COMPLEX ANALYSIS

An Introduction to the Theory of Analytic Functions of One Complex Variable

Third Edition

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1. FUNDAMENTAL THEOREMS

Many important properties of analytic functions are very difficult to prove without use of complex integration. For instance, it is only recently that it became possible to prove, without resorting to complex integrals or equivalent tools, that the derivative of an analytic function is continuous, or that the higher derivatives exist. At present the integration-free proofs are, to say the least, much more difficult than the classical proofs.†

As in the real case we distinguish between *definite* and *indefinite integrals*. An indefinite integral is a function whose derivative equals a given analytic function in a region; in many elementary cases indefinite integrals can be found by inversion of known derivation formulas. The definite integrals are taken over differentiable or piecewise differentiable arcs and are not limited to analytic functions. They can be defined by a limit process which mimics the definition of a real definite integral. Actually, we shall prefer to define complex definite integrals in terms of real integrals. This will save us from repeating existence proofs which are essentially the same as in the real case. Naturally, the reader must be thoroughly familiar with the theory of definite integrals of real continuous functions.

1.1. Line Integrals. The most immediate generalization of a real integral is to the definite integral of a complex function over a real interval. If $f(t) = u(t) + iv(t)$ is a continuous function,

† Without use of integration R. L. Plunkett proved the continuity of the derivative (*Bull. Am. Math. Soc.* **65**, 1959). E. H. Connell and P. Porcelli proved the existence of all derivatives (*Bull. Am. Math. Soc.* **67**, 1961). Both proofs lean on a topological theorem due to G. T. Whyburn.

defined in an interval (a,b), we set by definition

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt. \tag{1}$$

This integral has most of the properties of the real integral. In particular, if $c = \alpha + i\beta$ is a complex constant we obtain

$$\int_a^b cf(t)\,dt = c\int_a^b f(t)\,dt, \tag{2}$$

for both members are equal to

$$\int_a^b (\alpha u - \beta v)\,dt + i\int_a^b (\alpha v + \beta u)\,dt.$$

When $a \leqq b$, the fundamental inequality

$$\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt \tag{3}$$

holds for arbitrary complex $f(t)$. To see this we choose $c = e^{-i\theta}$ with a real θ in (2) and find

$$\text{Re}\left[e^{-i\theta}\int_a^b f(t)\,dt\right] = \int_a^b \text{Re}\,[e^{-i\theta}f(t)]\,dt \leq \int_a^b |f(t)|\,dt.$$

For $\theta = \arg \int_a^b f(t)\,dt$ the expression on the left reduces to the absolute value of the integral, and (3) results.†

We consider now a piecewise differentiable arc γ with the equation $z = z(t)$, $a \leqq t \leqq b$. If the function $f(z)$ is defined and continuous on γ, then $f(z(t))$ is also continuous and we can set

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt. \tag{4}$$

This is our *definition* of the complex line integral of $f(z)$ extended over the arc γ. In the right-hand member of (4), if $z'(t)$ is not continuous throughout, the interval of integration has to be subdivided in the obvious manner. Whenever a line integral over an arc γ is considered, let it be tacitly understood that γ is piecewise differentiable.

The most important property of the integral (4) is its invariance under a change of parameter. A change of parameter is determined by an increasing function $t = t(\tau)$ which maps an interval $\alpha \leqq \tau \leqq \beta$ onto $a \leqq t \leqq b$; we assume that $t(\tau)$ is piecewise differentiable. By the rule

† θ is not defined if $\int_a^b f\,dt = 0$, but then there is nothing to prove.

for changing the variable of integration we have

$$\int_a^b f(z(t))z'(t)\,dt = \int_\alpha^\beta f(z(t(\tau)))z'(t(\tau))t'(\tau)\,d\tau.$$

But $z'(t(\tau))t'(\tau)$ is the derivative of $z(t(\tau))$ with respect to τ, and hence the integral (4) has the same value whether γ be represented by the equation $z = z(t)$ or by the equation $z = z(t(\tau))$.

In Chap. 3, Sec. 2.1, we defined the opposite arc $-\gamma$ by the equation $z = z(-t)$, $-b \leqq t \leqq -a$. We have thus

$$\int_{-\gamma} f(z)\,dz = \int_{-b}^{-a} f(z(-t))(-z'(-t))\,dt,$$

and by a change of variable the last integral can be brought to the form

$$\int_b^a f(z(t))z'(t)\,dt.$$

We conclude that

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz. \tag{5}$$

The integral (4) has also a very obvious additive property. It is quite clear what is meant by subdividing an arc γ into a finite number of subarcs. A subdivision can be indicated by a symbolic equation

$$\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n,$$

and the corresponding integrals satisfy the relation

$$\int_{\gamma_1+\gamma_2+\cdots+\gamma_n} f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz + \cdots + \int_{\gamma_n} f\,dz. \tag{6}$$

Finally, the integral over a closed curve is also invariant under a shift of parameter. The old and the new initial point determine two subarcs γ_1, γ_2, and the invariance follows from the fact that the integral over $\gamma_1 + \gamma_2$ is equal to the integral over $\gamma_2 + \gamma_1$.

In addition to integrals of the form (4) we can also consider line integrals with respect to $\bar{z}$. The most convenient definition is by double conjugation

$$\int_\gamma f\,\overline{dz} = \overline{\int_\gamma \bar{f}\,dz}.$$

Using this notation, line integrals with respect to x or y can be introduced by

$$\int_\gamma f\,dx = \frac{1}{2}\left(\int_\gamma f\,dz + \int_\gamma f\,\overline{dz}\right)$$

$$\int_\gamma f\,dy = \frac{1}{2i}\left(\int_\gamma f\,dz - \int_\gamma f\,\overline{dz}\right).$$

With $f = u + iv$ we find that the integral (4) can be written in the form

$$\int_\gamma (u\,dx - v\,dy) + i \int_\gamma (u\,dy + v\,dx) \tag{7}$$

which separates the real and imaginary part.

