COMPLEX ANALYSIS

An Introduction to the Theory of Analytic Functions of One Complex Variable

Third Edition

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satisfies the hypothesis of the original theorem. Hence we obtain $|Sf(T^{-1}\zeta)| \leqq |\zeta|$, or $|Sf(z)| \leqq |Tz|$. Explicitly, this inequality can be written in the form

$$\left| \frac{M(f(z) - w_0)}{M^2 - \bar{w}_0 f(z)} \right| \leqq \left| \frac{R(z - z_0)}{R^2 - \bar{z}_0 z} \right|. \tag{36}$$

EXERCISES

1. Show by use of (36), or directly, that $|f(z)| \leqq 1$ for $|z| \leqq 1$ implies

$$\frac{|f'(z)|}{(1 - |f(z)|^2)} \leqq \frac{1}{1 - |z|^2}.$$

2. If $f(z)$ is analytic and $\operatorname{Im} f(z) \geqq 0$ for $\operatorname{Im} z > 0$, show that

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leqq \frac{|z - z_0|}{|z - \bar{z}_0|}$$

and

$$\frac{|f'(z)|}{\operatorname{Im} f(z)} \leqq \frac{1}{y} \qquad (z = x + iy).$$

3. In Ex. 1 and 2, prove that equality implies that $f(z)$ is a linear transformation.

4. Derive corresponding inequalities if $f(z)$ maps $|z| < 1$ into the upper half plane.

5. Prove by use of Schwarz's lemma that every one-to-one conformal mapping of a disk onto another (or a half plane) is given by a linear transformation.

***6.** If γ is a piecewise differentiable arc contained in $|z| < 1$ the integral

$$\int_\gamma \frac{|dz|}{1 - |z|^2}$$

is called the *noneuclidean length* (or hyperbolic length) of γ. Show that an analytic function $f(z)$ with $|f(z)| < 1$ for $|z| < 1$ maps every γ on an arc with smaller or equal noneuclidean length.

Deduce that a linear transformation of the unit disk onto itself preserves noneuclidean lengths, and check the result by explicit computation.

***7.** Prove that the arc of smallest noneuclidean length that joins two given points in the unit disk is a circular arc which is orthogonal to the unit circle. (Make use of a linear transformation that carries one end point to the origin, the other to a point on the positive real axis.)

The shortest noneuclidean length is called the *noneuclidean distance* between the end points. Derive a formula for the noneuclidean distance between z_1 and z_2. *Answer:*

$$\frac{1}{2}\log\frac{1+\left|\dfrac{z_1-z_2}{1-\bar{z}_1z_2}\right|}{1-\left|\dfrac{z_1-z_2}{1-\bar{z}_1z_2}\right|}.$$

*8. How should noneuclidean length in the upper half plane be defined?

4. THE GENERAL FORM OF CAUCHY'S THEOREM

In our preliminary treatment of Cauchy's theorem and the integral formula we considered only the case of a circular region. For the purpose of studying the local properties of analytic functions this was quite adequate, but from a more general point of view we cannot be satisfied with a result which is so obviously incomplete. The generalization can proceed in two directions. For one thing we can seek to characterize the regions in which Cauchy's theorem has universal validity. Secondly, we can consider an arbitrary region and look for the curves γ for which the assertion of Cauchy's theorem is true.

4.1. Chains and Cycles. In the first place we must generalize the notion of line integral. To this end we examine the equation

$$(37)\qquad \int_{\gamma_1+\gamma_2+\cdots+\gamma_n} f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz + \cdots + \int_{\gamma_n} f\,dz$$

which is valid when $\gamma_1, \gamma_2, \ldots, \gamma_n$ form a subdivision of the arc γ. Since the right-hand member of (37) has a meaning for any finite collection, nothing prevents us from considering an arbitrary formal sum $\gamma_1 + \gamma_2 + \cdots + \gamma_n$, which need not be an arc, and we define the corresponding integral by means of equation (37). Such formal sums of arcs are called *chains*. It is clear that nothing is lost and much may be gained by considering line integrals over arbitrary chains.

Just as there is nothing unique about the way in which an arc can be subdivided, it is clear that different formal sums can represent the same chain. The guiding principle is that two chains should be considered identical if they yield the same line integrals for all functions f. If this principle is analyzed, we find that the following operations do not change the identity of a chain: (1) permutation of two arcs, (2) subdivision of an arc, (3) fusion of subarcs to a single arc, (4) reparametrization of an arc, (5) cancellation of opposite arcs. On this basis it would be easy to formulate a logical equivalence relation which defines the identity of chains in a formal manner. Inasmuch as the situation does not involve any logical pitfalls, we shall dispense with this formalization.

The sum of two chains is defined in the obvious way by juxtaposition. It is clear that the additive property (37) of line integrals remains valid for arbitrary chains. When identical chains are added, it is convenient to denote the sum as a multiple. With this notation every chain can be written in the form

$$\gamma = a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n \tag{38}$$

where the a_j are positive integers and the γ_j are all different. For opposite arcs we are allowed to write $a(-\gamma) = -a\gamma$ and continue the reduction of (38) until no two γ_j are opposite. The coefficients will be arbitrary integers, and terms with zero coefficients can be added at will. The last device enables us to express any two chains in terms of the same arcs, and their sum is obtained by adding corresponding coefficients. The zero chain is either an empty sum or a sum with all coefficients equal to zero.

A chain is a *cycle* if it can be represented as a sum of closed curves. Very simple combinatorial considerations show that a chain is a cycle if and only if in any representation the initial and end points of the individual arcs are identical in pairs. Thus it is immediately possible to tell whether a chain is a cycle or not.

In the applications we shall consider chains which are contained in a given open set Ω. By this we mean that the chains have a representation by arcs in Ω and that only such representations will be considered. It is clear that all theorems which we have heretofore formulated only for closed curves in a region are in fact valid for arbitrary cycles in a region. In particular, *the integral of an exact differential over any cycle is zero.*

The index of a point with respect to a cycle is defined in exactly the same way as in the case of a single closed curve. It has the same properties, and in addition we can formulate the obvious but important additive law expressed by the equation $n(\gamma_1 + \gamma_2, a) = n(\gamma_1, a) + n(\gamma_2, a)$.

4.2. Simple Connectivity. There is little doubt that all readers will know what we mean if we speak about a region without holes. Such regions are said to be *simply connected,* and it is for simply connected regions that Cauchy's theorem is universally valid. The suggestive language we have used cannot take the place of a mathematical definition, but fortunately very little is needed to make the term precise. Indeed, a region without holes is obviously one whose complement consists of a single piece. We are thus led to the following definition:

Definition 1. *A region is simply connected if its complement with respect to the extended plane is connected.*

At this point we warn the reader that this definition is not the one that is commonly accepted, the main reason being that our definition cannot be used in more than two real dimensions. In the course of our work we shall find, however, that the property expressed by Definition 1 is equivalent to a number of other properties, more or less equally important. One of these states that any closed curve can be contracted to a point, and this condition is usually chosen as definition. Our choice has the advantage of leading very quickly to the essential results in complex integration theory.

