COMPLEX ANALYSIS

An Introduction to the Theory of Analytic Functions of One Complex Variable

Third Edition

Lars V. Ahlfors
Professor of Mathematics, Emeritus
Harvard University

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6. HARMONIC FUNCTIONS

The real and imaginary parts of an analytic function are conjugate harmonic functions. Therefore, all theorems on analytic functions are also theorems on pairs of conjugate harmonic functions. However, harmonic functions are important in their own right, and their treatment is not always simplified by the use of complex methods. This is particularly true when the conjugate harmonic function is not single-valued.

We assemble in this section some facts about harmonic functions that are intimately connected with Cauchy's theorem. The more delicate properties of harmonic functions are postponed to a later chapter.

6.1. Definition and Basic Properties. A real-valued function $u(z)$ or $u(x,y)$, defined and single-valued in a region Ω, is said to be *harmonic* in Ω, or a *potential function*, if it is continuous together with its partial derivatives of the first two orders and satisfies *Laplace's equation*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{54}$$

We shall see later that the regularity conditions can be weakened, but this is a point of relatively minor importance.

The sum of two harmonic functions and a constant multiple of a harmonic function are again harmonic; this is due to the linear character of Laplace's equation. The simplest harmonic functions are the linear functions $ax + by$. In polar coordinates (r,θ) equation (54) takes the form

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0.\dagger$$

This shows that $\log r$ is a harmonic function and that any harmonic function which depends only on r must be of the form $a \log r + b$. The argument θ is harmonic whenever it can be uniquely defined.

If u is harmonic in Ω, then

$$f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \tag{55}$$

is analytic, for writing $U = \frac{\partial u}{\partial x}$, $V = -\frac{\partial u}{\partial y}$ we have

$$\frac{\partial U}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial V}{\partial y}$$
$$\frac{\partial U}{\partial y} = \frac{\partial^2 u}{\partial x\,\partial y} = -\frac{\partial V}{\partial x}.$$

† This form cannot be used for $r = 0$.

This, it should be remembered, is the most natural way of passing from harmonic to analytic functions.

From (55) we pass to the differential

$$f\,dz = \left(\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\right) + i\left(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\right). \tag{56}$$

In this expression the real part is the differential of u,

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

If u has a conjugate harmonic function v, then the imaginary part can be written as

$$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy.$$

In general, however, there is no single-valued conjugate function, and in these circumstances it is better not to use the notation dv. Instead we write

$$^*du = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$$

and call $^*du$ the *conjugate differential of* du. We have by (56)

$$f\,dz = du + i\,^*du. \tag{57}$$

By Cauchy's theorem the integral of $f\,dz$ vanishes along any cycle which is homologous to zero in Ω. On the other hand, the integral of the exact differential du vanishes along all cycles. It follows by (57) that

$$\int_\gamma {}^*du = \int_\gamma -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy = 0 \tag{58}$$

for all cycles γ which are homologous to zero in Ω.

The integral in (58) has an important interpretation which cannot be left unmentioned. If γ is a regular curve with the equation $z = z(t)$, the direction of the tangent is determined by the angle $\alpha = \arg z'(t)$, and we can write $dx = |dz| \cos \alpha$, $dy = |dz| \sin \alpha$. The normal which points to the right of the tangent has the direction $\beta = \alpha - \pi/2$, and thus $\cos \alpha = -\sin \beta$, $\sin \alpha = \cos \beta$. The expression

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} \cos \beta + \frac{\partial u}{\partial y} \sin \beta$$

is a directional derivative of u, the right-hand *normal derivative* with respect to the curve γ. We obtain $^*du = (\partial u/\partial n)\,|dz|$, and (58) can be written in the form

$$\int_\gamma \frac{\partial u}{\partial n} |dz| = 0. \tag{59}$$

This is the classical notation. Its main advantage is that $\partial u/\partial n$ actually represents a rate of change in the direction perpendicular to γ. For instance, if γ is the circle $|z| = r$, described in the positive sense, $\partial u/\partial n$ can be replaced by the partial derivative $\partial u/\partial r$. It has the disadvantage that (59) is not expressed as an ordinary line integral, but as an integral with respect to arc length. For this reason the classical notation is less natural in connection with homology theory, and we prefer to use the notation $^*du$.

In a simply connected region the integral of $^*du$ vanishes over all cycles, and u has a single-valued conjugate function v which is determined up to an additive constant. In the multiply connected case the conjugate function has *periods*

$$\int_{\gamma_i} {}^*du = \int_{\gamma_i} \frac{\partial u}{\partial n} |dz|$$

corresponding to the cycles in a homology basis.

There is an important generalization of (58) which deals with a pair of harmonic functions. If u_1 and u_2 are harmonic in Ω, we claim that

$$\int_\gamma u_1 \, {}^*du_2 - u_2 \, {}^*du_1 = 0 \tag{60}$$

for every cycle γ which is homologous to zero in Ω. According to Theorem 16, Sec. 4.6, it is sufficient to prove (60) for $\gamma = \partial R$, where R is a rectangle contained in Ω. In R, u_1 and u_2 have single-valued conjugate functions v_1, v_2 and we can write

$$u_1 \, {}^*du_2 - u_2 \, {}^*du_1 = u_1 \, dv_2 - u_2 \, dv_1 = u_1 \, dv_2 + v_1 \, du_2 - d(u_2 v_1).$$

Here $d(u_2v_1)$ is an exact differential, and $u_1dv_2 + v_1du_2$ is the imaginary part of

$$(u_1 + iv_1)(du_2 + i \, dv_2).$$

The last differential can be written in the form $F_1 f_2 \, dz$ where $F_1(z)$ and $f_2(z)$ are analytic on R. The integral of $F_1 f_2 \, dz$ vanishes by Cauchy's theorem, and so does therefore the integral of its imaginary part. We conclude that (60) holds for $\gamma = \partial R$, and we have proved:

Theorem 19. *If u_1 and u_2 are harmonic in a region Ω, then*

$$\int_\gamma u_1 \, {}^*du_2 - u_2 \, {}^*du_1 = 0 \tag{60}$$

for every cycle γ which is homologous to zero in Ω.