Of course we could just as well have started by defining integrals of the form

$$\int_\gamma p\,dx + q\,dy,$$

in which case formula (7) would serve as definition of the integral (4). It is a matter of taste which one prefers.

An essentially different line integral is obtained by integration with respect to *arc length*. Two notations are in common use, and the definition is

$$\int_\gamma f\,ds = \int_\gamma f|dz| = \int_\gamma f(z(t))|z'(t)|\,dt. \tag{8}$$

This integral is again independent of the choice of parameter. In contrast to (5) we have now

$$\int_{-\gamma} f|dz| = \int_\gamma f|dz|$$

while (6) remains valid in the same form. The inequality

$$\left|\int_\gamma f\,dz\right| \leqq \int_\gamma |f| \cdot |dz| \tag{9}$$

is a consequence of (3).

For $f = 1$ the integral (8) reduces to $\int_\gamma |dz|$ which is by definition the *length* of γ. As an example we compute the length of a circle. From the parametric equation $z = z(t) = a + \rho e^{it}$, $0 \leqq t \leqq 2\pi$, of a full circle we obtain $z'(t) = i\rho e^{it}$ and hence

$$\int_0^{2\pi} |z'(t)|\,dt = \int_0^{2\pi} \rho\,dt = 2\pi\rho$$

as expected.

1.2. Rectifiable Arcs. The length of an arc can also be defined as the least upper bound of all sums

$$|z(t_1) - z(t_0)| + |z(t_2) - z(t_1)| + \cdots + |z(t_n) - z(t_{n-1})| \tag{10}$$

where $a = t_0 < t_1 < \cdots < t_n = b$. If this least upper bound is finite we say that the arc is *rectifiable*. It is quite easy to show that piecewise differentiable arcs are rectifiable, and that the two definitions of length coincide.

Because $|x(t_k) - x(t_{k-1})| \leqq |z(t_k) - z(t_{k-1})|$, $|y(t_k) - y(t_{k-1})| \leqq |z(t_k) - z(t_{k-1})|$ and $|z(t_k) - z(t_{k-1})| \leqq |x(t_k) - x(t_{k-1})| + |y(t_k) - y(t_{k-1})|$ it is clear that the sums (10) and the corresponding sums

$$|x(t_1) - x(t_0)| + \cdots + |x(t_n) - x(t_{n-1})|$$
$$|y(t_1) - y(t_0)| + \cdots + |y(t_n) - y(t_{n-1})|$$

are bounded at the same time. When the latter sums are bounded, one says that the functions $x(t)$ and $y(t)$ are of *bounded variation.* *An arc $z = z(t)$ is rectifiable if and only if the real and imaginary parts of $z(t)$ are of bounded variation.*

If γ is rectifiable and $f(z)$ continuous on γ it is possible to define integrals of type (8) as a limit

$$\int_\gamma f\,ds = \lim \sum_{k=1}^{n} f(z(t_k))|z(t_k) - z(t_{k-1})|.$$

Here the limit is of the same kind as that encountered in the definition of a definite integral.

In the elementary theory of analytic functions it is seldom necessary to consider arcs which are rectifiable, but not piecewise differentiable. However, the notion of rectifiable arc is one that every mathematician should know.

1.3. Line Integrals as Functions of Arcs. General line integrals of the form $\int_\gamma p\,dx + q\,dy$ are often studied as functions (or *functionals*) of the arc γ. It is then assumed that p and q are defined and continuous in a region Ω and that γ is free to vary in Ω. An important class of integrals is characterized by the property that the integral over an arc depends only on its end points. In other words, if γ_1 and γ_2 have the same initial point and the same end point, we require that $\int_{\gamma_1} p\,dx + q\,dy = \int_{\gamma_2} p\,dx + q\,dy.$ To say that an integral depends only on the end points is equivalent to saying that the integral over any closed curve is zero. Indeed, if γ is a closed curve, then γ and $-\gamma$ have the same end points, and if the integral depends only on the end points, we obtain

$$\int_\gamma = \int_{-\gamma} = -\int_\gamma$$

and consequently $\int_\gamma = 0$. Conversely, if γ_1 and γ_2 have the same end points, then $\gamma_1 - \gamma_2$ is a closed curve, and if the integral over any closed curve vanishes, it follows that $\int_{\gamma_1} = \int_{\gamma_2}$.

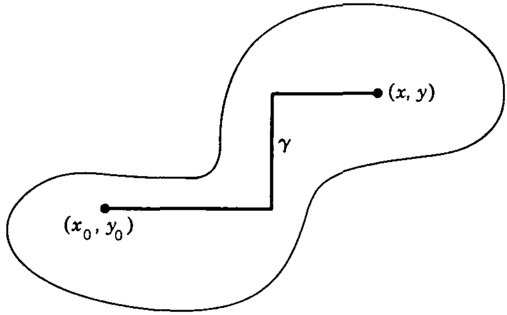

FIG. 4-1

The following theorem gives a necessary and sufficient condition under which a line integral depends only on the end points.

Theorem 1. *The line integral* $\int_\gamma p\,dx + q\,dy$*, defined in* Ω*, depends only on the end points of* γ *if and only if there exists a function* $U(x,y)$ *in* Ω *with the partial derivatives* $\partial U/\partial x = p$, $\partial U/\partial y = q$.