It is easy to see that a disk, a half plane, and a parallel strip are simply connected. The last example shows the importance of taking the complement with respect to the extended plane, for the complement of the strip in the finite plane is evidently not connected. The definition can be applied to regions on the Riemann sphere, and this is evidently the most symmetric situation. For our purposes it is nevertheless better to agree that all regions lie in the finite plane unless the contrary is explicitly stated. According to this convention the outside of a circle is not simply connected, for its complement consists of a closed disk and the point at infinity.

Theorem 14. *A region Ω is simply connected if and only if $n(\gamma,a) = 0$ for all cycles γ in Ω and all points a which do not belong to Ω.*

This alternative condition is also very suggestive. It states that a closed curve in a simply connected region cannot wind around any point which does not belong to the region. It seems quite evident that this condition is not fulfilled in the case of a region with a hole.

The necessity of the condition is almost trivial. Let γ be any cycle in Ω. If the complement of Ω is connected, it must be contained in one of the regions determined by γ, and inasmuch as ∞ belongs to the complement this must be the unbounded region. Consequently $n(\gamma,a) = 0$ for all finite points in the complement.

For the precise proof of the sufficiency an explicit construction is needed. We assume that the complement of Ω can be represented as the union $A \cup B$ of two disjoint closed sets. One of these sets contains ∞, and the other is consequently bounded; let A be the bounded set. The sets A and B have a shortest distance $\delta > 0$. Cover the whole plane with a net of squares Q of side $< \delta/\sqrt{2}$. We are free to choose the net so that a certain point $a \in A$ lies at the center of a square. The boundary

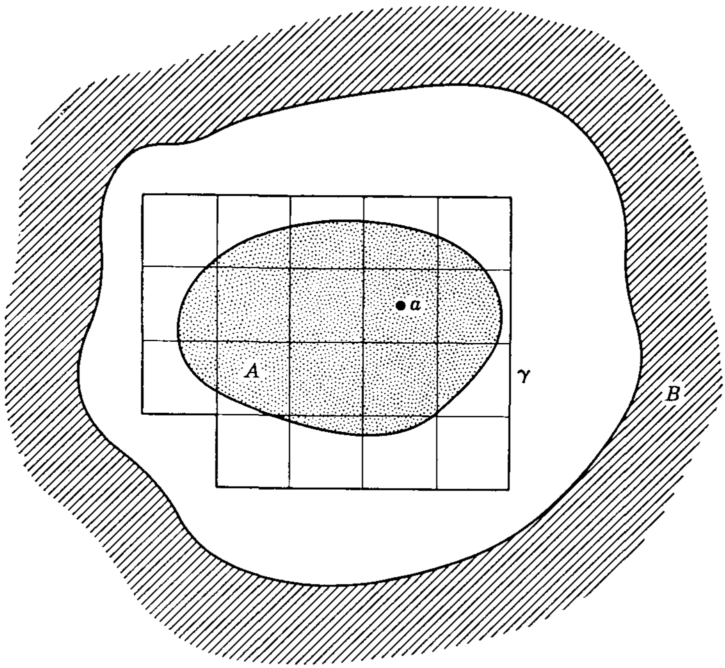

FIG. 4-9. Curve with index 1.

curve of Q is denoted by ∂Q; we assume explicitly that the squares Q are closed and that the interior of Q lies to the left of the directed line segments which make up ∂Q.

Consider now the cycle

$$\gamma = \sum_j \partial Q_j \tag{39}$$

where the sum ranges over all squares Q_j in the net which have a point in common with A (Fig. 4-9). Because a is contained in one and only one of these squares, it is evident that $n(\gamma,a) = 1$. Furthermore, it is clear that γ does not meet B. But if the cancellations are carried out, it is equally clear that γ does not meet A. Indeed, any side which meets A is a common side of two squares included in the sum (39), and since the directions are opposite the side does not appear in the reduced expression of γ. Hence γ is contained in Ω, and our theorem is proved.

We remark now that Cauchy's theorem is certainly not valid for regions which are not simply connected. In fact, if there is a cycle γ in Ω such that $n(\gamma,a) \neq 0$ for some a outside of Ω, then $1/(z - a)$ is analytic in

Ω while its integral

$$\int_\gamma \frac{dz}{z-a} = 2\pi i n(\gamma,a) \neq 0.$$

4.3. Homology. The characterization of simple connectivity by Theorem 14 singles out a property that is common to all cycles in a simply connected region, but which a cycle in an arbitrary region or open set may or may not have. This property plays an important role in topology and therefore has a special name.

Definition 2. *A cycle γ in an open set Ω is said to be homologous to zero with respect to Ω if $n(\gamma,a) = 0$ for all points a in the complement of Ω.*

In symbols we write $\gamma \sim 0 \pmod{\Omega}$. When it is clear to what open set we are referring, Ω need not be mentioned. The notation $\gamma_1 \sim \gamma_2$ shall be equivalent to $\gamma_1 - \gamma_2 \sim 0$. Homologies can be added and subtracted, and $\gamma \sim 0 \pmod{\Omega}$ implies $\gamma \sim 0 \pmod{\Omega'}$ for all $\Omega' \supset \Omega$.

Again, our terminology does not quite agree with standard usage. It was Emil Artin who discovered that the characterization of homology by vanishing winding numbers ties in precisely with what is needed for the general version of Cauchy's theorem. This idea has led to a remarkable simplification of earlier proofs.

4.4. The General Statement of Cauchy's Theorem. The definitive form of Cauchy's theorem is now very easy to state.

Theorem 15. *If $f(z)$ is analytic in Ω, then*

$$\int_\gamma f(z)\, dz = 0 \tag{40}$$

for every cycle γ which is homologous to zero in Ω.

In a different formulation, we are claiming that if γ is such that (40) holds for a certain collection of analytic functions, namely those of the form $1/(z-a)$ with a not in Ω, then it holds for all analytic functions in Ω.

In combination with Theorem 14 we have the following corollary:

Corollary 1. *If $f(z)$ is analytic in a simply connected region Ω, then* (40) *holds for all cycles γ in Ω.*

Before proving the theorem, we make an observation which ties up with the considerations in Section 1.3. As pointed out in that connection, the validity of (40) for all closed curves γ in a region means that the line integral of $f\,dz$ is independent of the path, or that $f\,dz$ is an exact differential. By Theorem 1 there is then a single-valued analytic function $F(z)$ such that $F'(z) = f(z)$ (the pleonastic term "single-valued" is used for emphasis only). In a simply connected region every analytic function is thus a derivative.

A particular application of this fact occurs very frequently:

Corollary 2. *If $f(z)$ is analytic and $\neq 0$ in a simply connected region Ω, then it is possible to define single-valued analytic branches of* $\log f(z)$ *and* $\sqrt[n]{f(z)}$ *in* Ω.