For $u_1 = 1$, $u_2 = u$ the formula reduces to (58). In the classical notation (60) would be written as

$$\int_\gamma \left(u_1 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial u_1}{\partial n} \right) |dz| = 0.$$

6.2. The Mean-value Property. Let us apply Theorem 19 with $u_1 = \log r$ and u_2 equal to a function u, harmonic in $|z| < \rho$. For Ω we choose the punctured disk $0 < |z| < \rho$, and for γ we take the cycle $C_1 - C_2$ where C_i is a circle $|z| = r_i < \rho$ described in the positive sense. On a circle $|z| = r$ we have $^*du = r(\partial u/\partial r)\, d\theta$ and hence (60) yields

$$\log r_1 \int_{C_1} r_1 \frac{\partial u}{\partial r}\, d\theta - \int_{C_1} u\, d\theta = \log r_2 \int_{C_2} r_2 \frac{\partial u}{\partial r}\, d\theta - \int_{C_2} u\, d\theta.$$

In other words, the expression

$$\int\limits_{|z|=r} u\, d\theta - \log r \int\limits_{|z|=r} r \frac{\partial u}{\partial r}\, d\theta$$

is constant, and this is true even if u is only known to be harmonic in an annulus. By (58) we find in the same way that

$$\int\limits_{|z|=r} r \frac{\partial u}{\partial r}\, d\theta$$

is constant in the case of an annulus and zero if u is harmonic in the whole disk. Combining these results we obtain:

Theorem 20. *The arithmetic mean of a harmonic function over concentric circles $|z| = r$ is a linear function of* $\log r$,

$$\frac{1}{2\pi} \int\limits_{|z|=r} u\, d\theta = \alpha \log r + \beta, \tag{61}$$

and if u is harmonic in a disk $\alpha = 0$ and the arithmetic mean is constant.

In the latter case $\beta = u(0)$, by continuity, and changing to a new origin we find

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta. \tag{62}$$

It is clear that (62) could also have been derived from the corresponding formula for analytic functions, Sec. 3.4, (34). It leads directly to the *maximum principle* for harmonic functions:

Theorem 21. *A nonconstant harmonic function has neither a maximum nor a minimum in its region of definition. Consequently, the maximum and the minimum on a closed bounded set E are taken on the boundary of E.*

The proof is the same as for the maximum principle of analytic functions and will not be repeated. It applies also to the minimum for the reason that $-u$ is harmonic together with u. In the case of analytic functions the corresponding procedure would have been to apply the maximum principle to $1/f(z)$ which is illegitimate unless $f(z) \neq 0$. Observe that the maximum principle for analytic functions follows from the maximum principle for harmonic functions by applying the latter to $\log |f(z)|$ which is harmonic when $f(z) \neq 0$.

EXERCISES

1. If u is harmonic and bounded in $0 < |z| < \rho$, show that the origin is a removable singularity in the sense that u becomes harmonic in $|z| < \rho$ when $u(0)$ is properly defined.

2. Suppose that $f(z)$ is analytic in the annulus $r_1 < |z| < r_2$ and continuous on the closed annulus. If $M(r)$ denotes the maximum of $|f(z)|$ for $|z| = r$, show that

$$M(r) \leqq M(r_1)^\alpha M(r_2)^{1-\alpha}$$

where $\alpha = \log (r_2/r) : \log (r_2/r_1)$ (Hadamard's three-circle theorem). Discuss cases of equality. *Hint:* Apply the maximum principle to a linear combination of $\log |f(z)|$ and $\log |z|$.

6.3. Poisson's Formula. The maximum principle has the following important consequence: If $u(z)$ is continuous on a closed bounded set E and harmonic on the interior of E, then it is uniquely determined by its values on the boundary of E. Indeed, if u_1 and u_2 are two such functions with the same boundary values, then $u_1 - u_2$ is harmonic with the boundary values 0. By the maximum and minimum principle the difference $u_1 - u_2$ must then be identically zero on E.

There arises the problem of finding u when its boundary values are given. At this point we shall solve the problem only in the simplest case, namely for a closed disk.

Formula (62) determines the value of u at the center of the disk. But this is all we need, for there exists a linear transformation which carries any point to the center. To be explicit, suppose that $u(z)$ is harmonic in the closed disk $|z| \leqq R$. The linear transformation

$$z = S(\zeta) = \frac{R(R\zeta + a)}{R + \bar{a}\zeta}$$

maps $|\zeta| \leqq 1$ onto $|z| \leqq R$ with $\zeta = 0$ corresponding to $z = a$. The function $u(S(\zeta))$ is harmonic in $|\zeta| \leqq 1$, and by (62) we obtain

$$u(a) = \frac{1}{2\pi} \int_{|\zeta|=1} u(S(\zeta))\, d \arg \zeta.$$

From

$$\zeta = \frac{R(z - a)}{R^2 - \bar{a}z}$$

we compute

$$d \arg \zeta = -i \frac{d\zeta}{\zeta} = -i \left(\frac{1}{z - a} + \frac{\bar{a}}{R^2 - \bar{a}z} \right) dz = \left(\frac{z}{z - a} + \frac{\bar{a}z}{R^2 - \bar{a}z} \right) d\theta.$$

On substituting $R^2 = z\bar{z}$ the coefficient of $d\theta$ in the last expression can be rewritten as

$$\frac{z}{z - a} + \frac{\bar{a}}{\bar{z} - \bar{a}} = \frac{R^2 - |a|^2}{|z - a|^2}$$

or, equivalently, as

$$\frac{1}{2}\left(\frac{z + a}{z - a} + \frac{\bar{z} + \bar{a}}{\bar{z} - \bar{a}} \right) = \operatorname{Re} \frac{z + a}{z - a}.$$

We obtain the two forms

$$(63) \qquad u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z - a|^2} u(z)\, d\theta = \frac{1}{2\pi} \int_{|z|=R} \operatorname{Re} \frac{z + a}{z - a} u(z)\, d\theta$$

of *Poisson's formula.* In polar coordinates,

$$u(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR \cos(\theta - \varphi) + r^2} u(Re^{i\theta})\, d\theta.$$

In the derivation we have assumed that $u(z)$ is harmonic in the closed disk. However, the result remains true under the weaker condition that $u(z)$ is harmonic in the open disk and continuous in the closed disk. Indeed, if $0 < r < 1$, then $u(rz)$ is harmonic in the closed disk, and we obtain

$$u(ra) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z - a|^2} u(rz)\, d\theta.$$

Now all we need to do is to let r tend to 1. Because $u(z)$ is uniformly continuous on $|z| \leqq R$ it is true that $u(rz) \to u(z)$ uniformly for $|z| = R$, and we conclude that (63) remains valid.

We shall formulate the result as a theorem:

Theorem 22. *Suppose that $u(z)$ is harmonic for $|z| < R$, continuous for $|z| \leqq R$. Then*

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z - a|^2} u(z)\, d\theta \tag{64}$$

for all $|a| < R$.

The theorem leads at once to an explicit expression for the conjugate function of u. Indeed, formula (63) gives

$$u(z) = \mathrm{Re}\left[\frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta}\right]. \tag{65}$$

The bracketed expression is an analytic function of z for $|z| < R$. It follows that $u(z)$ is the real part of

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} + iC \tag{66}$$

where C is an arbitrary real constant. This formula is known as Schwarz's formula.