The sufficiency follows at once, for if the condition is fulfilled we can write, with the usual notations,

$$\int_\gamma p\,dx + q\,dy = \int_a^b \left(\frac{\partial U}{\partial x}x'(t) + \frac{\partial U}{\partial y}y'(t)\right)dt = \int_a^b \frac{d}{dt}U(x(t),y(t))\,dt$$
$$= U(x(b),y(b)) - U(x(a),y(a)),$$

and the value of this difference depends only on the end points. To prove the necessity we choose a fixed point $(x_0,y_0) \in \Omega$, join it to (x,y) by a polygon γ, contained in Ω, whose sides are parallel to the coordinate axes (Fig. 4-1) and define a function by

$$U(x,y) = \int_\gamma p\,dx + q\,dy.$$

Since the integral depends only on the end points, the function is well defined. Moreover, if we choose the last segment of γ horizontal, we can keep y constant and let x vary without changing the other segments. On the last segment we can choose x for parameter and obtain

$$U(x,y) = \int^x p(x,y)\,dx + \text{const.},$$

the lower limit of the integral being irrelevant. From this expression it follows at once that $\partial U/\partial x = p$. In the same way, by choosing the last segment vertical, we can show that $\partial U/\partial y = q$.

It is customary to write $dU = (\partial U/\partial x)\,dx + (\partial U/\partial y)\,dy$ and to say that an expression $p\,dx + q\,dy$ which can be written in this form is an *exact differential.* Thus an integral depends only on the end points if and only if the integrand is an exact differential. Observe that p, q and U can be either real or complex. The function U, if it exists, is uniquely determined up to an additive constant, for if two functions have the same partial derivatives their difference must be constant.

When is $f(z)\,dz = f(z)\,dx + if(z)\,dy$ an exact differential? According to the definition there must exist a function $F(z)$ in Ω with the partial derivatives

$$\frac{\partial F(z)}{\partial x} = f(z)$$
$$\frac{\partial F(z)}{\partial y} = if(z).$$

If this is so, $F(z)$ fulfills the Cauchy-Riemann equation

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y};$$

since $f(z)$ is by assumption continuous (otherwise $\int_\gamma f\,dz$ would not be defined) $F(z)$ is analytic with the derivative $f(z)$ (Chap. 2, Sec. 1.2).

The integral $\int_\gamma f\,dz$, with continuous f, depends only on the end points of γ if and only if f is the derivative of an analytic function in Ω.

Under these circumstances we shall prove later that $f(z)$ is itself analytic.

As an immediate application of the above result we find that

$$\int_\gamma (z-a)^n\,dz = 0 \tag{11}$$

for all closed curves γ, provided that the integer n is $\geqq 0$. In fact, $(z-a)^n$ is the derivative of $(z-a)^{n+1}/(n+1)$, a function which is analytic in the whole plane. If n is negative, but $\neq -1$, the same result holds for all closed curves which do not pass through a, for in the complementary region of the point a the indefinite integral is still analytic and single-valued. For $n = -1$, (11) does not always hold. Consider a circle C with the center a, represented by the equation $z = a + \rho e^{it}$, $0 \leqq t \leqq 2\pi$. We obtain

$$\int_C \frac{dz}{z-a} = \int_0^{2\pi} i\,dt = 2\pi i.$$

This result shows that it is impossible to define a single-valued branch of $\log(z - a)$ in an annulus $\rho_1 < |z - a| < \rho_2$. On the other hand, if the closed curve γ is contained in a half plane which does not contain a, the integral vanishes, for in such a half plane a single-valued and analytic branch of $\log(z - a)$ can be defined.

EXERCISES

1. Compute

$$\int_\gamma x\,dz$$

where γ is the directed line segment from 0 to $1 + i$.

2. Compute

$$\int_{|z|=r} x\,dz,$$

for the positive sense of the circle, in two ways: first, by use of a parameter, and second, by observing that $x = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}\left(z + \frac{r^2}{z}\right)$ on the circle.

3. Compute

$$\int_{|z|=2} \frac{dz}{z^2 - 1}$$

for the positive sense of the circle.

4. Compute

$$\int_{|z|=1} |z - 1| \cdot |dz|.$$

5. Suppose that $f(z)$ is analytic on a closed curve γ (i.e., f is analytic in a region that contains γ). Show that

$$\int_\gamma \overline{f(z)} f'(z)\,dz$$

is purely imaginary. (The continuity of $f'(z)$ is taken for granted.)

6. Assume that $f(z)$ is analytic and satisfies the inequality $|f(z) - 1| < 1$ in a region Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 0$$

for every closed curve in Ω. (The continuity of $f'(z)$ is taken for granted.)

7. If $P(z)$ is a polynomial and C denotes the circle $|z - a| = R$, what is the value of $\int_C P(z)\,d\bar{z}$? *Answer:* $-2\pi i R^2 P'(a)$.

8. Describe a set of circumstances under which the formula

$$\int_\gamma \log z\, dz = 0$$

is meaningful and true.

1.4. Cauchy's Theorem for a Rectangle. There are several forms of Cauchy's theorem, but they differ in their topological rather than in their analytical content. It is natural to begin with a case in which the topological considerations are trivial.

We consider, specifically, a rectangle R defined by inequalities $a \leqq x \leqq b, c \leqq y \leqq d$. Its perimeter can be considered as a simple closed curve consisting of four line segments whose direction we choose so that R lies to the left of the directed segments. The order of the vertices is thus (a,c), (b,c), (b,d), (a,d). We refer to this closed curve as the *boundary curve* or *contour* of R, and we denote it by ∂R.†

We emphasize that R is chosen as a closed point set and, hence, is not a region. In the theorem that follows we consider a function which is analytic on the rectangle R. We recall to the reader that such a function is by definition defined and analytic in an open set which contains R.

The following is a preliminary version of *Cauchy's theorem:*

Theorem 2. *If the function $f(z)$ is analytic on R, then*

$$\int_{\partial R} f(z)\, dz = 0. \tag{12}$$

The proof is based on the method of bisection. Let us introduce the notation

$$\eta(R) = \int_{\partial R} f(z)\, dz$$

which we will also use for any rectangle contained in the given one. If R is divided into four congruent rectangles $R^{(1)}$, $R^{(2)}$, $R^{(3)}$, $R^{(4)}$, we find that

$$\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)}), \tag{13}$$

for the integrals over the common sides cancel each other. It is important to note that this fact can be verified explicitly and does not make illicit use of geometric intuition. Nevertheless, a reference to Fig. 4-2 is helpful.

