DIFFERENTIAL EQUATIONS AND LAPLACE TRANSFORMS

UNIT.I: First order , higher degree differential equations solvable for x- solvable for y- solvable for $\frac{dy}{dx}$ - Clairauts form- conditions of integrability of $Mdx + Ndy = 0$ -Simple problems.

Unit .II: Particular Integrals of second order differential equations with constant coefficients- Linear equations with constant coefficients – Linear equations with variable coefficients- Method of variation of parameters (Omit third and higher order equations).

Unit.III: Formation of partial differential equations – General ,Particular and Complete integrals- Solutions of Partial differential equations of the standard forms- Lagrange's method – Charpit's method and a few standard forms.

Unit. IV: Partial differential equations of second order homogeneous equations with constant coefficients – Particular integrals of $F(D, D^1)z = f(x, y)$ where $f(x, y)$ is one of the form $e^{ax+by}, \sin(ax+by), \cos(ax+by), x^r y^s$ and $e^{ax+by} f(x, y)$.

UNIT.V: Laplace transforms-standard formulae- Basic theorems and simple applications- Inverse Laplace transforms –Use of Laplace transform in solving ordinary diffrerntial equations with constant coefficients.

TEXT BOOKS:

1. **M.D.Raisinghania,** Ordinary and partial differential equations, Sulthan chand and co.
2. **M.K.Venkataraman,** Engineering Mathematics,S.V.Publications,1985,Revised edition.

DIFFERENTIAL EQUATIONS

Definition: A differential equation is an equation in which differential coefficients occurs. Differential equations are of two types.

(i) Ordinary differential equations (ii) Partial differential equations

Definition: An ordinary differential equation is an equation in which a single independent variable enters either explicitly or implicitly .For example

$$\frac{dy}{dx} = 2\cos x, \frac{d^2y}{dx^2} + m^2 y = 0 \text{ and } x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = \sin x$$

are all ordinary differential equations.

Definition: A partial differential equation is one in which at least two (two or more) independent variables occur and the partial differential coefficients occurring in them have reference to any of these variables .For example

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \text{ and } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)$$

are all partial differential equations.

Definition: The order of an ordinary differential equation is the order of the highest derivative occurring in it.

Definition: The degree of the differential equation is the degree of the highest derivative when it is cleared of radicals and fractions.

For example 1) $2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 5y = x$,the order of the differential equation is two and the degree is also two.

2) $$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = a\frac{d^2y}{dx^2}$$

the order and degree of the differential equation are both two.

Equations of the first order but of higher degree:

TYPE:A Equations solvable for $\frac{dy}{dx}$

We shall denote $\frac{dy}{dx}$ hereafter by p .

Let the equation of the first order and of the n^{th} degree in p be

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + ... + P_n = 0 \quad \text{-------------------} \quad (1)$$

Where $P_1, P_2, ...P_n$ denote functions of x and y.
Suppose the first number of (1) can be resolved into factors of the first degree of the form

$$(p-R_1)(p-R_2).......(p-R_n)$$

Any relation between x and y which makes any of these factors vanish is a solution of (1).
Let the primitives of $p-R_1=0, p-R_2=0$ etc. be

$$\phi_1(x,y,c_1)=0, \phi_2(x,y,c_2)=0..........\phi_n(x,y,c_n)=0$$

respectively, where $c_1, c_2, ...c_n$ are arbitrary constants. Without any loss of generality, we replace $c_1, c_2, ...c_n$ by c, where c is an arbitrary constant. Hence the solution of (1) is

$$\phi_1(x,y,c)\phi_2(x,y,c).........\phi_n(x,y,c)=0$$

Problems:

1) Solve $x^2p^2+3xyp+2y^2=0$

Solution: Solving for p,

$$x^2p^2+xyp+2xyp+2y^2=0 \Rightarrow xp(xp+y)+2y(xp+y)=0$$

$$\Rightarrow (xp+y)(xp+2y)=0 \Rightarrow (xp+y)=0 \;\&\; (xp+2y)=0$$

$$(xp+y)=0 \Rightarrow p=-\frac{y}{x} \;\&\; (xp+2y)=0 \Rightarrow p=-\frac{2y}{x}$$

$p=\frac{dy}{dx}=-\frac{y}{x}$ gives

$$\frac{dy}{y}=-\frac{dx}{x} \Rightarrow \log y=-\log x+\log c \Rightarrow \log y+\log x=\log c$$

$$\Rightarrow \log xy=\log c \Rightarrow xy=c..(1)$$

$$p=\frac{dy}{dx}=-\frac{2y}{x} \Rightarrow \frac{dy}{y}=-2\frac{dx}{x} \Rightarrow \log y=-2\log x+\log c \Rightarrow \log y=-\log x^2+\log c$$

$$\Rightarrow \log y+\log x^2=\log c \Rightarrow \log yx^2=\log c \Rightarrow yx^2=c........(2)$$