As a special case of (64), note that $u = 1$ yields

$$\int_{|z|=R} \frac{R^2 - |z|^2}{|z - a|^2} d\theta = 2\pi \tag{67}$$

for all $|a| < R$.

6.4. Schwarz's Theorem. Theorem 22 serves to express a given harmonic function through its values on a circle. But the right-hand side of formula (64) has a meaning as soon as u is defined on $|z| = R$, provided it is sufficiently regular, for instance piecewise continuous. As in (65) the integral can again be written as the real part of an analytic function, and consequently it is a harmonic function. The question is, does it have the boundary values $u(z)$ on $|z| = R$?

There is reason to clarify the notations. Choosing $R = 1$ we define, for any piecewise continuous function $U(\theta)$ in $0 \leqq \theta \leqq 2\pi$,

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\frac{e^{i\theta}+z}{e^{i\theta}-z}\, U(\theta)\, d\theta$$

and call this the *Poisson integral* of U. Observe that $P_U(z)$ is not only a function of z, but also a function of the function U; as such it is called a *functional.* The functional is *linear* inasmuch as

$$P_{U+V} = P_U + P_V$$

and

$$P_{cU} = cP_U$$

for constant c. Moreover, $U \geqq 0$ implies $P_U(z) \geqq 0$; because of this property P_U is said to be a *positive* linear functional.

We deduce from (67) that $P_c = c$. From this property, together with the linear and positive character of the functional, it follows that any inequality $m \leqq U \leqq M$ implies $m \leqq P_U \leqq M$.

The question of boundary values is settled by the following fundamental theorem that was first proved by H. A. Schwarz:

Theorem 23. *The function $P_U(z)$ is harmonic for $|z| < 1$, and*

(68) $$\lim_{z \to e^{i\theta_0}} P_U(z) = U(\theta_0)$$

provided that U is continuous at θ_0.

We have already remarked that P_U is harmonic. To study the boundary behavior, let C_1 and C_2 be complementary arcs of the unit circle, and denote by U_1 the function which coincides with U on C_1 and vanishes on C_2, by U_2 the corresponding function for C_2. Clearly, $P_U = P_{U_1} + P_{U_2}$.

Since P_{U_1} can be regarded as a line integral over C_1 it is, by the same reasoning as before, harmonic everywhere except on the closed arc C_1. The expression

$$\operorname{Re}\frac{e^{i\theta}+z}{e^{i\theta}-z} = \frac{1-|z|^2}{|e^{i\theta}-z|^2}$$

vanishes on $|z| = 1$ for $z \neq e^{i\theta}$. It follows that P_{U_1} is zero on the open arc C_2, and since it is continuous $P_{U_1}(z) \to 0$ as $z \to e^{i\theta} \in C_2$.

In proving (68) we may suppose that $U(\theta_0) = 0$, for if this is not the case we need only replace U by $U - U(\theta_0)$. Given $\varepsilon > 0$ we can find C_1 and C_2 such that $e^{i\theta_0}$ is an interior point of C_2 and $|U(\theta)| < \varepsilon/2$ for $e^{i\theta} \in C_2$. Under this condition $|U_2(\theta)| < \varepsilon/2$ for all θ, and hence $|P_{U_2}(z)| < \varepsilon/2$ for all $|z| < 1$. On the other hand, since U_1 is continuous and vanishes

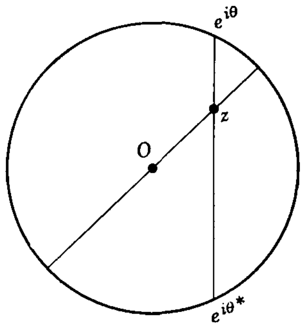

FIG. 4-15

at $e^{i\theta_0}$ there exists a δ such that $|P_{U_1}(z)| < \varepsilon/2$ for $|z - e^{i\theta_0}| < \delta$. It follows that $|P_U(z)| \leqq |P_{U_1}| + |P_{U_2}| < \varepsilon$ as soon as $|z| < 1$ and $|z - e^{i\theta_0}| < \delta$, which is precisely what we had to prove.

There is an interesting geometric interpretation of Poisson's formula, also due to Schwarz. Given a fixed z inside the unit circle we determine for each $e^{i\theta}$ the point $e^{i\theta^*}$ which is such that $e^{i\theta}$, z and $e^{i\theta^*}$ are in a straight line (Fig. 4-15). It is clear geometrically, or by simple calculation, that

$$1 - |z|^2 = |e^{i\theta} - z|\,|e^{i\theta^*} - z|. \tag{69}$$

But the ratio $(e^{i\theta} - z)/(e^{i\theta^*} - z)$ is negative, so we must have

$$1 - |z|^2 = -\,(e^{i\theta} - z)(e^{-i\theta^*} - \bar{z}).$$

We regard θ^* as a function of θ and differentiate. Since z is constant we obtain

$$\frac{e^{i\theta}\,d\theta}{e^{i\theta} - z} = \frac{e^{-i\theta^*}\,d\theta^*}{e^{-i\theta^*} - \bar{z}}$$

and, on taking absolute values,

$$\frac{d\theta^*}{d\theta} = \left|\frac{e^{i\theta^*} - z}{e^{i\theta} - z}\right|. \tag{70}$$

It follows by (69) and (70) that

$$\frac{1 - |z|^2}{|e^{i\theta} - z|^2} = \frac{d\theta^*}{d\theta}$$

and hence

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} U(\theta)\,d\theta^* = \frac{1}{2\pi}\int_0^{2\pi} U(\theta^*)\,d\theta.$$

In other words, to find $P_U(z)$, replace each value of $U(\theta)$ by the value at the point opposite to z, and take the average over the circle.

EXERCISES

1. Assume that $U(\xi)$ is piecewise continuous and bounded for all real ξ. Show that

$$P_U(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x - \xi)^2 + y^2} U(\xi)\, d\xi$$

represents a harmonic function in the upper half plane with boundary values $U(\xi)$ at points of continuity (Poisson's integral for the half plane).

2. Prove that a function which is harmonic and bounded in the upper half plane, continuous on the real axis, can be represented as a Poisson integral (Ex. 1).

Remark. The point at ∞ presents an added difficulty, for we cannot immediately apply the maximum and minimum principle to $u - P_u$. A good try would be to apply the maximum principle to $u - P_u - \varepsilon y$ for $\varepsilon > 0$, with the idea of letting ε tend to 0. This almost works, for the function tends to 0 for $y \to 0$ and to $-\infty$ for $y \to \infty$, but we lack control when $|x| \to \infty$. Show that the reasoning can be carried out successfully by application to $u - P_u - \varepsilon \operatorname{Im} (\sqrt{iz})$.