In fact, we know that there exists an analytic function $F(z)$ in Ω such that $F'(z) = f'(z)/f(z)$. The function $f(z)e^{-F(z)}$ has the derivative zero and is therefore a constant. Choosing a point $z_0 \in \Omega$ and one of the infinitely many values $\log f(z_0)$, we find that

$$e^{F(z)-F(z_0)+\log f(z_0)} = f(z),$$

and consequently we can set $\log f(z) = F(z) - F(z_0) + \log f(z_0)$. To define $\sqrt[n]{f(z)}$ we merely write it in the form $\exp((1/n) \log f(z))$.

4.5. Proof of Cauchy's Theorem.† We begin with a construction that parallels the one in the proof of Theorem 14. Assume first that Ω is bounded, but otherwise arbitrary. Given $\delta > 0$, we cover the plane by a net of squares of side δ, and we denote by Q_j, $j \in J$, the closed squares in the net which are contained in Ω; because Ω is bounded the set J is finite, and if δ is sufficiently small it is not empty. The union of the squares Q_j, $j \in J$, consists of closed regions whose oriented boundaries make up the cycle

$$\Gamma_\delta = \sum_{j \in J} \partial Q_j.$$

Clearly, Γ_δ is a sum of oriented line segments which are sides of exactly one Q_j. We denote by Ω_δ the interior of the union $\bigcup Q_j$ (Fig. 4-10).

Let γ be a cycle which is homologous to zero in Ω; we choose δ so small that γ is contained in Ω_δ. Consider a point $\zeta \in \Omega - \Omega_\delta$. It belongs to at least one Q which is not a Q_j. There is a point $\zeta_0 \in Q$ which is not in Ω. It is possible to join ζ and ζ_0 by a line segment which lies in Q and therefore does not meet Ω_δ. Since γ, considered as a point set, is contained in Ω_δ it follows that $n(\gamma,\zeta) = n(\gamma,\zeta_0) = 0$. In particular, $n(\gamma,\zeta) = 0$ for all points ζ on Γ_δ.

† This proof follows a suggestion by A. F. Beardon, who has kindly permitted its use in this connection.

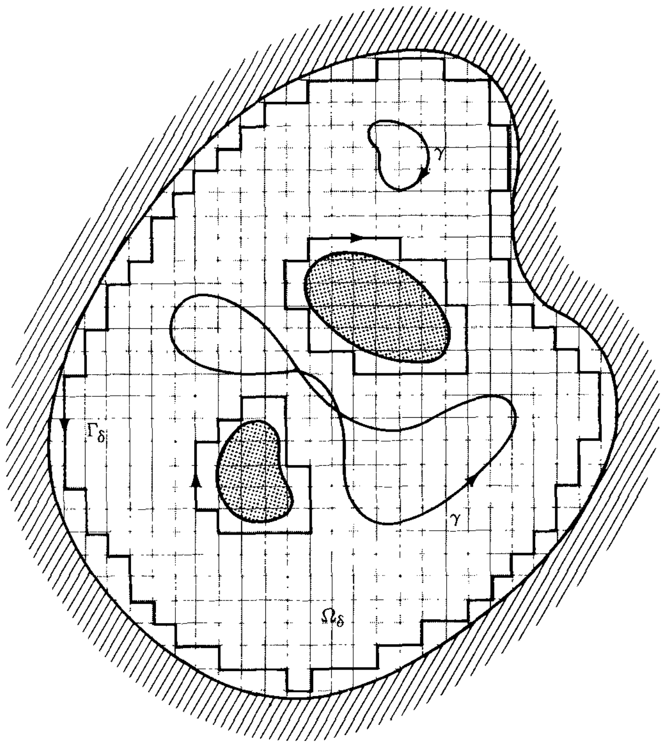

FIG. 4-10

Suppose now that f is analytic in Ω. If z lies in the interior of Q_{j_0}, say, then

$$\frac{1}{2\pi i}\int_{\partial Q_j}\frac{f(\zeta)\,d\zeta}{\zeta - z} = \begin{cases} f(z) & \text{if } j = j_0 \\ 0 & \text{if } j \neq j_0 \end{cases}$$

and hence

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_\delta}\frac{f(\zeta)\,d\zeta}{\zeta - z}. \tag{41}$$

Since both sides are continuous functions of z, this equation will hold for all $z \in \Omega_\delta$.

As a consequence we obtain

$$\int_\gamma f(z)\,dz = \int_\gamma\left(\frac{1}{2\pi i}\int_{\Gamma_\delta}\frac{f(\zeta)\,d\zeta}{\zeta - z}\right)dz. \tag{42}$$

The integrand of the iterated integral is a continuous function of both integration variables, namely the parameters of Γ_δ and γ. Therefore, the order of integration can be reversed. In other words,

$$\int_\gamma \left(\frac{1}{2\pi i}\int_{\Gamma_\delta}\frac{f(\zeta)\,d\zeta}{\zeta - z}\right) dz = \int_{\Gamma_\delta}\left(\frac{1}{2\pi i}\int_\gamma \frac{dz}{\zeta - z}\right) f(\zeta)\,d\zeta.$$

On the right the inside integral is $-n(\gamma,\zeta) = 0$. Hence the integral (42) is zero, and we have proved the theorem for bounded Ω.

If Ω is unbounded, we replace it by its intersection Ω' with a disk $|z| < R$ which is large enough to contain γ. Any point a in the complement of Ω' is either in the complement of Ω or lies outside the disk. In either case $n(\gamma,a) = 0$, so that $\gamma \sim 0 \pmod{\Omega'}$. The proof is applicable to Ω', and we conclude that the theorem is valid for arbitrary Ω.

4.6. Locally Exact Differentials. A differential $p\,dx + q\,dy$ is said to be *locally exact* in Ω if it is exact in some neighborhood of each point in Ω. It is not difficult to see (Ex. 1, p. 148) that this is so if and only if

$$\int_\gamma p\,dx + q\,dy = 0 \tag{43}$$

for all $\gamma = \partial R$ where R is a rectangle contained in Ω. This condition is certainly fulfilled if $p\,dx + q\,dy = f(z)\,dz$ with f analytic in Ω, and by Theorem 15, (43) is then true for any cycle $\gamma \sim 0 \pmod{\Omega}$.

Theorem 16. *If $p\,dx + q\,dy$ is locally exact in Ω, then*

$$\int_\gamma p\,dx + q\,dy = 0$$

for every cycle $\gamma \sim 0$ in Ω.

There does not seem to be any direct way of modifying the proof of Theorem 15 so that it would cover this more general situation. We shall therefore end up by presenting two different proofs of Cauchy's general theorem. As in the earlier editions of this book, we shall follow Artin's proof of Theorem 16. The separate proof of Cauchy's theorem has been included because of its special appeal.

For the proof of Theorem 16 we show first that γ can be replaced by a polygon σ with horizontal and vertical sides such that every locally exact differential has the same integral over σ as over γ. This property implies, in particular, $n(\sigma,a) = n(\gamma,a)$ for a in the complement of Ω, and hence $\sigma \sim 0$. It will thus be sufficient to prove the theorem for polygons with sides parallel to the axes.

