + \ldots + \alpha_r(W + v_r) = W + 0$$

$$\Rightarrow (W + \alpha_1 v_1) + (W + \alpha_2 v_2) + \ldots + (W + \alpha_r v_r) = W$$

$$\Rightarrow W + \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r = W$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r \in W.$$

Now, since $\{w_1, w_2, \ldots, w_n\}$ is a basis for W

$$\alpha_1 v_1 + \ldots + \alpha_r v_r = \beta_1 w_1 + \ldots + \beta_m w_m.$$

$$\therefore \quad \alpha_1 v_1 + \ldots + \alpha_r v_r - \beta_1 w_1 - \ldots - \beta_m w_m = 0.$$

$$\therefore \quad \alpha_1 = \alpha_2 = \alpha_r = \beta_1 = \beta_2 = \ldots = \beta_m = 0$$

$\therefore \quad S'$ is a linearly independent set.

Now, let $W + v \in \frac{V}{W}$.

Let $v = \alpha_1 v_1 + \ldots + \alpha_r v_r + \beta_1 w_1 + \ldots + \beta_m w_m$

Then
$$W + v = W + (\alpha_1 v_1 + \ldots + \alpha_r v_r + \beta_1 w_1 + \ldots + \beta_m w_m)$$
$$= W + (\alpha_1 v_1 + \ldots + \alpha_r v_r)$$
(since $\beta_1 w_1 + \ldots + \beta_m w_m \in W$)
$$= (W + \alpha_1 v_1) + \ldots + (W + \alpha_r v_r)$$
$$= \alpha_1(W + v_1) + \ldots + \alpha_r(W + v_r).$$

Hence S' spans $\frac{V}{W}$ so that S' is a basis for $\frac{V}{W}$.

$$\therefore \quad \dim \frac{V}{W} = r = n - m = \dim V - \dim W.$$

Theorem 5.28. Let V be a finite-dimensional vector space over a field F. Let A and B be subspaces of V.

Then $\quad dim\,(A+B) = dim\,A + dim\,B - dim\,(A \cap B)$

Proof. A and B are subspaces of V. Hence $A \cap B$ is subspace of V.

Let $dim\,(A \cap B) = r$.

Let $S = \{v_1, v_2, \ldots\ldots, v_r\}$ be a basis for $A \cap B$

Since $A \cap B$ is a subspace of A and B, S is a part of a basis for A and B.

Let $\{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots u_s\}$ be a basis for A and $\{v_1, v_2, \ldots, v_r, w_1, w_2, \ldots, w_t\}$ be a basis for B.

We shall prove that $S' = \{v_1, \ldots v_r, u_1, \ldots\ldots u_s, w_1, \ldots, w_t\}$ is a basis for $A+B$.

Let $\alpha_1 v_1 + \ldots + \alpha_r v_r + \beta_1 u_1 + \ldots + \beta_s u_s + \gamma_1 w_1 + \ldots + \gamma_t v_t = \mathbf{0}$.

Then $\quad \beta_1 u_1 + \ldots + \beta_s u_s = -(\gamma_1 w_1 + \ldots + \gamma_t w_t) - (\alpha_1 v_1 + \ldots + \alpha_r v_r) \in B$

Hence $\beta_1 u_1 + \ldots + \beta_s u_s \in B$.

Also $\beta_1 u_1 + \ldots\ldots + \beta_s u_s \in A$.

Hence $\beta_1 u_1 + \ldots + \beta_s u_s \in A \cap B$.

$\therefore \quad \beta_1 u_1 + \ldots + \beta_s u_s = \delta_1 v_1 + \ldots + \delta_r v_r$.

$\therefore \quad \beta_1 u_1 + \ldots + \beta_s u_s - \delta_1 v_1 - \ldots - \delta_r v_r = \mathbf{0}$.

$\therefore \quad \beta_1 = \ldots = \beta_s = \delta_1 = \ldots = \delta_r = 0$

(since $\{u_1, \ldots, u_s, v_1, \ldots, v_r\}$ is linearly independent)

Similarly we can prove $\gamma_1 = \gamma_2 = \ldots = \gamma_t = 0$.

$\therefore \quad \alpha_i = \beta_j = \gamma_k = 0$ for $1 \le i \le r$, $1 \le j \le s$; $1 \le k \le t$

Thus S' is a linearly independent set.