† This is standard notation, and we shall use it repeatedly. Note that by earlier convention ∂R is also the boundary of R as a point set (Chap. 3, Sec. 1.2).

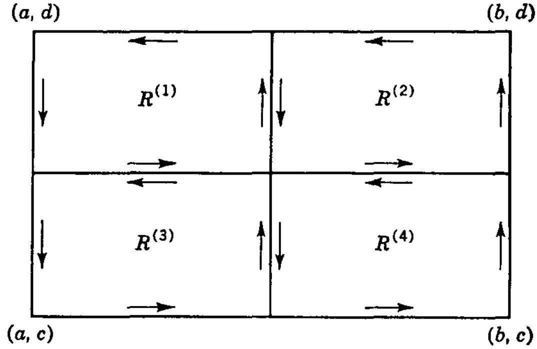

FIG. 4-2. Bisection of rectangle.

It follows from (13) that at least one of the rectangles $R^{(k)}$, $k = 1, 2, 3, 4$, must satisfy the condition

$$|\eta(R^{(k)})| \geqq \tfrac{1}{4}|\eta(R)|.$$

We denote this rectangle by R_1; if several $R^{(k)}$ have this property, the choice shall be made according to some definite rule.

This process can be repeated indefinitely, and we obtain a sequence of nested rectangles $R \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$ with the property

$$|\eta(R_n)| \geqq \tfrac{1}{4}|\eta(R_{n-1})|$$

and hence

$$(14) \qquad |\eta(R_n)| \geqq 4^{-n}|\eta(R)|.$$

The rectangles R_n converge to a point $z^* \in R$ in the sense that R_n will be contained in a prescribed neighborhood $|z - z^*| < \delta$ as soon as n is sufficiently large. First of all, we choose δ so small that $f(z)$ is defined and analytic in $|z - z^*| < \delta$. Secondly, if $\varepsilon > 0$ is given, we can choose δ so that

$$\left|\frac{f(z) - f(z^*)}{z - z^*} - f'(z^*)\right| < \varepsilon$$

or

$$(15) \qquad |f(z) - f(z^*) - (z - z^*)f'(z^*)| < \varepsilon|z - z^*|$$

for $|z - z^*| < \delta$. We assume that δ satisfies both conditions and that R_n is contained in $|z - z^*| < \delta$.

We make now the observation that

$$\int_{\partial R_n} dz = 0$$

$$\int_{\partial R_n} z\,dz = 0.$$

These trivial special cases of our theorem have already been proved in Sec. 1.1. We recall that the proof depended on the fact that 1 and z are the derivatives of z and $z^2/2$, respectively.

By virtue of these equations we are able to write

$$\eta(R_n) = \int_{\partial R_n} [f(z) - f(z^*) - (z - z^*)f'(z^*)]\, dz,$$

and it follows by (15) that

$$|\eta(R_n)| \leqq \varepsilon \int_{\partial R_n} |z - z^*| \cdot |dz|. \tag{16}$$

In the last integral $|z - z^*|$ is at most equal to the length d_n of the diagonal of R_n. If L_n denotes the length of the perimeter of R_n, the integral is hence $\leqq d_n L_n$. But if d and L are the corresponding quantities for the original rectangle R, it is clear that $d_n = 2^{-n}d$ and $L_n = 2^{-n}L$. By (16) we have hence

$$|\eta(R_n)| \leqq 4^{-n}\, dL\, \varepsilon,$$

and comparison with (14) yields

$$|\eta(R)| \leqq dL\, \varepsilon.$$

Since ε is arbitrary, we can only have $\eta(R) = 0$, and the theorem is proved.

This beautiful proof, which could hardly be simpler, is due to É. Goursat who discovered that the classical hypothesis of a continuous $f'(z)$ is redundant. At the same time the proof is simpler than the earlier proofs inasmuch as it leans neither on double integration nor on differentiation under the integral sign.

The hypothesis in Theorem 2 can be weakened considerably. We shall prove at once the following stronger theorem which will find very important use.

Theorem 3. *Let $f(z)$ be analytic on the set R' obtained from a rectangle R by omitting a finite number of interior points ζ_j. If it is true that*

$$\lim_{z \to \zeta_j} (z - \zeta_j) f(z) = 0$$

for all j, then

$$\int_{\partial R} f(z)\, dz = 0.$$

It is sufficient to consider the case of a single exceptional point ζ, for evidently R can be divided into smaller rectangles which contain at most one ζ_j.

We divide R into nine rectangles, as shown in Fig. 4-3, and apply

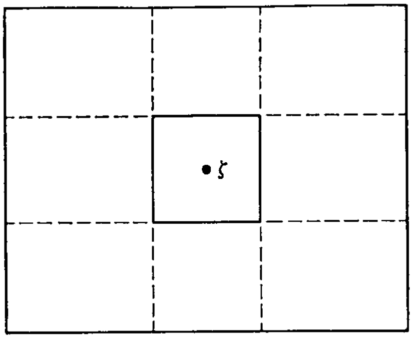

FIG. 4-3

Theorem 2 to all but the rectangle R_0 in the center. If the corresponding equations (12) are added, we obtain, after cancellations,

$$\int_{\partial R} f\,dz = \int_{\partial R_0} f\,dz. \tag{17}$$

If $\varepsilon > 0$ we can choose the rectangle R_0 so small that

$$|f(z)| \leqq \frac{\varepsilon}{|z - \zeta|}$$

on ∂R_0. By (17) we have thus

$$\left| \int_{\partial R} f\,dz \right| \leqq \varepsilon \int_{\partial R_0} \frac{|dz|}{|z - \zeta|}.$$

If we assume, as we may, that R_0 is a square of center ζ, elementary estimates show that

$$\int_{\partial R_0} \frac{|dz|}{|z - \zeta|} < 8.$$

Thus we obtain

$$\left| \int_{\partial R} f\,dz \right| < 8\varepsilon,$$

and since ε is arbitrary the theorem follows.