We construct σ as an approximation of γ. Let the distance from γ to the complement of Ω be ρ. If γ is given by $z = z(t)$, the function $z(t)$ is uniformly continuous on the closed interval $[a,b]$. We determine $\delta > 0$ so

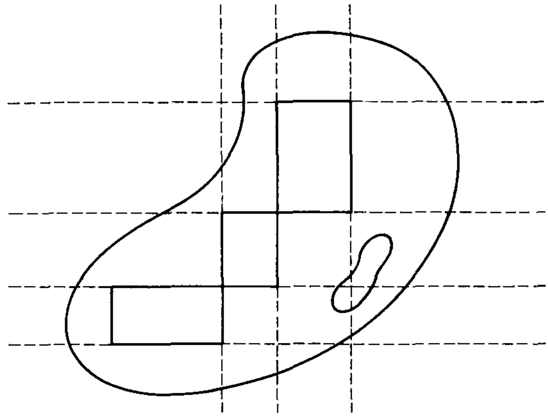

FIG. 4-11

that $|z(t) - z(t')| < \rho$ for $|t - t'| < \delta$ and divide $[a,b]$ into subintervals of length $< \delta$. The corresponding aubarcs γ_i of γ have the property that each is contained in a disk of radius ρ which lies entirely in Ω. The end points of γ_i can be joined within that disk by a polygon σ_i consisting of a horizontal and a vertical segment. Since the differential is exact in the disk,

$$\int_{\sigma_i} p\,dx + q\,dy = \int_{\gamma_i} p\,dx + q\,dy,$$

and if $\sigma = \Sigma\sigma_i$, we obtain

$$\int_{\sigma} p\,dx + q\,dy = \int_{\gamma} p\,dx + q\,dy,$$

as desired.

To continue the proof we extend all segments that make up σ to infinite lines (Fig. 4-11). They divide the plane into some finite rectangles R_i and some unbounded regions R'_j which may be regarded as infinite rectangles.

Choose a point a_i from the interior of each R_i, and form the cycle

$$\sigma_0 = \sum_i n(\sigma,a_i)\,\partial R_i \tag{44}$$

where the sum ranges over all finite rectangles; the coefficients $n(\sigma,a_i)$ are well determined, for no a_i lies on σ. In the discussion that follows we shall also make use of points a'_j chosen from the interior of each R'_j.

It is clear that $n(\partial R_i,a_k) = 1$ if $k = i$ and 0 if $k \neq i$; similarly, $n(\partial R_i,a'_j) = 0$ for all j. With this in mind it follows from (44) that $n(\sigma_0,a_i) = n(\sigma,a_i)$ and $n(\sigma_0,a'_j) = 0$. It is also true that $n(\sigma,a'_j) = 0$, for the interior of R'_j belongs to the unbounded region determined by σ. We have thus shown that $n(\sigma - \sigma_0, a) = 0$ for all $a = a_i$ and $a = a'_j$.

From this property of $\sigma - \sigma_0$ we wish to conclude that σ_0 is identical with σ up to segments that cancel against each other. Let σ_{ik} be the common side of two adjacent rectangles R_i, R_k; we choose the orientation so that R_i lies to the left of σ_{ik}. Suppose that the reduced expression of $\sigma - \sigma_0$ contains the multiple $c\sigma_{ik}$. Then the cycle $\sigma - \sigma_0 - c\,\partial R_i$ does not contain σ_{ik}, and it follows that a_i and a_k must have the same index with respect to this cycle. On the other hand, these indices are $-c$ and 0, respectively; we conclude that $c = 0$. The same reasoning applies if σ_{ij} is the common side of a finite rectangle R_i and an infinite rectangle R'_j. Thus every side of a finite rectangle occurs with coefficient zero in $\sigma - \sigma_0$, proving that

$$\sigma = \sum_i n(\sigma, a_i)\, \partial R_i. \tag{45}$$

We prove now that all the R_i whose corresponding coefficient $n(\sigma, a_i)$ is different from zero are actually contained in Ω. Suppose that a point a in the closed rectangle R_i were not in Ω. Then $n(\sigma, a) = 0$ because $\sigma \sim 0$ (mod Ω). On the other hand, the line segment between a and a_i does not intersect σ, and hence $n(\sigma, a_i) = n(\sigma, a) = 0$. We conclude by the local exactness that the integral of $p\,dx + q\,dy$ over any ∂R_i which occurs effectively in (45) is zero. Consequently,

$$\int_\sigma p\,dx + q\,dy = 0,$$

and Theorem 16 is proved.

4.7. Multiply Connected Regions. A region which is not simply connected is called multiply connected. More precisely, Ω is said to have the finite connectivity n if the complement of Ω has exactly n components and infinite connectivity if the complement has infinitely many components. In a less precise but more suggestive language, a region of connectivity n arises by punching n holes in the Riemann sphere.

In the case of finite connectivity, let $A_1, A_2, \ldots, A_n$ be the components of the complement of Ω, and assume that ∞ belongs to A_n. If γ is an arbitrary cycle in Ω, we can prove, just as in Theorem 14, that $n(\gamma, a)$ is constant when a varies over any one of the components A_i and that $n(\gamma, a) = 0$ in A_n. Moreover, duplicating the construction used in the proof of the same theorem we can find cycles γ_i, $i = 1, \ldots, n-1$, such that $n(\gamma_i, a) = 1$ for $a \in A_i$ and $n(\gamma_i, a) = 0$ for all other points outside of Ω.

For a given cycle γ in Ω, let c_i be the constant value of $n(\gamma,a)$ for $a \in A_i$. We find that any point outside of Ω has the index zero with respect to the cycle $\gamma - c_1\gamma_1 - c_2\gamma_2 - \cdots - c_{n-1}\gamma_{n-1}$. In other words,

$$\gamma \sim c_1\gamma_1 + c_2\gamma_2 + \cdots + c_{n-1}\gamma_{n-1}.$$

Every cycle is thus homologous to a linear combination of the cycles $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$. This linear combination is uniquely determined, for if two linear combinations were homologous to the same cycle their difference would be a linear combination which is homologous to zero. But it is clear that the cycle $c_1\gamma_1 + c_2\gamma_2 + \cdots + c_{n-1}\gamma_{n-1}$ winds c_i times around the points in A_i; hence it cannot be homologous to zero unless all the c_i vanish.

In view of these circumstances the cycles $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ are said to form a *homology basis* for the region Ω. It is not the only homology basis, but by an elementary theorem in linear algebra we may conclude that every homology basis has the same number of elements. We find that every region with a finite homology basis has finite connectivity, and the number of basis elements is one less than the connectivity.

By Theorem 18 we obtain, for any analytic function $f(z)$ in Ω,

$$\int_\gamma f\,dz = c_1 \int_{\gamma_1} f\,dz + c_2 \int_{\gamma_2} f\,dz + \cdots + c_{n-1} \int_{\gamma_{n-1}} f\,dz.$$

The numbers

$$P_i = \int_{\gamma_i} f\,dz$$

depend only on the function, and not on γ. They are called *modules of periodicity* of the differential $f\,dz$, or, with less accuracy, the *periods* of the indefinite integral. We have found that the integral of $f(z)$ over any cycle is a linear combination of the periods with integers as coefficients, and the integral along an arc from z_0 to z is determined up to additive multiples of the periods. The vanishing of the periods is a necessary and sufficient condition for the existence of a single-valued indefinite integral.