Clearly S' spans $A+B$.

$\therefore \quad S'$ is a basis for $A+B$.

Hence $dim\,(A+B) = r+s+t$.

Also $dim\,A = r+s$; $dim\,B = r+t$ and $dim\,(A \cap B) = r$.

$$\therefore \quad dim\,A + dim\,B - dim\,A \cap B = (r+s)+(r+t)-r$$
$$= r+s+t$$
$$= dim\,(A+B).$$

Aliter. By theorem 5.7, $\dfrac{A+B}{A} = \dfrac{B}{A \cap B}$.

Hence $dim\left[\dfrac{A+B}{A}\right] = dim\left[\dfrac{B}{A \cap B}\right]$.

$\therefore \quad dim\,(A+B) - dim\,A = dim\,B - dim(A \cap B)$.

$\therefore \quad dim\,(A+B) = dim\,A + dim\,B - dim\,(A \cap B)$.

Corollary. If $V = A \oplus B$, $dim\,V = dim\,A + dim\,B$.

Proof. $V = A \oplus B \Rightarrow A + B = V$ and $A \cap B = \{\mathbf{0}\}$

$\therefore \quad dim\,(A \cap B) = 0$.

Hence $dim\,V = dim\,A + dim\,B$.

Exercises

1. Find the dimension of the subspace spanned by the following vectors in $V_3(\mathbf{R})$.
   (a) $(1, 1, 1), (-1, -1, -1)$.
   (b) $(1, 0, 2), (2, 0, 1), (1, 0, 1)$
   (c) $(1, 2, -3), (0, 0, 1), (-1, 2, 1)$.
   (d) $(1, 1, 2), (-1, 1, 0)$.
2. Find the dimension of the subspace spanned by the following vectors in $V_4(\mathbf{R})$
   (a) e_1, e_2, e_3, e_4
   (b) e_1, e_2
   (c) $e_1; e_2, e_3$
   (d) e_1
3. In $V_3(\mathbf{R})$, find $dim\,(A+B)$ and $dim\,(A \cap B)$ where
   (a) A is the subspace spanned by $(1, 1, 1)$ and B is the subspace spanned by $(-1, -1, -1)$
   (b) A is the subspace spanned by $(1, 1, 1)$ and B is the subspace spanned by $(1, 2, 1)$.

(c) A is the subspace spanned by $(1, 1, 1)$ and $(1, 2, 1)$ and B is the subspace spanned by $(0, 0, 1)$.

(d) A is the subspace spanned by $(1, 1, 1)$ and $(1, 2, 1)$ and B is the subspace spanned by $(1, -1, 1)$ and $(-1, 1, -1)$

4. Let V_1 and V_2 be subspaces of V such that $V_1 \cap V_2$ is the zero space.
Prove that $dim\ V_1 + dim\ V_2 \leq dim\ V$.

5. Let V_1 and V_2 be subspaces of V such that every vector $v \in V$ can be represented as $v = v_1 + v_2$ where $v_1 \in V_1$ and $v_2 \in V_2$. Prove that $dim\ V_1 + dim\ V_2 \geq dim\ V$.

6. If A and B are finite dimensional subspaces of V such that $A \subseteq B$ and $dim\ A = dim\ B$ then show that $A = B$.

7. Let S be a subspace of a finite-dimensional vector space V. If $dim\ V = dim\ S$ then prove that $S = V$.

8. Let W_1 and W_2 be two subspaces of a finite-dimensional vector space V.
If $dim V = dim\ W_1 + dim\ W_2$ and $W_1 \cap W_2 = \{0\}$ prove that, $V = W_1 \oplus W_2$.

Answers.

1. (a) 1	(b) 2	(c) 3	(d) 2
2. (a) 4	(b) 2	(c) 3	(d) 1
3. (a) 1; 1	(b) 2; 0	(c) 3; 0	(d) 3; 0

5.7. Rank and Nullity

Definition. Let $T : V \to W$ be a linear transformation. Then the dimension of $T(V)$ is called the ***rank*** of T. The dimension of $ker\ T$ is called the ***nullity*** of T.

Theorem 5.29. Let $T : V \to W$ be a linear transformation. Then $dim\ V = rank\ T + nullity\ T$.