We observe that the hypothesis of the theorem is certainly fulfilled if $f(z)$ is analytic and *bounded* on R'.

1.5. Cauchy's Theorem in a Disk. It is not true that the integral of an analytic function over a closed curve is always zero.

Indeed, we have found that

$$\int_C \frac{dz}{z-a} = 2\pi i$$

when C is a circle about a. In order to make sure that the integral vanishes, it is necessary to make a special assumption concerning the region Ω in which $f(z)$ is known to be analytic and to which the curve γ is restricted. We are not yet in a position to formulate this condition, and for this reason we must restrict attention to a very special case. In what follows we assume that Ω is an open disk $|z - z_0| < \rho$ to be denoted by Δ.

Theorem 4. *If $f(z)$ is analytic in an open disk Δ, then*

$$\int_\gamma f(z)\,dz = 0 \tag{18}$$

for every closed curve γ in Δ.

The proof is a repetition of the argument used in proving the second half of Theorem 1. We define a function $F(z)$ by

$$F(z) = \int_\sigma f\,dz \tag{19}$$

where σ consists of the horizontal line segment from the center (x_0,y_0) to (x,y_0) and the vertical segment from (x,y_0) to (x,y); it is immediately seen that $\partial F/\partial y = if(z)$. On the other hand, by Theorem 2 σ can be replaced by a path consisting of a vertical segment followed by a horizontal segment. This choice defines the same function $F(z)$, and we obtain $\partial F/\partial x = f(z)$. Hence $F(z)$ is analytic in Δ with the derivative $f(z)$, and $f(z)\,dz$ is an exact differential.

Clearly, the same proof would go through for any region which contains the rectangle with the opposite vertices z_0 and z as soon as it contains z. A rectangle, a half plane, or the inside of an ellipse all have this property, and hence Theorem 4 holds for any of these regions. By this method we cannot, however, reach full generality.

For the applications it is very important that the conclusion of Theorem 4 remains valid under the weaker condition of Theorem 3. We state this as a separate theorem.

Theorem 5. *Let $f(z)$ be analytic in the region Δ' obtained by omitting a finite number of points ζ_j from an open disk Δ. If $f(z)$ satisfies the condition $\lim_{z\to\zeta_j}(z - \zeta_j)f(z) = 0$ for all j, then* (18) *holds for any closed curve γ in Δ'.*

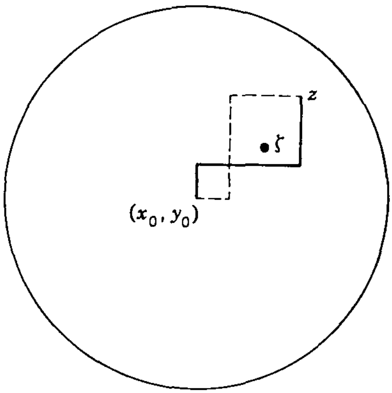

FIG. 4-4

The proof must be modified, for we cannot let σ pass through the exceptional points. Assume first that no ζ_j lies on the lines $x = x_0$ and $y = y_0$. It is then possible to avoid the exceptional points by letting σ consist of three segments (Fig. 4-4). By an obvious application of Theorem 3 we find that the value of $F(z)$ in (18) is independent of the choice of the middle segment; moreover, the last segment can be either vertical or horizontal. We conclude as before that $F(z)$ is an indefinite integral of $f(z)$, and the theorem follows.

In case there are exceptional points on the lines $x = x_0$ and $y = y_0$ the reader will easily convince himself that a similar proof can be carried out, provided that we use four line segments in the place of three.

2. CAUCHY'S INTEGRAL FORMULA

Through a very simple application of Cauchy's theorem it becomes possible to represent an analytic function $f(z)$ as a line integral in which the variable z enters as a parameter. This representation, known as *Cauchy's integral formula,* has numerous important applications. Above all, it enables us to study the local properties of an analytic function in great detail.

2.1. The Index of a Point with Respect to a Closed Curve. As a preliminary to the derivation of Cauchy's formula we must define a notion which in a precise way indicates how many times a closed curve winds around a fixed point not on the curve. If the curve is piecewise differentiable, as we shall assume without serious loss of generality, the definition can be based on the following lemma:

Lemma 1. *If the piecewise differentiable closed curve γ does not pass through the point a, then the value of the integral*

$$\int_\gamma \frac{dz}{z-a}$$

is a multiple of $2\pi i$.

This lemma may seem trivial, for we can write

$$\int_\gamma \frac{dz}{z-a} = \int_\gamma d \log (z-a) = \int_\gamma d \log |z-a| + i \int_\gamma d \arg (z-a).$$

When z describes a closed curve, $\log |z-a|$ returns to its initial value and $\arg (z-a)$ increases or decreases by a multiple of 2π. This would seem to imply the lemma, but more careful thought shows that the reasoning is of no value unless we define $\arg (z-a)$ in a unique way.

The simplest proof is computational. If the equation of γ is $z = z(t)$, $\alpha \leqq t \leqq \beta$, let us consider the function

$$h(t) = \int_\alpha^t \frac{z'(t)}{z(t)-a}\, dt.$$

It is defined and continuous on the closed interval $[\alpha,\beta]$, and it has the derivative

$$h'(t) = \frac{z'(t)}{z(t)-a}$$

whenever $z'(t)$ is continuous. From this equation it follows that the derivative of $e^{-h(t)}(z(t)-a)$ vanishes except perhaps at a finite number of points, and since this function is continuous it must reduce to a constant. We have thus

$$e^{h(t)} = \frac{z(t)-a}{z(\alpha)-a}.$$

Since $z(\beta) = z(\alpha)$ we obtain $e^{h(\beta)} = 1$, and therefore $h(\beta)$ must be a multiple of $2\pi i$. This proves the lemma.