In order to illustrate, let us consider the extremely simple case of an annulus, defined by $r_1 < |z| < r_2$. The complement has the components $|z| \leqq r_1$ and $|z| \geqq r_2$; we include the degenerate cases $r_1 = 0$ and $r_2 = \infty$. The annulus is doubly connected, and a homology basis is formed by any circle $|z| = r$, $r_1 < r < r_2$. If this circle is denoted by C, any cycle in the annulus satisfies $\gamma \sim nC$ where $n = n(\gamma,0)$. The integral of an analytic function over a cycle is a multiple of the single period

$$P = \int_C f\,dz$$

whose value is of course independent of the radius r.

EXERCISES

1. Prove without use of Theorem 16 that $p\,dx + q\,dy$ is locally exact in Ω if and only if

$$\int_{\partial R} p\,dx + q\,dy = 0$$

for every rectangle $R \subset \Omega$ with sides parallel to the axes.

2. Prove that the region obtained from a simply connected region by removing m points has the connectivity $m + 1$, and find a homology basis.

3. Show that the bounded regions determined by a closed curve are simply connected, while the unbounded region is doubly connected.

4. Show that single-valued analytic branches of $\log z$, z^α and z^z can be defined in any simply connected region which does not contain the origin.

5. Show that a single-valued analytic branch of $\sqrt{1 - z^2}$ can be defined in any region such that the points ± 1 are in the same component of the complement. What are the possible values of

$$\int \frac{dz}{\sqrt{1 - z^2}}$$

over a closed curve in the region?

5. THE CALCULUS OF RESIDUES

The results of the preceding section have shown that the determination of line integrals of analytic functions over closed curves can be reduced to the determination of periods. Under certain circumstances it turns out that the periods can be found without or with very little computation. We are thus in possession of a method which in many cases permits us to evaluate integrals without resorting to explicit calculation. This is of great value for practical purposes as well as for the further development of the theory.

In order to make this method more systematic a simple formalism, known as the calculus of residues, was introduced by Cauchy, the founder of complex integration theory. From the point of view adopted in this book the use of residues amounts essentially to an application of the results proved in Sec. 4 under particularly simple circumstances.

5.1. The Residue Theorem. Our first task is to review earlier results in the light of the more general theorems of Sec. 4. Clearly, all results which were derived as consequences of Cauchy's theorem for a disk remain valid in arbitrary regions for all cycles which are homologous to zero. For instance, and this application is typical, Cauchy's integral formula can now be expressed in the following form:

If $f(z)$ is analytic in a region Ω, then

$$n(\gamma,a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-a}$$

for every cycle γ which is homologous to zero in Ω.

The proof is a repetition of the proof of Theorem 6. In this connection we point out that there is of course no longer any need to give a separate proof of Theorem 15 in the presence of removable singularities. Indeed, our discussion of the local behavior has already shown that all removable singularities can simply be ignored.

We turn now to the discussion of a function $f(z)$ which is analytic in a region Ω except for isolated singularities. For a first orientation, let us assume that there are only a finite number of singular points, denoted by $a_1, a_2, \ldots, a_n$. The region obtained by excluding the points a_j will be denoted by Ω'.

To each a_j there exists a $\delta_j > 0$ such that the doubly connected region $0 < |z - a_j| < \delta_j$ is contained in Ω'. Draw a circle C_j about a_j of radius $< \delta_j$, and let

$$P_j = \int_{C_j} f(z)\,dz \tag{46}$$

be the corresponding period of $f(z)$. The particular function $1/(z - a_j)$ has the period $2\pi i$. Therefore, if we set $R_j = P_j/2\pi i$, the combination

$$f(z) - \frac{R_j}{z - a_j}$$

has a vanishing period. The constant R_j which produces this result is called the *residue* of $f(z)$ at the point a_j. We repeat the definition in the following form:

Definition 3. *The residue of $f(z)$ at an isolated singularity a is the unique complex number R which makes $f(z) - R/(z - a)$ the derivative of a single-valued analytic function in an annulus $0 < |z - a| < \delta$.*

It is helpful to use such self-explanatory notations as $R = \text{Res}_{z=a} f(z)$.

Let γ be a cycle in Ω' which is homologous to zero with respect to Ω. Then γ satisfies the homology

$$\gamma \sim \sum_j n(\gamma,a_j)C_j$$

with respect to Ω'; indeed, we can easily verify that the points a_j as well as all points outside of Ω have the same order with respect to both cycles.

By virtue of the homology we obtain, with the notation (46),

$$\int_\gamma f\,dz = \sum_j n(\gamma,a_j)P_j,$$

and since $P_j = 2\pi i \cdot R_j$ finally

$$\frac{1}{2\pi i}\int_\gamma f\,dz = \sum_j n(\gamma,a_j)R_j.$$

This is the *residue theorem*, except for the restrictive assumption that there are only a finite number of singularities. In the general case we need only prove that $n(\gamma,a_j) = 0$ except for a finite number of points a_j, for then the same proof can be applied. The assertion follows by routine reasoning. The set of all points a with $n(\gamma,a) = 0$ is open and contains all points outside of a large circle. The complement is consequently a compact set, and as such it cannot contain more than a finite number of the isolated points a_j. Therefore $n(\gamma,a_j) \neq 0$ only for a finite number of the singularities, and we have proved:

Theorem 17. *Let $f(z)$ be analytic except for isolated singularities a_j in a region Ω. Then*

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_j n(\gamma,a_j)\ \mathrm{Res}_{z=a_j} f(z) \tag{47}$$

for any cycle γ which is homologous to zero in Ω and does not pass through any of the points a_j.

In the applications it is frequently the case that each $n(\gamma,a_j)$ is either 0 or 1. Then we have simply

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_j \mathrm{Res}_{z=a_j} f(z)$$

where the sum is extended over all singularities enclosed by γ.

The residue theorem is of little value unless we have at our disposal a simple procedure to determine the residues. For essential singularities there is no such procedure of any practical value, and thus it is not surprising that the residue theorem is comparatively seldom used in the presence of essential singularities. With respect to poles the situation is entirely different. We need only look at the expansion

$$f(z) = B_h(z-a)^{-h} + \cdots + B_1(z-a)^{-1} + \varphi(z)$$

to recognize that the residue equals the coefficient B_1. Indeed, when the term $B_1(z-a)^{-1}$ is omitted, the remainder is evidently a derivative.

Since the principal part at a pole is always either given or can be easily found, we have thus a very simple method for finding the residues.

For simple poles the method is even more immediate, for then the residue equals the value of the function $(z - a)f(z)$ for $z = a$. For instance, let it be required to find the residues of the function

$$\frac{e^z}{(z-a)(z-b)}$$

at the poles a and $b \neq a$. The residue at a is obviously $e^a/(a - b)$, and the residue at b is $e^b/(b - a)$. If $b = a$, the situation is slightly more complicated. We must then expand e^z by Taylor's theorem in the form $e^z = e^a + e^a(z - a) + f_2(z)(z - a)^2$. Dividing by $(z - a)^2$ we find that the residue of $e^z/(z - a)^2$ at $z = a$ is e^a.