Proof. We know that $V/ker\ T = T(V)$.

$\therefore\ dim\ V - dim\ (ker\ T) = dim\ (T(V))$

$\therefore\ dim\ V - nullity\ T = rank\ T$

$\therefore\ dim\ V = nullity\ T + rank\ T$

Note. $ker\ T$ is also called ***null space*** of T.

Example. Let V denote the set of all polynomials of $degree \leq n$ in $\mathbf{R}[x]$. Let $T : V \to V$ be defined by $T(f) = \dfrac{df}{dx}$. We know that T is a linear transformation. Since $\dfrac{df}{dx} = 0 \Leftrightarrow f$ is constant, $ker\ T$ consists of all constant polynomials. The dimension of this subspace of V is 1. Hence $nullity\ T$ is 1. Since $dim\ V = n + 1$, $rank\ T = n$.

Exercises

1. Find the *rank* and *nullity* of the linear transformations given in section 5.3.

2. Let V be a finite-dimensional vector space over a field F. Let $T : V \to V$ be a linear transformation such that $rank\ T = nullity T$. Show that $dim\ V$ is even. Give an example of such a transformation.

Answers.

1.
1. $nullity\ T = dim\ V$; $rank\ T = 0$.
2. $nullity\ T = 0$; $rank\ T = dim\ V$.
3. $nullity\ T = dim\ W$; $rank\ T = dim\ V - dim\ W$.
4. $nullity\ T = 2$; $rank\ T = 1$;
5. $nullity\ T = 1$; $rank\ T = n$.
6. $nullity\ T = 0$; $rank\ T = n + 1$.

Definition. A linear transformation $T : V \to W$ is called ***non-singular*** if T is 1-1; otherwise T is called ***singular***.

Exercises

1. Let V and W be finite dimensional vector spaces over a field F and $dim\ V > dim\ W$. Then show that any linear transformation $T : V \to W$ is singular.

2. Let V be a finite-dimensional vector space over a field F. Then any non-singular linear transformation $T : V \to V$ is onto.

3. Let $T : V \to W$ be a linear transformation. Show that T is a non-singular iff

$$rank\ T = dim\ V.$$

4. Let $T_1 : V \to V$ and $T_2 : V \to V$ be linear transformations. Prove that

    (a) $rank\,(T_2T_1) \leq rank\,T_2$.
    (b) $nullity\,(T_2T_1) \geq nullity\,T_1$.
    (c) $rank\,(T_2T_1) = rank\,T_2$ iff T_1 is non-singular.

5. Let $T : V \to W$ be a linear transformation which is both 1-1 and onto. Show that $T^{-1} : W \to V$ is a linear transformation.

6. Determine which of the following statements are true and which are false.

    (a) If $T : V \to W$ is a linear transformation then

        (i) $rank\,T \leq dim\,V$
        (ii) $nullity\,T \leq dim\,V$.
        (iii) $rank\,T \leq dim\,W$.
        (iv) If T is onto $rank\,T = dim\,W$.
        (v) If T is non-singular $rank\,T = dim\,V$
        (vi) $rank\,T = dim\,V \Rightarrow nullity\,T = 0$.

    (b) Every linear transformation $T : V_4(\mathbf{R}) \to V_3(\mathbf{R})$ is singular.
    (c) If $T : V \to W$ is non-singular and $\{v_1, \ldots\ldots, v_n\}$ is a basis then $\{T(v_1), \ldots\ldots, T(v_n)\}$ is a basis for W.

Answers.

6.(a) (i) T(ii) T (iii) T (iv) T (v) T (vi) T
(b) T (c) F.

5.8. Matrix of a Linear Transformation

Let V and W be finite dimensional vector spaces over a field F. Let $dim\,V = m$ and $dim\,W = n$. Fix an ordered basis $\{v_1, v_2, \ldots, v_m\}$ for V and an ordered basis $\{w_1, w_2, \ldots, w_n\}$ for W.