3. In Ex. 1, assume that U has a jump at 0, for instance $U(+0) = 0$, $U(-0) = 1$. Show that $P_U(z) - \frac{1}{\pi} \arg z$ tends to 0 as $z \to 0$. Generalize to arbitrary jumps and to the case of the circle.

4. If C_1 and C_2 are complementary arcs on the unit circle, set $U = 1$ on C_1, $U = 0$ on C_2. Find $P_U(z)$ explicitly and show that $2\pi P_U(z)$ equals the length of the arc, opposite to C_1, cut off by the straight lines through z and the end points of C_1.

5. Show that the mean-value formula (62) remains valid for $u = \log |1 + z|$, $z_0 = 0$, $r = 1$, and use this fact to compute

$$\int_0^{\pi} \log \sin \theta \, d\theta.$$

6. If $f(z)$ is analytic in the whole plane and if $z^{-1} \operatorname{Re} f(z) \to 0$ when $z \to \infty$, show that f is a constant. *Hint:* Use (66).

7. If $f(z)$ is analytic in a neighborhood of ∞ and if $z^{-1} \operatorname{Re} f(z) \to 0$ when $z \to \infty$, show that $\lim_{z \to \infty} f(z)$ exists. (In other words, the isolated singularity at ∞ is removable.)

Hint: Show first, by use of Cauchy's integral formula, that $f = f_1 + f_2$ where $f_1(z) \to 0$ for $z \to \infty$ and $f_2(z)$ is analytic in the whole plane.

*8. If $u(z)$ is harmonic for $0 < |z| < \rho$ and $\lim_{z \to 0} zu(z) = 0$, prove that u can be written in the form $u(z) = \alpha \log |z| + u_0(z)$ where α is a constant and u_0 is harmonic in $|z| < \rho$.

Hint: Choose α as in (61). Then show that u_0 is the real part of an analytic function $f_0(z)$ and use the preceding exercise to conclude that the singularity at 0 is removable.

6.5. The Reflection Principle. An elementary aspect of the *symmetry principle*, or *reflection principle*, has been discussed already in connection with linear transformations (Chap. 3, Sec. 3.3). There are many more general variants first formulated by H. A. Schwarz.

The principle of reflection is based on the observation that if $u(z)$ is a harmonic function, then $u(\bar{z})$ is likewise harmonic, and if $f(z)$ is an analytic function, then $\overline{f(\bar{z})}$ is also analytic. More precisely, if $u(z)$ is harmonic and $f(z)$ analytic in a region then $u(\bar{z})$ is harmonic and $\overline{f(\bar{z})}$ analytic as functions of z in the region Ω^* obtained by reflecting Ω in the real axis; that is, $z \in \Omega^*$ if and only if $\bar{z} \in \Omega$. The proofs of these statements consist in trivial verifications.

Consider the case of a symmetric region: $\Omega^* = \Omega$. Because Ω is connected it must intersect the real axis along at least one open interval. Assume now that $f(z)$ is analytic in Ω and real on at least one interval of the real axis. Since $f(z) - \overline{f(\bar{z})}$ is analytic and vanishes on an interval it must be identically zero, and we conclude that $f(z) = \overline{f(\bar{z})}$ in Ω. With the notation $f = u + iv$ we have thus $u(z) = u(\bar{z})$, $v(z) = -v(\bar{z})$.

This is important, but it is a rather weak result, for we are assuming that $f(z)$ is already known to be analytic in all of Ω. Let us denote the intersection of Ω with the upper half plane by Ω^+, and the intersection of Ω with the real axis by σ. Suppose that $f(z)$ is defined on $\Omega^+ \cup \sigma$, analytic in Ω^+, continuous and real on σ. Under these conditions we want to show that $f(z)$ is the restriction to Ω^+ of a function which is analytic in all of Ω and satisfies the symmetry condition $f(z) = \overline{f(\bar{z})}$. In other words, part of our theorem asserts that $f(z)$ has an *analytic continuation* to Ω.

Even in this formulation the assumptions are too strong. Indeed, the main thing is that the imaginary part $v(z)$ vanishes on σ, and nothing at all need to be assumed about the real part. In the definitive statement of the reflection principle the emphasis should therefore be on harmonic functions.

Theorem 24. *Let Ω^+ be the part in the upper half plane of a symmetric region Ω, and let σ be the part of the real axis in Ω. Suppose that $v(x)$ is continuous in $\Omega^+ \cup \sigma$, harmonic in Ω^+, and zero on σ. Then v has a harmonic extension to Ω which satisfies the symmetry relation $v(\bar{z}) = -v(z)$.*

In the same situation, if v is the imaginary part of an analytic function $f(z)$ in Ω^+, then $f(z)$ has an analytic extension which satisfies $f(z) = \overline{f(\bar{z})}$.

For the proof we construct the function $V(z)$ which is equal to $v(z)$ in Ω^+, 0 on σ, and equal to $-v(\bar{z})$ in the mirror image of Ω^+. We have to show that V is harmonic on σ. For a point $x_0 \in \sigma$ consider a disk with center x_0 contained in Ω, and let P_V denote the Poisson integral with respect to this disk formed with the boundary values V. The difference $V - P_V$ is harmonic in the upper half of the disk. It vanishes on the half circle, by Theorem 23, and also on the diameter, because V tends to zero by definition and P_V vanishes by obvious symmetry. The maximum and minimum principle implies that $V = P_V$ in the upper half disk, and the same proof can be repeated for the lower half. We conclude that V is harmonic in the whole disk, and in particular at x_0.

For the remaining part of the theorem, let us again consider a disk with center on σ. We have already extended v to the whole disk, and v has a conjugate harmonic function $-u_0$ in the same disk which we may normalize so that $u_0 = \operatorname{Re} f(z)$ in the upper half. Consider

$$U_0(z) = u_0(z) - u_0(\bar{z}).$$

On the real diameter it is clear that $\partial U_0/\partial x = 0$ and also

$$\frac{\partial U_0}{\partial y} = 2\frac{\partial u_0}{\partial y} = -2\frac{\partial v}{\partial x} = 0.$$

It follows that the analytic function $\partial U_0/\partial x - i\,\partial U_0/\partial y$ vanishes on the real axis, and hence identically. Therefore U_0 is a constant, and this constant is evidently zero. We have proved that $u_0(z) = u_0(\bar{z})$.

The construction can be repeated for arbitrary disks. It is clear that the u_0 coincide in overlapping disks. The definition can be extended to all of Ω, and the theorem follows.