We can now define *the index of the point a with respect to the curve* γ by the equation

$$n(\gamma,a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}.$$

With a suggestive terminology the index is also called the *winding number* of γ with respect to a.

It is clear that $n(-\gamma,a) = -n(\gamma,a)$.

The following property is an immediate consequence of Theorem 4:

(i) *If γ lies inside of a circle, then $n(\gamma,a) = 0$ for all points a outside of the same circle.*

As a point set γ is closed and bounded. Its complement is open and can be represented as a union of disjoint regions, the components of the complement. We shall say, for short, that γ *determines* these regions. If the complementary regions are considered in the extended plane, there is exactly one which contains the point at infinity. Consequently, γ determines one and only one unbounded region.

(ii) *As a function of a the index $n(\gamma,a)$ is constant in each of the regions determined by γ, and zero in the unbounded region.*

Any two points in the same region determined by γ can be joined by a polygon which does not meet γ. For this reason it is sufficient to prove that $n(\gamma,a) = n(\gamma,b)$ if γ does not meet the line segment from a to b. Outside of this segment the function $(z - a)/(z - b)$ is never real and $\leqq 0$. For this reason the principal branch of $\log [(z - a)/(z - b)]$ is analytic in the complement of the segment. Its derivative is equal to $(z - a)^{-1} - (z - b)^{-1}$, and if γ does not meet the segment we must have

$$\int_\gamma \left(\frac{1}{z - a} - \frac{1}{z - b}\right) dz = 0;$$

hence $n(\gamma,a) = n(\gamma,b)$. If $|a|$ is sufficiently large, γ is contained in a disk $|z| < \rho < |a|$ and we conclude by (i) that $n(\gamma,a) = 0$. This proves that $n(\gamma,a) = 0$ in the unbounded region.

We shall find the case $n(\gamma,a) = 1$ particularly important, and it is desirable to formulate a geometric condition which leads to this consequence. For simplicity we take $a = 0$.

Lemma 2. *Let z_1, z_2 be two points on a closed curve γ which does not pass through the origin. Denote the subarc from z_1 to z_2 in the direction of the curve by γ_1, and the subarc from z_2 to z_1 by γ_2. Suppose that z_1 lies in the lower half plane and z_2 in the upper half plane. If γ_1 does not meet the negative real axis and γ_2 does not meet the positive real axis, then $n(\gamma,0) = 1$.*

For the proof we draw the half lines L_1 and L_2 from the origin through z_1 and z_2 (Fig. 4-5). Let ζ_1, ζ_2 be the points in which L_1, L_2 intersect a circle C about the origin. If C is described in the positive sense, the arc C_1 from ζ_1 to ζ_2 does not intersect the negative axis, and the arc C_2 from ζ_2 to ζ_1 does not intersect the positive axis. Denote the directed line segments from z_1 to ζ_1 and from z_2 to ζ_2 by δ_1, δ_2. Introducing the closed curves $\sigma_1 = \gamma_1 + \delta_2 - C_1 - \delta_1$, $\sigma_2 = \gamma_2 + \delta_1 - C_2 - \delta_2$ we find that $n(\gamma,0) = n(C,0) + n(\sigma_1,0) + n(\sigma_2,0)$ because of cancellations. But σ_1 does not meet the negative axis. Hence the origin belongs to the

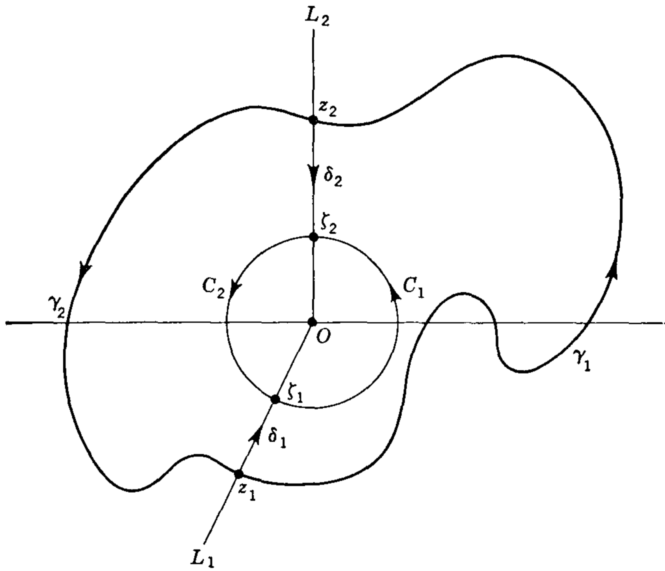

FIG. 4-5

unbounded region determined by σ_1, and we obtain $n(\sigma_1,0) = 0$. For a similar reason $n(\sigma_2,0) = 0$, and we conclude that $n(\gamma,0) = n(C,0) = 1$.

*EXERCISES

These are not routine exercises. They serve to illustrate the topological use of winding numbers.

1. Give an alternate proof of Lemma 1 by dividing γ into a finite number of subarcs such that there exists a single-valued branch of $\arg(z - a)$ on each subarc. Pay particular attention to the compactness argument that is needed to prove the existence of such a subdivision.

2. It is possible to define $n(\gamma,a)$ for any continuous closed curve γ that does not pass through a, whether piecewise differentiable or not. For this purpose γ is divided into subarcs $\gamma_1, \ldots, \gamma_n$, each contained in a disk that does not include a. Let σ_k be the directed line segment from the initial to the terminal point of γ_k, and set $\sigma = \sigma_1 + \cdots + \sigma_n$. We define $n(\gamma,a)$ to be the value of $n(\sigma,a)$.