Remark. In presentations of Cauchy's theorem, the integral formula and the residue theorem which follow more classical lines, there is no mention of homology, nor is the notion of index used explicitly. Instead, the curve γ to which the theorems are applied is supposed to form the complete boundary of a subregion of Ω, and the orientation is chosen so that the subregion lies to the left of Ω. In rigorous texts considerable effort is spent on proving that these intuitive notions have a precise meaning. The main objection to this procedure is the necessity to allot time and attention to rather delicate questions which are peripheral in comparison with the main issues.

With the general point of view that we have adopted it is still possible, and indeed quite easy, to isolate the classical case. All that is needed is to accept the following definition:

Definition 4. *A cycle γ is said to bound the region Ω if and only if $n(\gamma,a)$ is defined and equal to 1 for all points $a \in \Omega$ and either undefined or equal to zero for all points a not in Ω.*

If γ bounds Ω, and if $\Omega + \gamma$ is contained in a larger region Ω', then it is clear that γ is homologous to zero with respect to Ω'. The following statements are therefore trivial consequences of Theorems 15 and 17:

If γ bounds Ω and $f(z)$ is analytic on the set $\Omega + \gamma$, then

$$\int_\gamma f(z)\,dz = 0$$

and

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

for all $z \in \Omega$.

If $f(z)$ is analytic on $\Omega + \gamma$ except for isolated singularities in Ω, then

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_j \operatorname{Res}_{z=a_j} f(z)$$

where the sum ranges over the singularities $a_j \in \Omega$.

We observe that a cycle γ which bounds Ω must contain the set theoretic boundary of Ω. Indeed, if z_0 lies on the boundary of Ω, then every neighborhood of z_0 contains points from Ω and points not in Ω. If such a neighborhood were free from points of γ, $n(\gamma,z)$ would be defined and constant in the neighborhood. This contradicts the definition, and hence every neighborhood of z_0 must meet γ; since γ is closed, z_0 must lie on γ.

The converse of the preceding statement is not true, for a point on γ may well have a neighborhood which does not meet Ω. Normally, one would try to choose γ so that it is identical with the boundary of Ω, but for Cauchy's theorem and related considerations this assumption is not needed.

5.2. The Argument Principle. Cauchy's integral formula can be considered as a special case of the residue theorem. Indeed, the function $f(z)/(z - a)$ has a simple pole at $z = a$ with the residue $f(a)$, and when we apply (47), the integral formula results.

Another application of the residue theorem occurred in the proof of Theorem 10 which served to determine the number of zeros of an analytic function, For a zero of order h we can write $f(z) = (z - a)^h f_h(z)$, with $f_h(a) \neq 0$, and obtain $f'(z) = h(z - a)^{h-1}f_h(z) + (z - a)^h f_h'(z)$. Consequently $f'(z)/f(z) = h/(z - a) + f_h'(z)/f_h(z)$, and we see that f'/f has a simple pole with the residue h. In the formula (32) this residue is accounted for by a corresponding repetition of terms.

We can now generalize Theorem 10 to the case of a meromorphic function. If f has a pole of order h, we find by the same calculation as above, with $-h$ replacing h, that f'/f has the residue $-h$. The following theorem results:

Theorem 18. *If $f(z)$ is meromorphic in Ω with the zeros a_j and the poles b_k, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_j n(\gamma,a_j) - \sum_k n(\gamma,b_k) \tag{48}$$

for every cycle γ which is homologous to zero in Ω and does not pass through any of the zeros or poles.

It is understood that multiple zeros and poles have to be repeated as many times as their order indicates; the sums in (48) are finite.

Theorem 18 is usually referred to as the *argument principle.* The name refers to the interpretation of the left-hand member of (48) as $n(\Gamma,0)$ where Γ is the image cycle of γ. If Γ lies in a disk which does not contain the origin, then $n(\Gamma,0) = 0$. This observation is the basis for the following corollary, known as *Rouché's theorem:*

Corollary. *Let γ be homologous to zero in Ω and such that $n(\gamma,z)$ is either 0 or 1 for any point z not on γ. Suppose that $f(z)$ and $g(z)$ are analytic in Ω and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on γ. Then $f(z)$ and $g(z)$ have the same number of zeros enclosed by γ.*

The assumption implies that $f(z)$ and $g(z)$ are zero-free on γ. Moreover, they satisfy the inequality

$$\left|\frac{g(z)}{f(z)} - 1\right| < 1$$

on γ. The values of $F(z) = g(z)/f(z)$ on γ are thus contained in the open disk of center 1 and radius 1. When Theorem 18 is applied to $F(z)$, we have thus $n(\Gamma,0) = 0$, and the assertion follows.

A typical application of Rouché's theorem would be the following. Suppose that we wish to find the number of zeros of a function $f(z)$ in the disk $|z| \leqq R$. Using Taylor's theorem we can write

$$f(z) = P_{n-1}(z) + z^n f_n(z)$$

where P_{n-1} is a polynomial of degree $n - 1$. For a suitably chosen n it may happen that we can prove the inequality $R^n|f_n(z)| < |P_{n-1}(z)|$ on $|z| = R$. Then $f(z)$ has the same number of zeros in $|z| \leqq R$ as $P_{n-1}(z)$, and this number can be determined by approximate solution of the polynomial equation $P_{n-1}(z) = 0$.

Theorem 18 can be generalized in the following manner. If $g(z)$ is analytic in Ω, then $g(z)\dfrac{f'(z)}{f(z)}$ has the residue $hg(a)$ at a zero a of order h and the residue $-hg(a)$ at a pole. We obtain thus the formula

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}\,dz = \sum_j n(\gamma,a_j)g(a_j) - \sum_k n(\gamma,b_k)g(b_k). \tag{49}$$

This result is important for the study of the inverse function. With the notations of Theorem 11 we know that the equation $f(z) = w$, $|w - w_0| < \delta$ has n roots $z_j(w)$ in the disk $|z - z_0| < \varepsilon$. If we apply

(49) with $g(z) = z$, we obtain

$$\sum_{j=1}^{n} z_j(w) = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{f'(z)}{f(z) - w} z\, dz. \tag{50}$$

For $n = 1$ the inverse function $f^{-1}(w)$ can thus be represented explicitly by

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{f'(z)}{f(z) - w} z\, dz.$$

If (49) is applied with $g(z) = z^m$, equation (50) is replaced by

$$\sum_{j=1}^{n} z_j(w)^m = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{f'(z)}{f(z) - w} z^m\, dz.$$

The right-hand member represents an analytic function of w for $|w - w_0| < \delta$. Thus the power sums of the roots $z_j(w)$ are single-valued analytic functions of w. But it is well known that the elementary symmetric functions can be expressed as polynomials in the power sums. Hence they are also analytic, and we find that the $z_j(w)$ are the roots of a polynomial equation

$$z^n + a_1(w)z^{n-1} + \cdots + a_{n-1}(w)z + a_n(w) = 0$$

whose coefficients are analytic functions of w in $|w - w_0| < \delta$.