The theorem has obvious generalizations. The domain Ω can be taken to be symmetric with respect to a circle C rather than with respect to a straight line, and when z tends to C it may be assumed that $f(z)$ approaches another circle C'. Under such conditions $f(z)$ has an analytic continuation which maps symmetric points with respect to C onto symmetric points with respect to C'.

EXERCISES

1. If $f(z)$ is analytic in the whole plane and real on the real axis, purely imaginary on the imaginary axis, show that $f(z)$ is odd.

2. Show that every function f which is analytic in a symmetric region Ω can be written in the form $f_1 + if_2$ where f_1, f_2 are analytic in Ω and real on the real axis.

3. If $f(z)$ is analytic in $|z| \leqq 1$ and satisfies $|f| = 1$ on $|z| = 1$, show that $f(z)$ is rational.

4. Use (66) to derive a formula for $f'(z)$ in terms of $u(z)$.

5. If $u(z)$ is harmonic and $0 \leqq u(z) \leqq Ky$ for $y > 0$, prove that $u = ky$ with $0 \leqq k \leqq K$. [Reflect over the real axis, complete to an analytic function $f(z) = u + iv$, and use Ex. 4 to show that $f'(z)$ is bounded.]

5 SERIES AND PRODUCT DEVELOPMENTS

Very general theorems have their natural place in the theory of analytic functions, but it must also be kept in mind that the whole theory originated from a desire to be able to manipulate explicit analytic expressions. Such expressions take the form of infinite series, infinite products, and other limits. In this chapter we deal partly with the rules that govern such limits, partly with quite explicit representations of elementary transcendental functions and other specific functions.

1. POWER SERIES EXPANSIONS

In a preliminary way we have considered power series in Chap. 2, mainly for the purpose of defining the exponential and trigonometric functions. Without use of integration we were not able to prove that every analytic function has a power series expansion. This question will now be resolved in the affirmative, essentially as an application of Cauchy's theorem.

The first subsection deals with more general properties of sequences of analytic functions.

1.1. Weierstrass's Theorem. The central theorem concerning the convergence of analytic functions asserts that the limit of a uniformly convergent sequence of analytic functions is an analytic function. The precise assumptions must be carefully stated, and they should not be too restrictive.

We are considering a sequence $\{f_n(z)\}$ where each $f_n(z)$ is defined and analytic in a region Ω_n. The limit function $f(z)$ must also be considered in some region Ω, and clearly, if $f(z)$ is to be defined in Ω, each point of Ω must belong to all Ω_n for n greater than a certain n_0. In the general case n_0 will not be the same for all points of Ω, and for this reason it would not make sense to require that the convergence be uniform in Ω. In fact, in the most typical case the regions Ω_n form an increasing sequence, $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$, and Ω is the union of the Ω_n. In these circumstances no single function $f_n(z)$ is defined in all of Ω; yet the limit $f(z)$ may exist at all points of Ω, although the convergence cannot be uniform.

As a very simple example take $f_n(z) = z/(2z^n + 1)$ and let Ω_n be the disk $|z| < 2^{-1/n}$. It is practically evident that $\lim_{n\to\infty} f_n(z) = z$ in the disk $|z| < 1$ which we choose as our region Ω. In order to study the uniformity of the convergence we form the difference

$$f_n(z) - z = -2z^{n+1}/(2z^n + 1).$$

For any given value of z we can make $|z^n| < \varepsilon/4$ by taking $n > \log(4/\varepsilon)/\log(1/|z|)$. If $\varepsilon < 1$ we have then $2|z|^{n+1} < \varepsilon/2$ and $|1 + 2z^n| > \frac{1}{2}$ so that $|f_n(z) - z| < \varepsilon$. It follows that the convergence is uniform in any closed disk $|z| \leqq r < 1$, or on any subset of such a closed disk.

With another formulation, in the preceding example the sequence $\{f_n(z)\}$ tends to the limit function $f(z)$ uniformly on every compact subset of the region Ω. In fact, on a compact set $|z|$ has a maximum $r < 1$ and the set is thus contained in the closed disk $|z| \leqq r$. This is the typical situation. We shall find that we can frequently prove uniform convergence on every compact subset of Ω; on the other hand, this is the natural condition in the theorem that we are going to prove.

Theorem 1. *Suppose that $f_n(z)$ is analytic in the region Ω_n, and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region Ω, uniformly on every compact subset of Ω. Then $f(z)$ is analytic in Ω. Moreover, $f_n'(z)$ converges uniformly to $f'(z)$ on every compact subset of Ω.*

The analyticity of $f(z)$ follows most easily by use of Morera's theorem (Chap. 4, Sec. 2.3). Let $|z - a| \leqq r$ be a closed disk contained in Ω; the assumption implies that this disk lies in Ω_n for all n greater than a certain n_0.† If γ is any closed curve contained in $|z - a| < r$, we have

$$\int_\gamma f_n(z)\,dz = 0$$

† In fact, the regions Ω_n form an open covering of $|z - a| \leqq r$. The disk is compact and hence has a finite subcovering. This means that it is contained in a fixed Ω_{n_0}.

for $n > n_0$, by Cauchy's theorem. Because of the uniform convergence on γ we obtain

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz = 0,$$

and by Morera's theorem it follows that $f(z)$ is analytic in $|z - a| < r$. Consequently $f(z)$ is analytic in the whole region Ω.

An alternative and more explicit proof is based on the integral formula

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\zeta)\,d\zeta}{\zeta - z},$$

where C is the circle $|\zeta - a| = r$ and $|z - a| < r$. Letting n tend to ∞ we obtain by uniform convergence

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

and this formula shows that $f(z)$ is analytic in the disk. Starting from the formula

$$f_n'(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\zeta)\,d\zeta}{(\zeta - z)^2}$$

the same reasoning yields

$$\lim_{n\to\infty} f_n'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2} = f'(z),$$

and simple estimates show that the convergence is uniform for $|z - a| \leqq \rho < r$. Any compact subset of Ω can be covered by a finite number of such closed disks, and therefore the convergence is uniform on every compact subset. The theorem is proved, and by repeated applications it follows that $f_n^{(k)}(z)$ converges uniformly to $f^{(k)}(z)$ on every compact subset of Ω.

Theorem 1 is due to Weierstrass, in an equivalent formulation. Its application to series whose terms are analytic functions is particularly important. The theorem can then be expressed as follows:

If a series with analytic terms,

$$f(z) = f_1(z) + f_2(z) + \cdots + f_n(z) + \cdots,$$

converges uniformly on every compact subset of a region Ω, then the sum $f(z)$ is analytic in Ω, and the series can be differentiated term by term.