To justify the definition, prove the following:

(*a*) the result is independent of the subdivision;

(*b*) if γ is piecewise differentiable the new definition is equivalent to the old;

(*c*) the properties (i) and (ii) of the text continue to hold.

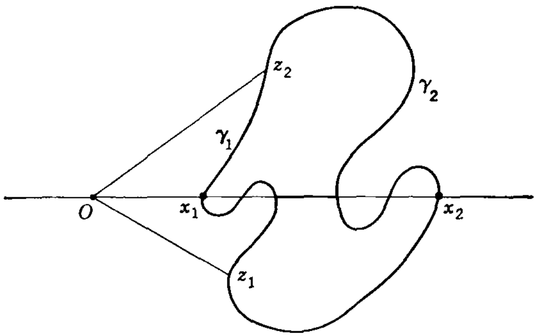

FIG. 4-6. Part of the Jordan curve theorem.

3. The *Jordan curve theorem* asserts that every Jordan curve in the plane determines exactly two regions. The notion of winding number leads to a quick proof of one part of the theorem, namely that the complement of a Jordan curve γ has at least two components. This will be so if there exists a point a with $n(\gamma,a) \neq 0$.

We may assume that Re $z > 0$ on γ, and that there are points z_1, $z_2 \epsilon \gamma$ with Im $z_1 < 0$, Im $z_2 > 0$. These points may be chosen so that there are no other points of γ on the line segments from 0 to z_1 and from 0 to z_2. Let γ_1 and γ_2 be the arcs of γ from z_1 to z_2 (excluding the end points).

Let σ_1 be the closed curve that consists of the line segment from 0 to z_1 followed by γ_1 and the segment from z_2 to 0, and let σ_2 be constructed in the same way with γ_2 in the place of γ_1. Then $\sigma_1 - \sigma_2 = \gamma$ or $-\gamma$.

The positive real axis intersects both γ_1 and γ_2 (why?). Choose the notation so that the intersection x_2 farthest to the right is with γ_2 (Fig. 4-6).

Prove the following:

(*a*) $n(\sigma_1,x_2) = 0$, hence $n(\sigma_1,z) = 0$ for $z \epsilon \gamma_2$;

(*b*) $n(\sigma_1,x) = n(\sigma_2,x) = 1$ for small $x > 0$ (Lemma 2);

(*c*) the first intersection x_1 of the positive real axis with γ lies on γ_1;

(*d*) $n(\sigma_2,x_1) = 1$, hence $n(\sigma_2,z) = 1$ for $z \epsilon \gamma_1$;

(*e*) there exists a segment of the positive real axis with one end point on γ_1, the other on γ_2, and no other points on γ. The points x between the end points satisfy $n(\gamma,x) = 1$ or -1.

2.2. The Integral Formula. Let $f(z)$ be analytic in an open disk Δ. Consider a closed curve γ in Δ and a point $a \epsilon \Delta$ which does not lie on γ. We apply Cauchy's theorem to the function

$$F(z) = \frac{f(z) - f(a)}{z - a}.$$

This function is analytic for $z \neq a$. For $z = a$ it is not defined, but it satisfies the condition

$$\lim_{z \to a} F(z)(z - a) = \lim_{z \to a} (f(z) - f(a)) = 0$$

which is the condition of Theorem 5. We conclude that

$$\int_\gamma \frac{f(z) - f(a)}{z - a}\, dz = 0.$$

This equation can be written in the form

$$\int_\gamma \frac{f(z)\, dz}{z - a} = f(a) \int_\gamma \frac{dz}{z - a},$$

and we observe that the integral in the right-hand member is by definition $2\pi i \cdot n(\gamma,a)$. We have thus proved:

Theorem 6. *Suppose that $f(z)$ is analytic in an open disk Δ, and let γ be a closed curve in Δ. For any point a not on γ*

$$n(\gamma,a) \cdot f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)\, dz}{z - a}, \tag{20}$$

where $n(\gamma,a)$ is the index of a with respect to γ.

In this statement we have suppressed the requirement that a be a point in Δ. We have done so in view of the obvious interpretation of the formula (20) for the case that a is not in Δ. Indeed, in this case $n(\gamma,a)$ and the integral in the right-hand member are both zero.

It is clear that Theorem 6 remains valid for any region Ω to which Theorem 5 can be applied. The presence of exceptional points ζ_j is permitted, provided none of them coincides with a.

The most common application is to the case where $n(\gamma,a) = 1$. We have then

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)\, dz}{z - a}, \tag{21}$$

and this we interpret as a *representation formula.* Indeed, it permits us to compute $f(a)$ as soon as the values of $f(z)$ on γ are given, together with the fact that $f(z)$ is analytic in Δ. In (21) we may let a take different values, provided that the order of a with respect to γ remains equal to 1. We may thus treat a as a variable, and it is convenient to change the notation and rewrite (21) in the form

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{\zeta - z}. \tag{22}$$

It is this formula which is usually referred to as *Cauchy's integral formula.* We must remember that it is valid only when $n(\gamma,z) = 1$, and that we have proved it only when $f(z)$ is analytic in a disk.

EXERCISES

1. Compute

$$\int_{|z|=1} \frac{e^z}{z}\,dz.$$

2. Compute

$$\int_{|z|=2} \frac{dz}{z^2+1}$$

by decomposition of the integrand in partial fractions.

3. Compute

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

under the condition $|a| \neq \rho$. *Hint:* make use of the equations $z\bar{z} = \rho^2$ and

$$|dz| = -i\rho\,\frac{dz}{z}.$$

2.3. Higher Derivatives. The representation formula (22) gives us an ideal tool for the study of the local properties of analytic functions. In particular we can now show that an analytic function has derivatives of all orders, which are then also analytic.