EXERCISES

1. How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$? *Hint:* Look for the biggest term when $|z| = 1$ and apply Rouché's theorem.

2. How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2?

3. How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane? *Hint:* Sketch the image of the imaginary axis and apply the argument principle to a large half disk.

5.3. Evaluation of Definite Integrals. The calculus of residues provides a very efficient tool for the evaluation of definite integrals. It is particularly important when it is impossible to find the indefinite integral explicitly, but even if the ordinary methods of calculus can be applied the use of residues is frequently a laborsaving device. The fact that the calculus of residues yields complex rather than real integrals is no disadvantage, for clearly the evaluation of a complex integral is equivalent to the evaluation of two definite integrals.

There are, however, some serious limitations, and the method is far from infallible. In the first place, the integrand must be closely connected with some analytic function. This is not very serious, for usually we are only required to integrate elementary functions, and they can all be extended to the complex domain. It is much more serious that the technique of complex integration applies only to closed curves, while a real integral is always extended over an interval. A special device must be used in order to reduce the problem to one which concerns integration over a closed curve. There are a number of ways in which this can be accomplished, but they all apply under rather special circumstances. The technique can be learned at the hand of typical examples, but even complete mastery does not guarantee success.

1. All integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta \tag{51}$$

where the integrand is a rational function of $\cos\theta$ and $\sin\theta$ can be easily evaluated by means of residues. Of course these integrals can also be computed by explicit integration, but this technique is usually very laborious. It is very natural to make the substitution $z = e^{i\theta}$ which immediately transforms (51) into the line integral

$$-i \int_{|z|=1} R\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right]\frac{dz}{z}.$$

It remains only to determine the residues which correspond to the poles of the integrand inside the unit circle.

As an example, let us compute

$$\int_0^{\pi} \frac{d\theta}{a + \cos\theta}, \qquad a > 1.$$

This integral is not extended over $(0,2\pi)$, but since $\cos\theta$ takes the same values in the intervals $(0,\pi)$ and $(\pi,2\pi)$ is is clear that the integral from 0 to π is one-half of the integral from 0 to 2π. Taking this into account we find that the integral equals

$$-i \int_{|z|=1} \frac{dz}{z^2 + 2az + 1}$$

The denominator can be factored into $(z - \alpha)(z - \beta)$ with

$$\alpha = -a + \sqrt{a^2 - 1}, \qquad \beta = -a - \sqrt{a^2 - 1}.$$

Evidently $|\alpha| < 1$, $|\beta| > 1$, and the residue at α is $1/(\alpha - \beta)$. The value of the integral is found to be $\pi/\sqrt{a^2 - 1}$.

2. An integral of the form

$$\int_{-\infty}^{\infty} R(x)\,dx$$

converges if and only if in the rational function $R(x)$ the degree of the denominator is at least two units higher than the degree of the numerator, and if no pole lies on the real axis. The standard procedure is to integrate the complex function $R(z)$ over a closed curve consisting of a line segment $(-\rho,\rho)$ and the semicircle from ρ to $-\rho$ in the upper half plane. If ρ is large enough this curve encloses all poles in the upper half plane, and the corresponding integral is equal to $2\pi i$ times the sum of the residues in the upper half plane. As $\rho \to \infty$ obvious estimates show that the integral over the semicircle tends to 0, and we obtain

$$\int_{-\infty}^{\infty} R(x)\,dx = 2\pi i \sum_{y>0} \operatorname{Res} R(z).$$

3. The same method can be applied to an integral of the form

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,dx \tag{52}$$

whose real and imaginary parts determine the important integrals

$$\int_{-\infty}^{\infty} R(x)\cos x\,dx, \qquad \int_{-\infty}^{\infty} R(x)\sin x\,dx. \tag{53}$$

Since $|e^{iz}| = e^{-y}$ is bounded in the upper half plane, we can again conclude that the integral over the semicircle tends to zero, provided that the rational function $R(z)$ has a zero of at least order two at infinity. We obtain

$$\int_{-\infty}^{\infty} R(x)e^{ix}\,dx = 2\pi i \sum_{y>0} \operatorname{Res} R(z)e^{iz}.$$

It is less obvious that the same result holds when $R(z)$ has only a simple zero at infinity. In this case it is not convenient to use semicircles. For one thing, it is not so easy to estimate the integral over the semicircle, and secondly, even if we were successful we would only have proved that the integral

$$\int_{-\rho}^{\rho} R(x)e^{ix}\,dx$$

over a symmetric interval has the desired limit for $\rho \to \infty$. In reality we are of course required to prove that

$$\int_{-X_1}^{X_2} R(x)e^{ix}\,dx$$

has a limit when X_1 and X_2 tend independently to ∞. In the earlier examples this question did not arise because the convergence of the integral was assured beforehand.

For the proof we integrate over the perimeter of a rectangle with the vertices X_2, $X_2 + iY$, $-X_1 + iY$, $-X_1$ where $Y > 0$. As soon as X_1, X_2 and Y are sufficiently large, this rectangle contains all the poles in the upper half plane. Under the hypothesis $|zR(z)|$ is bounded. Hence the integral over the right vertical side is, except for a constant factor, less than

$$\int_0^Y e^{-y} \frac{dy}{|z|} < \frac{1}{X_2} \int_0^Y e^{-y}\, dy.$$

The last integral can be evaluated explicitly and is found to be < 1. Hence the integral over the right vertical side is less than a constant times $1/X_2$, and a corresponding result is found for the left vertical side. The integral over the upper horizontal side is evidently less than $e^{-Y}(X_1 + X_2)/Y$ multiplied with a constant. For fixed X_1, X_2 it tends to 0 as $Y \to \infty$, and we conclude that

$$\left| \int_{-X_1}^{X_2} R(x)e^{ix}\, dx - 2\pi i \sum_{y>0} \operatorname{Res} R(z)e^{iz} \right| < A\left(\frac{1}{X_1} + \frac{1}{X_2}\right)$$

where A denotes a constant. This inequality proves that

$$\int_{-\infty}^{\infty} R(x)e^{ix}\, dx = 2\pi i \sum_{y>0} \operatorname{Res} R(z)e^{iz}$$

under the sole condition that $R(\infty) = 0$.

In the discussion we have assumed, tacitly, that $R(z)$ has no poles on the real axis since otherwise the integral (52) has no meaning. However, one of the integrals (53) may well exist, namely, if $R(z)$ has simple poles which coincide with zeros of $\sin x$ or $\cos x$. Let us suppose, for instance, that $R(z)$ has a simple pole at $z = 0$. Then the second integral (53) has a meaning and calls for evaluation.

We use the same method as before, but we use a path which avoids the origin by following a small semicircle of radius δ in the lower half plane (Fig. 4-12). It is easy to see that this closed curve encloses the poles in the upper half plane, the pole at the origin, and no others, as soon as X_1, X_2, Y are sufficiently large and δ is sufficiently small. Suppose that the residue at 0 is B, so that we can write $R(z)e^{iz} = B/z + R_0(z)$ where $R_0(z)$ is analytic at the origin. The integral of the first term over the semicircle is πiB, while the integral of the second term tends to 0 with δ.