The task of proving uniform convergence on a compact point set A can be facilitated by use of the maximum principle. In fact, with the notations of Theorem 1, the difference $|f_m(z) - f_n(z)|$ attains its maximum in A on the boundary of A. For this reason uniform convergence on the boundary of A implies uniform convergence on A. For instance, if the functions $f_n(z)$ are analytic in the disk $|z| < 1$, and if it can be shown that the sequence converges uniformly on each circle $|z| = r_m$ where $\lim_{m\to\infty} r_m = 1$, then Weierstrass's theorem applies and we can conclude that the limit function is analytic.

The following theorem is due to A. Hurwitz:

Theorem 2. *If the functions $f_n(z)$ are analytic and $\neq 0$ in a region Ω, and if $f_n(z)$ converges to $f(z)$, uniformly on every compact subset of Ω, then $f(z)$ is either identically zero or never equal to zero in Ω.*

Suppose that $f(z)$ is not identically zero. The zeros of $f(z)$ are in any case isolated. For any point $z_0 \in \Omega$ there is therefore a number $r > 0$ such that $f(z)$ is defined and $\neq 0$ for $0 < |z - z_0| \leqq r$. In particular, $|f(z)|$ has a positive minimum on the circle $|z - z_0| = r$, which we denote by C. It follows that $1/f_n(z)$ converges uniformly to $1/f(z)$ on C. Since it is also true that $f_n'(z) \to f'(z)$, uniformly on C, we may conclude that

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_C \frac{f_n'(z)}{f_n(z)}\,dz = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\,dz.$$

But the integrals on the left are all zero, for they give the number of roots of the equation $f_n(z) = 0$ inside of C. The integral on the right is therefore zero, and consequently $f(z_0) \neq 0$ by the same interpretation of the integral. Since z_0 was arbitrary, the theorem follows.

EXERCISES

1. Using Taylor's theorem applied to a branch of $\log(1 + z/n)$, prove that

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z$$

uniformly on all compact sets.

2. Show that the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

converges for $\operatorname{Re} z > 1$, and represent its derivative in series form.

3. Prove that

$$(1 - 2^{1-z})\zeta(z) = 1^{-z} - 2^{-z} + 3^{-z} - \cdots$$

and that the latter series represents an analytic function for $\operatorname{Re} z > 0$.

4. As a generalization of Theorem 2, prove that if the $f_n(z)$ have at most m zeros in Ω, then $f(z)$ is either identically zero or has at most m zeros.

5. Prove that

$$\sum_{n=1}^{\infty} \frac{nz^n}{1-z^n} = \sum_{n=1}^{\infty} \frac{z^n}{(1-z^n)^2}$$

for $|z| < 1$. (Develop in a double series and reverse the order of summation.)

1.2. The Taylor Series. We show now that every analytic function can be developed in a convergent Taylor series. This is an almost immediate consequence of the finite Taylor development given in Chap. 4, Sec. 3.1, Theorem 8, together with the corresponding representation of the remainder term. According to this theorem, if $f(z)$ is analytic in a region Ω containing z_0, we can write

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + f_{n+1}(z)(z - z_0)^{n+1}$$

with

$$f_{n+1}(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}(\zeta - z)}.$$

In the last formula C is any circle $|z - z_0| = \rho$ such that the closed disk $|z - z_0| \leqq \rho$ is contained in Ω.

If M denotes the maximum of $|f(z)|$ on C, we obtain at once the estimate

$$|f_{n+1}(z)(z - z_0)^{n+1}| \leqq \frac{M|z - z_0|^{n+1}}{\rho^n(\rho - |z - z_0|)}$$

We conclude that the remainder term tends uniformly to zero in every disk $|z - z_0| \leqq r < \rho$. On the other hand, ρ can be chosen arbitrarily close to the shortest distance from z_0 to the boundary of Ω. We have proved:

Theorem 3. *If $f(z)$ is analytic in the region Ω, containing z_0, then the representation*

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \cdots + \frac{f^{((n)}(z_0)}{n!}(z - z_0)^n + \cdots$$

is valid in the largest open disk of center z_0 contained in Ω.

The radius of convergence of the Taylor series is thus at least equal to the shortest distance from z_0 to the boundary of Ω. It may well be larger, but if it is there is no guarantee that the series still represents $f(z)$ at all points which are simultaneously in Ω and in the circle of convergence.

We recall that the developments

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

served as definitions of the functions they represent. Of course, as we have remarked before, every convergent power series is its own Taylor series. We gave earlier a direct proof that power series can be differentiated term by term. This is also a direct consequence of Weierstrass's theorem.

If we want to represent a fractional power of z or $\log z$ through a power series, we must first of all choose a well-defined branch, and secondly we have to choose a center $z_0 \neq 0$. It amounts to the same thing if we develop the function $(1 + z)^\mu$ or $\log (1 + z)$ about the origin, choosing the branch which is respectively equal to 1 or 0 at the origin. Since this branch is single-valued and analytic in $|z| < 1$, the radius of convergence is at least 1. It is elementary to compute the coefficients, and we obtain

$$(1 + z)^\mu = 1 + \mu z + \binom{\mu}{2} z^2 + \cdots + \binom{\mu}{n} z^n + \cdots$$
$$\log (1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \cdots$$

where the binomial coefficients are defined by

$$\binom{\mu}{n} = \frac{\mu(\mu - 1) \cdots (\mu - n + 1)}{1 \cdot 2 \cdots n}.$$

If the logarithmic series had a radius of convergence greater than 1, then $\log (1 + z)$ would be bounded for $|z| < 1$. Since this is not the case, the radius of convergence must be exactly 1. Similarly, if the binomial series were convergent in a circle of radius >1, the function $(1 + z)^\mu$ and all its derivatives would be bounded in $|z| < 1$. Unless μ is a positive integer, one of the derivatives will be a negative power of $1 + z$, and hence unbounded. Thus the radius of convergence is precisely 1 except in the trivial case in which the binomial series reduces to a polynomial.

The series developments of the cyclometric functions arc tan z and arc sin z are most easily obtained by consideration of the derived series. From the expansion

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$

we obtain by integration

$$\text{arc tan } z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

where the branch is uniquely determined as

$$\text{arc tan } z = \int_0^z \frac{dz}{1+z^2}$$

for any path inside the unit circle. For justification we can either rely on uniform convergence or apply Theorem 1. The radius of convergence cannot be greater than that of the derived series, and hence it is exactly 1.