We consider a function $f(z)$ which is analytic in an arbitrary region Ω. To a point $a \in \Omega$ we determine a δ-neighborhood Δ contained in Ω, and in Δ a circle C about a. Theorem 6 can be applied to $f(z)$ in Δ. Since $n(C,a) = 1$ we have $n(C,z) = 1$ for all points z inside of C. For such z we obtain by (22)

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

Provided that the integral can be differentiated under the sign of integration we find

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2} \tag{23}$$

and

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}. \tag{24}$$

If the differentiations can be justified, we shall have proved the existence of all derivatives at the points inside of C. Since every point in Ω lies inside of some such circle, the existence will be proved in the whole region Ω. At the same time we shall have obtained a convenient representation formula for the derivatives.

For the justification we could either refer to corresponding theorems in the real case, or we could prove a general theorem concerning line integrals whose integrand depends analytically on a parameter. Actually, we shall prove only the following lemma which is all we need in the present case:

Lemma 3. *Suppose that $\varphi(\zeta)$ is continuous on the arc γ. Then the function*

$$F_n(z) = \int_\gamma \frac{\varphi(\zeta)\,d\zeta}{(\zeta - z)^n}$$

is analytic in each of the regions determined by γ, and its derivative is $F_n'(z) = nF_{n+1}(z)$.

We prove first that $F_1(z)$ is continuous. Let z_0 be a point not on γ, and choose the neighborhood $|z - z_0| < \delta$ so that it does not meet γ. By restricting z to the smaller neighborhood $|z - z_0| < \delta/2$ we attain that $|\zeta - z| > \delta/2$ for all $\zeta \in \gamma$. From

$$F_1(z) - F_1(z_0) = (z - z_0) \int_\gamma \frac{\varphi(\zeta)\,d\zeta}{(\zeta - z)(\zeta - z_0)}$$

we obtain at once

$$|F_1(z) - F_1(z_0)| < |z - z_0| \cdot \frac{2}{\delta^2} \int_\gamma |\varphi|\,|d\zeta|,$$

and this inequality proves the continuity of $F_1(z)$ at z_0.

From this part of the lemma, applied to the function $\varphi(\zeta)/(\zeta - z_0)$, we conclude that the difference quotient

$$\frac{F_1(z) - F_1(z_0)}{z - z_0} = \int_\gamma \frac{\varphi(\zeta)\,d\zeta}{(\zeta - z)(\zeta - z_0)}$$

tends to the limit $F_2(z_0)$ as $z \to z_0$. Hence it is proved that $F_1'(z) = F_2(z)$.

The general case is proved by induction. Suppose we have shown that $F_{n-1}'(z) = (n - 1)F_n(z)$. From the identity

$$F_n(z) - F_n(z_0)$$
$$= \left[\int_\gamma \frac{\varphi\,d\zeta}{(\zeta - z)^{n-1}(\zeta - z_0)} - \int_\gamma \frac{\varphi\,d\zeta}{(\zeta - z_0)^n}\right] + (z - z_0) \int_\gamma \frac{\varphi\,d\zeta}{(\zeta - z)^n(\zeta - z_0)}$$

we can conclude that $F_n(z)$ is continuous. Indeed, by the induction hypothesis, applied to $\varphi(\zeta)/(\zeta - z_0)$, the first term tends to zero for $z \to z_0$, and in the second term the factor of $z - z_0$ is bounded in a neighborhood of z_0. Now, if we divide the identity by $z - z_0$ and let z tend to z_0, the quotient in the first term tends to a derivative which by the induction hypothesis equals $(n-1)F_{n+1}(z_0)$. The remaining factor in the second term is continuous, by what we have already proved, and has the limit $F_{n+1}(z_0)$. Hence $F_n'(z_0)$ exists and equals $nF_{n+1}(z_0)$.

It is clear that Lemma 3 is just what is needed in order to deduce (23) and (24) in a rigorous way. We have thus proved that an analytic function has derivatives of all orders which are analytic and can be represented by the formula (24).

Among the consequences of this result we like to single out two classical theorems. The first is known as *Morera's theorem*, and it can be stated as follows:

If $f(z)$ is defined and continuous in a region Ω, and if $\int_\gamma f\,dz = 0$ for all closed curves γ in Ω, then $f(z)$ is analytic in Ω.

The hypothesis implies, as we have already remarked in Sec. 1.3, that $f(z)$ is the derivative of an analytic function $F(z)$. We know now that $f(z)$ is then itself analytic.

A second classical result goes under the name of *Liouville's theorem:*

A function which is analytic and bounded in the whole plane must reduce to a constant.

For the proof we make use of a simple estimate derived from (24). Let the radius of C be r, and assume that $|f(\zeta)| \leqq M$ on C. If we apply (24) with $z = a$, we obtain at once

$$|f^{(n)}(a)| \leqq Mn!r^{-n}. \tag{25}$$

For Liouville's theorem we need only the case $n = 1$. The hypothesis means that $|f(\zeta)| \leqq M$ on all circles. Hence we can let r tend to ∞, and (25) leads to $f'(a) = 0$ for all a. We conclude that the function is constant.

Liouville's theorem leads to an almost trivial proof of the *fundamental theorem of algebra.* Suppose that $P(z)$ is a polynomial of degree > 0. If $P(z)$ were never zero, the function $1/P(z)$ would be analytic in the whole plane. We know that $P(z) \to \infty$ for $z \to \infty$, and therefore $1/P(z)$ tends to zero. This implies boundedness (the absolute value is continuous on the Riemann sphere and has thus a finite maximum), and by Liouville's theorem $1/P(z)$ would be constant. Since this is not so, the equation $P(z) = 0$ must have a root.

The inequality (25) is known as *Cauchy's estimate.* It shows above