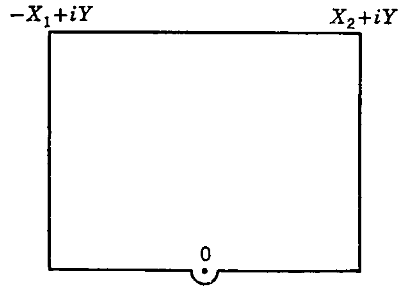

FIG. 4-12

It is clear that we are led to the result

$$\lim_{\delta\to 0} \int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} R(x)e^{ix}\,dx = 2\pi i\left[\sum_{y>0} \text{Res } R(z)e^{iz} + \tfrac{1}{2}B\right].$$

The limit on the left is called the *Cauchy principal value* of the integral; it exists although the integral itself has no meaning. On the right-hand side we observe that one-half of the residue at 0 has been included; this is as if one-half of the pole were counted as lying in the upper half plane.

In the general case where several poles lie on the real axis we obtain

$$\text{pr.v.} \int_{-\infty}^{\infty} R(x)e^{ix}\,dx = 2\pi i \sum_{y>0} \text{Res } R(z)e^{iz} + \pi i \sum_{y=0} \text{Res } R(z)e^{iz}$$

where the notations are self-explanatory. It is an essential hypothesis that all the poles on the real axis be simple, and as before we must assume that $R(\infty) = 0$.

As the simplest example we have

$$\text{pr.v.}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \pi i.$$

Separating the real and imaginary part we observe that the real part of the equation is trivial by the fact that the integrand is odd. In the imaginary part it is not necessary to take the principal value, and since the integrand is even we find

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

We remark that integrals containing a factor $\cos^n x$ or $\sin^n x$ can be evaluated by the same technique. Indeed, these factors can be written as linear combinations of terms $\cos mx$ and $\sin mx$, and the corresponding integrals can be reduced to the form (52) by a change of variable:

$$\int_{-\infty}^{\infty} R(x)e^{imx}\,dx = \frac{1}{m}\int_{-\infty}^{\infty} R\left(\frac{x}{m}\right) e^{ix}\,dx.$$

4. The next category of integrals have the form

$$\int_0^\infty x^\alpha R(x)\,dx$$

where the exponent α is real and may be supposed to lie in the interval $0 < \alpha < 1$. For convergence $R(z)$ must have a zero of at least order two at ∞ and at most a simple pole at the origin.

The new feature is the fact that $R(z)z^\alpha$ is not single-valued. This, however, is just the circumstance which makes it possible to find the integral from 0 to ∞.

The simplest procedure is to start with the substitution $x = t^2$ which transforms the integral into

$$2\int_0^\infty t^{2\alpha+1}R(t^2)\,dt.$$

For the function $z^{2\alpha}$ we may choose the branch whose argument lies between $-\pi\alpha$ and $3\pi\alpha$; it is well defined and analytic in the region obtained by omitting the negative imaginary axis. As long as we avoid the negative imaginary axis, we can apply the residue theorem to the function $z^{2\alpha+1}R(z^2)$. We use a closed curve consisting of two line segments along the positive and negative axis and two semicircles in the upper half plane, one very large and one very small (Fig. 4-13). Under our assumptions it is easy to show that the integrals over the semicircles tend to zero. Hence the residue theorem yields the value of the integral

$$\int_{-\infty}^{\infty} z^{2\alpha+1}R(z^2)\,dz = \int_0^\infty (z^{2\alpha+1} + (-z)^{2\alpha+1})R(z^2)\,dz.$$

However, $(-z)^{2\alpha} = e^{2\pi i\alpha}z^{2\alpha}$, and the integral equals

$$(1 - e^{2\pi i\alpha})\int_0^\infty z^{2\alpha+1}R(z^2)\,dz.$$

Since the factor in front is $\neq 0$, we are finally able to determine the value of the desired integral.

The evaluation calls for determination of the residues of $z^{2\alpha+1}R(z^2)$ in the upper half plane. These are the same as the residues of $z^\alpha R(z)$ in the whole plane. For practical purposes it may be preferable not to use any preliminary substitution and integrate the function $z^\alpha R(z)$ over the closed curve shown in Fig. 4-14. We have then to use the branch of z^α whose argument lies between 0 and $2\pi\alpha$. This method needs some justification, for it does not conform to the hypotheses of the residue theorem. The justification is trivial.

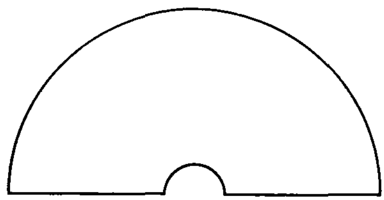

FIG. 4-13

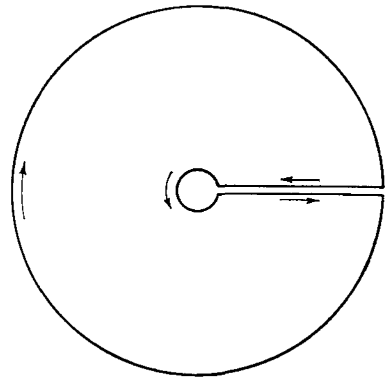

FIG. 4-14

5. As a final example we compute the special integral

$$\int_0^\pi \log \sin \theta \, d\theta.$$

Consider the function $1 - e^{2iz} = -2ie^{iz} \sin z$. From the representation $1 - e^{2iz} = 1 - e^{-2y}(\cos 2x + i \sin 2x)$, we find that this function is real and negative only for $x = n\pi$, $y \leqq 0$. In the region obtained by omitting these half lines the principal branch of $\log(1 - e^{2iz})$ is hence single-valued and analytic. We apply Cauchy's theorem to the rectangle whose vertices are $0, \pi, \pi + iY$, and iY; however, the points 0 and π have to be avoided, and we do this by following small circular quadrants of radius δ.

Because of the periodicity the integrals over the vertical sides cancel against each other. The integral over the upper horizontal side tends to 0 as $Y \to \infty$. Finally, the integrals over the quadrants can also be seen to approach zero as $\delta \to 0$. Indeed, since the imaginary part of the logarithm is bounded we need only consider the real part. From the fact that $|1 - e^{2iz}|/|z| \to 2$ for $z \to 0$ we see that $\log |1 - e^{2iz}|$ becomes infinite like $\log \delta$, and since $\delta \log \delta \to 0$ the integral over the quadrant near the origin will tend to zero.

The same proof applies near the vertex π, and we obtain

$$\int_0^\pi \log(-2ie^{ix} \sin x) \, dx = 0$$

If we choose $\log e^{ix} = ix$, the imaginary part lies between 0 and π. Therefore, in order to obtain the principal branch with an imaginary part between $-\pi$ and π, we must choose $\log(-i) = -\pi i/2$. The equation can now be written in the form