If $\sqrt{1-z^2}$ is the branch with a positive real part, we have

$$\frac{1}{\sqrt{1-z^2}} = 1 + \frac{1}{2}z^2 + \frac{1 \cdot 3}{2 \cdot 4}z^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}z^6 + \cdots$$

for $|z| < 1$, and through integration we obtain

$$\text{arc sin } z = z + \frac{1}{2}\frac{z^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{z^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{z^7}{7} + \cdots.$$

The series represents the principal branch of arc sin z with a real part between $-\pi/2$ and $\pi/2$.

For combinations of elementary functions it is mostly not possible to find a general law for the coefficients. In order to find the first few coefficients we need not, however, calculate the successive derivatives. There are simple techniques which allow us to compute, with a reasonable amount of labor, all the coefficients that we are likely to need.

It is convenient to introduce the notation $[z^n]$ for any function which is analytic and has a zero of at least order n at the origin; less precisely, $[z^n]$ denotes a function which "contains the factor z^n." With this notation any function which is analytic at the origin can be written in the form

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n + [z^{n+1}],$$

where the coefficients are uniquely determined and equal to the Taylor coefficients of $f(z)$. Thus, in order to find the first n coefficients of the Taylor expansion, it is sufficient to determine a polynomial $P_n(z)$ such that $f(z) - P_n(z)$ has a zero of at least order $n + 1$ at the origin. The degree of $P_n(z)$ does not matter; it is true in any case that the coefficients of z^m, $m \leqq n$, are the Taylor coefficients of $f(z)$.

For instance, suppose that

$$f(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n + \cdots$$
$$g(z) = b_0 + b_1z + b_2z^2 + \cdots + b_nz^n + \cdots .$$

With an abbreviated notation we write

$$f(z) = P_n(z) + [z^{n+1}]; \qquad g(z) = Q_n(z) + [z^{n+1}].$$

It is then clear that $f(z)g(z) = P_n(z)Q_n(z) + [z^{n+1}]$, and the coefficients of the terms of degree $\leqq n$ in P_nQ_n are the Taylor coefficients of the product $f(z)g(z)$. Explicitly we obtain

$$f(z)g(z) = a_0b_0 + (a_0b_1 + a_1b_0)z + \cdots \\ + (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)z^n + \cdots .$$

In deriving this expansion we have not even mentioned the question of convergence, but since the development is identical with the Taylor development of $f(z)g(z)$, it follows by Theorem 3 that the radius of convergence is at least equal to the smaller of the radii of convergence of the given series $f(z)$ and $g(z)$. In the practical computation of P_nQ_n it is of course not necessary to determine the terms of degree higher than n.

In the case of a quotient $f(z)/g(z)$ the same method can be applied, provided that $g(0) = b_0 \neq 0$. By use of ordinary long division, continued until the remainder contains the factor z^{n+1}, we can determine a polynomial R_n such that $P_n = Q_nR_n + [z^{n+1}]$. Then $f - R_ng = [z^{n+1}]$, and since $g(0) \neq 0$ we find that $f/g = R_n + [z^{n+1}]$. The coefficients of R_n are the Taylor coefficients of $f(z)/g(z)$. They can be determined explicitly in determinant form, but the expressions are too complicated to be of essential help.

It is also important that we know how to form the development of a composite function $f(g(z))$. In this case, if $g(z)$ is developed around z_0, the expansion of $f(w)$ must be in powers of $w - g(z_0)$. To simplify, let us assume that $z_0 = 0$ and $g(0) = 0$. We can then set

$$f(w) = a_0 + a_1w + \cdots + a_nw^n + \cdots$$

and $g(z) = b_1z + b_2z^2 + \cdots + b_nz^n + \cdots$. Using the same notations as before we write $f(w) = P_n(w) + [w^{n+1}]$ and $g(z) = Q_n(z) + [z^{n+1}]$ with $Q_n(0) = 0$. Substituting $w = g(z)$ we have to observe that

$$P_n(Q_n + [z^{n+1}]) = P_n(Q_n(z)) + [z^{n+1}]$$

and that any expression of the form $[w^{n+1}]$ becomes a $[z^{n+1}]$. Thus we obtain $f(g(z)) = P_n(Q_n(z)) + [z^{n+1}]$, and the Taylor coefficients of $f(g(z))$ are the coefficients of $P_n(Q_n(z))$ for powers $\leqq n$.

Finally, we must be able to expand the inverse function of an analytic function $w = g(z)$. Here we may suppose that $g(0) = 0$, and we are looking for the branch of the inverse function $z = g^{-1}(w)$ which is analytic in a neighborhood of the origin and vanishes for $w = 0$. For the existence of the inverse function it is necessary and sufficient that $g'(0) \neq 0$; hence we assume that

$$g(z) = a_1z + a_2z^2 + \cdots = Q_n(z) + [z^{n+1}]$$

with $a_1 \neq 0$. Our problem is to determine a polynomial $P_n(w)$ such that $P_n(Q_n(z)) = z + [z^{n+1}]$. In fact, under the assumption $a_1 \neq 0$ the notations $[z^{n+1}]$ and $[w^{n+1}]$ are interchangeable, and from $z = P_n(Q_n(z)) + [z^{n+1}]$ we obtain $z = P_n(g(z) + [z^{n+1}]) + [z^{n+1}] = P_n(w) + [w^{n+1}]$. Hence $P_n(w)$ determines the coefficients of $g^{-1}(w)$.

In order to prove the existence of a polynomial P_n we proceed by induction. Clearly, we can take $P_1(w) = w/a_1$. If P_{n-1} is given, we set $P_n = P_{n-1} + b_nw^n$ and obtain

$$\begin{aligned} P_n(Q_n(z)) &= P_{n-1}(Q_n(z)) + b_na_1^nz^n + [z^{n+1}] \\ &= P_{n-1}(Q_{n-1}(z) + a_nz^n) + b_na_1^nz^n + [z^{n+1}] \\ &= P_{n-1}(Q_{n-1}(z)) + P'_{n-1}(Q_{n-1}(z))a_nz^n + b_na_1^nz^n + [z^{n+1}]. \end{aligned}$$

In the last member the first two terms form a known polynomial of the form $z + c_nz^n + [z^{n+1}]$, and we have only to take $b_n = -c_na_1^{-n}$.

For practical purposes the development of the inverse function is found by successive substitutions. To illustrate the method we determine the expansion of tan w from the series

$$w = \text{arc tan } z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots .$$

If we want the development to include fifth powers, we write

$$z = w + \frac{z^3}{3} - \frac{z^5}{5} + [z^7]$$

and substitute this expression in the terms to the right. With appropriate remainders we obtain

$$\begin{aligned} z = w + \frac{1}{3}\left(w + \frac{z^3}{3} + [w^5]\right)^3 - \frac{1}{5}(w + [w^3])^5 + [w^7] \\ = w + \frac{1}{3}w^3 + \frac{1}{3}w^2z^3 - \frac{1}{5}w^5 + [w^7] \\ = w + \frac{1}{3}w^3 + \frac{1}{3}w^2(w + [w^3])^3 - \frac{1}{5}w^5 + [w^7] = w + \frac{1}{3}w^3 + \frac{2}{15}w^5 + [w^7]. \end{aligned}$$

Thus the development of tan w begins with the terms

$$\tan w = w + \frac{1}{3} w^3 + \frac{2}{15} w^5 + \cdots .$$

EXERCISES

1. Develop $1/(1 + z^2)$ in powers of $z - a$, a being a real number. Find the general coefficient and for $a = 1$ reduce to simplest form.