Let $T : V \to W$ be a linear transformation. We have seen that T is completely specified by the elements $T(v_1), T(v_2), \ldots, T(v_m)$. Now, let

$$\begin{aligned} T(v_1) &= a_{11}w_1 + a_{12}w_2 + \ldots\ldots + a_{1n}w_n \\ T(v_2) &= a_{21}w_1 + a_{22}w_2 + \ldots\ldots + a_{2n}w_n \quad (1) \\ \ldots & \quad \ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots \\ T(v_m) &= a_{m1}w_1 + a_{m2}w_2 + \ldots\ldots + a_{mn}w_n \end{aligned}$$

Hence $T(v_1), T(v_2), \ldots\ldots, T(v_m)$ are completely specified by the mn elements a_{ij} of the field F. These a_{ij} can be convenie tly arranged in the form of m rows and n columns as ows.

$$\begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

Such an array of mn elements of F arranged in m rows and n columns is known as $m \times n$ ***matrix*** over the field F and is denoted by (a_{ij}). Thus to every linear transformation T there is associated with it an $m \times n$ matrix over F. Conversely any $m \times n$ matrix over F defines a linear transformation $T : V \to W$ given by the formula (1).

Note. The $m \times n$ matrix which we have associated with a linear transformation $T : V \to W$ depends on the choice of the basis for V and W.

For example, consider the linear transformation $T : V_2(\mathbf{R}) \to V_2(\mathbf{R})$ given by $T(a, b) = (a, a + b)$. Choose $\{e_1, e_2\}$ as a basis both for the domain and the range.

Then $T(e_1) = (1, 1) = e_1 + e_2$
$T(e_2) = (0, 1) = e_2$.

Hence the matrix representing T is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Now, we choose $\{e_1, e_2\}$ as a basis for the domain and $\{(1, 1), (1, -1)\}$ as a basis for the range.

Let $w_1 = (1, 1)$ and $w_2 = (1, -1)$.

Then $T(e_1) = (1, 1) = w_1$,

and $T(e_2) = (0, 1) = (1/2)w_1 - (1/2)w_2$.

Hence the matrix representing T is $\begin{bmatrix} 1 & 0 \\ 1/2 & -1/2 \end{bmatrix}$

Solved problems

Problem 1. Obtain the matrix representing the linear transformation $T : V_3(\mathbf{R}) \to V_3(\mathbf{R})$ given by $T(a, b, c) = (3a, a-b, 2a+b+c)$ w.r.t. the standard basis $\{e_1, e_2, e_3\}$.

Solution.

$$T(e_1) = T(1, 0, 0) = (3, 1, 2) = 3e_1 + e_2 + 2e_3$$
$$T(e_2) = T(0, 1, 0) = (0, -1, 1) = -e_2 + e_3$$
$$T(e_3) = T(0, 0, 1) = (0, 0, 1) = e_3$$

Thus the matrix representing T is $\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Problem 2. Find the linear transformation $T : V_3(\mathbf{R}) \to V_3(\mathbf{R})$ determined by the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ w.r.t. the standard basis $\{e_1, e_2, e_3\}$.

Solution.

$$T(e_1) = e_1 + 2e_2 + e_3 = (1, 2, 1).$$
$$T(e_2) = 0e_1 + e_2 + e_3 = (0, 1, 1)$$
$$T(e_3) = -e_1 + 3e_2 + 4e_3 = (-1, 3, 4).$$

Now, $(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$

$$= ae_1 + be_2 + ce_3.$$

$\therefore \quad T(a, b, c) = T(ae_1 + be_2 + ce_3)$

$$= aT(e_1) + bT(e_2) + cT(e_3)$$
$$= a(1, 2, 1) + b(0, 1, 1) + c(-1, 3, 4).$$

$\therefore \quad T(a, b, c) = (a - c, 2a + b + 3c, a + b + 4c)$

This is the required linear transformation.

Exercises

1. Obtain the matrices for the following linear transformations.
   (a) $T : V_2(\mathbf{R}) \to V_2(\mathbf{R})$ given by $T(a, b) = (-b, a)$ w.r.t.
      (i) standard basis
      (ii) the basis $\{(1, 2), (1, -1)\}$ for both domain and range.
   (b) $T : V_3(\mathbf{R}) \to V_2(\mathbf{R})$ given by $T(a, b, c) = (a + b, 2c - a)$ w.r.t.
      (i) standard basis
      (ii) $\{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$ as a basis for $V_3(\mathbf{R})$ and $\{(0, 1), (1, 0)\}$ for $V_2(\mathbf{R})$.
   (c) $T : V_3(\mathbf{R}) \to V_3(\mathbf{R})$ given by $T(a, b, c) = (3a + c, -2a + b, a + 2b + 4c)$ w.r.t.
      (i) the standard basis
      (ii) the basis $\{(1, 0, 1), (-1, 2, 1) (2, 1, 1)\}$ for both domain and range.
   (d) Let V be the set of all polynomials of *degree* $\leq n$ in $\mathbf{R}[x]$.
      $T : V \to V$ defined by $T(f) = \dfrac{df}{dx}$ w.r.t. the basis $\{1, x, x^2, \ldots, x^n\}$.

2. Obtain the linear transformation determined by the following matrices
   (a) $T : V_2(\mathbf{R}) \to V_2(\mathbf{R})$ given by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ w.r.t. the standard basis.
   (b) $T : V_3(\mathbf{R}) \to V_3(\mathbf{R})$ given by $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ w.r.t. the standard basis.
   (c) $T : V_2(\mathbf{R}) \to V_3(\mathbf{R})$ given by $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$ w.r.t. the standard basis.

Answers.

1. (a) (i) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} -1/3 & -5/3 \\ 2/3 & 1/3 \end{bmatrix}$

(b) (i) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -3 & 1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}$

(c) (i) $\begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 17/4 & -3/4 & -1/2 \\ 35/4 & 15/4 & -7/2 \\ 17/2 & -3/2 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & n & 0 \end{bmatrix}$

2. (a) $T(a, b) = (a \cos\theta + b \sin\theta, -a \sin\theta + b\cos\theta)$

(b) $T(x, y, x) = (ax + by + cz, bx + cy + az, cx + ay + bz)$

(c) $T(a, b) = (2a + b, a + b, -a - b)$.

Definition. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. We define the ***sum*** of these two matrices by $A + b = (a_{ij} + b_{ij})$.

Note that we have defined addition only for two matrices having the same number of rows and the same number of columns.

Definition. Let $A = (a_{ij})$ be an arbitrary matrix over a field F. Let $\alpha \in F$. We define $\alpha A = (\alpha a_{ij})$.

Theorem 5.30. The set $M_{m\times n}(F)$ of all $m \times n$ matrices over the field F is a vector space of dimension mn over F under matrix addition and scalar multiplication defined above.

Proof. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices over the field F. The addition of $m \times n$ matrices is a binary operation which is both commutative and associative. The $m \times n$ matrix whose entries are 0 is the *identity matrix* and $(-a_{ij})$ is the *inverse matrix* of (a_{ij}). Thus the set of all $m \times n$ matrices over the field F is an *abelian group* with respect to addition. The verification of the following axioms are straight forward.

(a) $\alpha(A + B) = \alpha A + \alpha B$

(b) $(\alpha + \beta)A = \alpha A + \beta A$

(c) $(\alpha\beta)A = \alpha(\beta A)$

(d) $1A = A$.

Hence the set of all $m \times n$ over F is a vector space over F.

Now, we shall prove that the dimension of this vector space is mn. Let E_{ij} be the matrix with entry 1 in the $(i, j)^{\text{th}}$ place and 0 in the other places. We have mn matrices of this form. Also any matrix $A = (a_{ij})$ can be written as $A = \sum a_{ij} E_{ij}$. Hence A is a linear combiation of the matrices E_{ij}. Further these mn matrices E_{ij} are linearly independent. Hence these mn matrices form a basis for the space of all $m \times n$ matrices. Therefore the dimension of the **vector** space is mn.

Theorem 5.31. Let V and W be two finite dimensional vector spaces over a field F. Let $dim\, V = m$ and $dim\, W = n$. Then $L(V, W)$ is a vector space of dimension mn over F.

Proof. By theorem 5.8, $L(V, W)$ is a vector space over F. Now, we shall prove that the vector space $L(V, W)$ is isomorphic to the vector space $M_{m\times n}(F)$. Since $M_{m\times n}(F)$ is of dimension mn, it follows that $L(V, W)$ is also of dimension mn.

Fix a basis $\{v_1, v_2, \ldots\ldots, v_m\}$ for V and a basis $\{w_1, w_2, \ldots\ldots, w_n\}$ for W.

We know that any linear transformation

$T \in L(V, W)$ can be represented by an $m \times n$ matrix over F.