2. The Legendre polynomials are defined as the coefficients $P_n(\alpha)$ in the development

$$(1 - 2\alpha z + z^2)^{-\frac{1}{2}} = 1 + P_1(\alpha)z + P_2(\alpha)z^2 + \cdots .$$

Find P_1, P_2, P_3, and P_4.

3. Develop log $(\sin z/z)$ in powers of z up to the term z^6.

4. What is the coefficient of z^7 in the Taylor development of tan z?

5. The Fibonacci numbers are defined by $c_0 = 0$, $c_1 = 1$,

$$c_n = c_{n-1} + c_{n-2}.$$

Show that the c_n are Taylor coefficients of a rational function, and determine a closed expression for c_n.

1.3. The Laurent Series. A series of the form

$$b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n} + \cdots \tag{1}$$

can be considered as an ordinary power series in the variable $1/z$. It will therefore converge outside of some circle $|z| = R$, except in the extreme case $R = \infty$; the convergence is uniform in every region $|z| \geqq \rho > R$, and hence the series represents an analytic function in the region $|z| > R$. If the series (1) is combined with an ordinary power series, we get a more general series of the form

$$\sum_{n=-\infty}^{+\infty} a_n z^n. \tag{2}$$

It will be termed convergent only if the parts consisting of nonnegative powers and negative powers are separately convergent. Since the first part converges in a disk $|z| < R_2$ and the second series in a region $|z| > R_1$, there is a common region of convergence only if $R_1 < R_2$, and (2) represents an analytic function in the annulus $R_1 < |z| < R_2$.

Conversely, we may start from an analytic function $f(z)$ whose region of definition contains an annulus $R_1 < |z| < R_2$, or more generally an annulus $R_1 < |z - a| < R_2$. We shall show that such a function can always be developed in a general power series of the form

$$f(z) = \sum_{n=-\infty}^{+\infty} A_n(z-a)^n.$$

The proof is extremely simple. All we have to show is that $f(z)$ can be written as a sum $f_1(z) + f_2(z)$ where $f_1(z)$ is analytic for $|z - a| < R_2$ and $f_2(z)$ is analytic for $|z - a| > R_1$ with a removable singularity at ∞. Under these circumstances $f_1(z)$ can be developed in nonnegative powers of $z - a$, and $f_2(z)$ can be developed in nonnegative powers of $1/(z - a)$.

To find the representation $f(z) = f_1(z) + f_2(z)$ define $f_1(z)$ by

$$f_1(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

for $|z - a| < r < R_2$ and $f_2(z)$ by

$$f_2(z) = -\frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

for $R_1 < r < |z - a|$. In both integrals the value of r is irrelevant as long as the inequality is fulfilled, for it is an immediate consequence of Cauchy's theorem that the value of the integral does not change with r provided that the circle does not pass over the point z. For this reason $f_1(z)$ and $f_2(z)$ are uniquely defined and represent analytic functions in $|z - a| < R_2$ and $|z - a| > R_1$ respectively. Moreover, by Cauchy's integral theorem $f(z) = f_1(z) + f_2(z)$.

The Taylor development of $f_1(z)$ is

$$f_1(z) = \sum_{n=0}^{\infty} A_n(z-a)^n$$

with

$$A_n = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)\,d\zeta}{(\zeta - a)^{n+1}}. \tag{3}$$

In order to find the development of $f_2(z)$ we perform the transformation $\zeta = a + 1/\zeta'$, $z = a + 1/z'$. This transformation carries $|\zeta - a| = r$ into $|\zeta'| = 1/r$ with negative orientation, and by simple calculations we obtain

$$f_2\left(a + \frac{1}{z'}\right) = \frac{1}{2\pi i} \int_{|\zeta'| = \frac{1}{r}} \frac{z'}{\zeta'} \frac{f\left(a + \frac{1}{\zeta'}\right) d\zeta'}{\zeta' - z'} = \sum_{n=1}^{\infty} B_n z'^n$$

with

$$B_n = \frac{1}{2\pi i} \int_{|\zeta'|=\frac{1}{r}} \frac{f\left(a + \frac{1}{\zeta'}\right) d\zeta'}{\zeta'^{n+1}} = \frac{1}{2\pi i} \int_{|\zeta - a| = r} f(\zeta)(\zeta - a)^{n-1} \, d\zeta.$$

This formula shows that we can write

$$f(z) = \sum_{n=-\infty}^{+\infty} A_n(z - a)^n$$

where all the coefficients A_n are determined by (3). Observe that the integral in (3) is independent of r as long as $R_1 < r < R_2$.

If $R_1 = 0$ the point a is an isolated singularity and $A_{-1} = B_1$ is the residue at a, for $f(z) - A_{-1}(z - a)^{-1}$ is the derivative of a single-valued function in $0 < |z - a| < R_2$.

EXERCISES

1. Prove that the Laurent development is unique.

2. Let Ω be a doubly connected region whose complement consists of the components E_1, E_2. Prove that every analytic function $f(z)$ in Ω can be written in the form $f_1(z) + f_2(z)$ where $f_1(z)$ is analytic outside of E_1 and $f_2(z)$ is analytic outside of E_2. (The precise proof requires a construction like the one in Chap. 4, Sec. 4.5.)

3. The expression

$$\{f,z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is called the *Schwarzian derivative* of f. If f has a multiple zero or pole, find the leading term in the Laurent development of $\{f,z\}$. *Answer:* If $f(z) = a(z - z_0)^m + \cdots$, then $\{f,z\} = \frac{1}{2}(1 - m^2)(z - z_0)^{-2} + \cdots$.

4. Show that the Laurent development of $(e^z - 1)^{-1}$ at the origin is of the form

$$\frac{1}{z} - \frac{1}{2} + \sum_{1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$$

where the numbers B_k are known as the Bernoulli numbers. Calculate B_1, B_2, B_3. (By Sec. 2.1, Ex. 5, the B_k are all positive.)

5. Express the Taylor development of $\tan z$ and the Laurent development of $\cot z$ in terms of the Bernoulli numbers.