The solution is $(xy-c)(yx^2-c)=0$

2) Solve $p^2-5p+6=0$

Solution: Solving for p, $p^2-5p+6=0 \Rightarrow (p-2)(p-3)=0 \Rightarrow p=2, p=3$

If $p=2 \Rightarrow \frac{dy}{dx}=2 \Rightarrow dy=2dx \Rightarrow y=2x+c$

If $p=3 \Rightarrow \frac{dy}{dx}=3 \Rightarrow dy=3dx \Rightarrow y=3x+c$

$\therefore$ The solution is $(y-2x-c)(y-3x-c)=0$

3) Solve $p^2-3p+2=0$

Solution: Solving for p, we get $p^2-3p+2=0 \Rightarrow (p-1)(p-2)=0 \Rightarrow p=1, p=2$

If $p=2 \Rightarrow \frac{dy}{dx}=2 \Rightarrow dy=2dx \Rightarrow y=2x+c$

If $p=1 \Rightarrow \frac{dy}{dx}=1 \Rightarrow dy=dx \Rightarrow y=x+c$

$\therefore$ The solution is $(y-2x-c)(y-x-c)=0$

4) Solve $p^2-p-6=0$

Solution: Solving for p, we get $p^2-p-6=0 \Rightarrow (p-3)(p+2)=0 \Rightarrow p=3, p=-2$

If $p=3 \Rightarrow \frac{dy}{dx}=3 \Rightarrow dy=3dx \Rightarrow y=3x+c$

If $p=-2 \Rightarrow \frac{dy}{dx}=-2 \Rightarrow dy=-2dx \Rightarrow y=-2x+c$

$\therefore$ The solution is $(y-3x-c)(y+2x-c)=0$

5) Solve $xyp^2+p(3x^2-2y^2)-6xy=0$

Solution: Solving for p ,we get

$$p=\frac{-(3x^2-2y^2)\pm\sqrt{(3x^2-2y^2)^2-4xy(-6xy)}}{2xy}$$

$$=\frac{(2y^2-3x^2)\pm\sqrt{9x^4+4y^4-12x^2y^2+24x^2y^2}}{2xy}$$

$$=\frac{(2y^2-3x^2)\pm\sqrt{9x^4+4y^4+12x^2y^2}}{2xy}$$

$$=\frac{(2y^2-3x^2)\pm\sqrt{(3x^2+2y^2)^2}}{2xy}$$

$$=\frac{(2y^2-3x^2)\pm(3x^2+2y^2)}{2xy}$$

$$p=\frac{2y^2-3x^2+3x^2+2y^2}{2xy} \quad \text{or} \quad p=\frac{2y^2-3x^2-3x^2-2y^2}{2xy}$$

$\Rightarrow p = \frac{2y}{x}$ or $p = -\frac{3x}{y}$

$$\therefore p = \frac{2y}{x} \Rightarrow \frac{dy}{dx} = \frac{2y}{x} \Rightarrow \frac{dy}{y} = \frac{2dx}{x}$$

$$\Rightarrow \log y = 2\log x + \log c \Rightarrow \log y = \log x^2 + \log c$$

$$\Rightarrow \log y = \log x^2 c \Rightarrow y = cx^2 \Rightarrow (y - cx^2) = 0$$

$$\therefore p = -\frac{3x}{y} \Rightarrow \frac{dy}{dx} = -\frac{3x}{y} \Rightarrow ydy = -3xdx$$

$$\Rightarrow \frac{y^2}{2} = -3\frac{x^2}{2} + c \Rightarrow y^2 + 3x^2 = 2c$$

$$\Rightarrow y^2 + 3x^2 = 2c \Rightarrow (y^2 + 3x^2 - c) = 0$$

$\therefore$ The solution is $(y - cx^2)(y^2 + 3x^2 - c) = 0$

6) Solve $p^2 - (\cos x + \sec x)p + 1 = 0$

Solution: This is quadratic in p .Solving this for p we get,

$$p = \frac{(\cos x + \sec x) \pm \sqrt{(\cos x + \sec x)^2 - 4.1.1}}{2.1}$$

$$= \frac{(\cos x + \sec x) \pm \sqrt{\cos^2 x + \sec^2 x + 2\cos x \sec x - 4\cos x \sec x}}{2}$$

$$= \frac{(\cos x + \sec x) \pm \sqrt{\cos^2 x + \sec^2 x + 2 - 4}}{2}$$

$$= \frac{(\cos x + \sec x) \pm \sqrt{\cos^2 x + \sec^2 x - 2\cos x \sec x}}{2}$$

$$p = \frac{(\cos x + \sec x) \pm \sqrt{(\cos x - \sec x)^2}}{2} \Rightarrow p = \frac{(\cos x + \sec x) \pm (\cos x - \sec x)}{2}$$

$$\Rightarrow p = \cos \mathrm{x} \quad \text{or} \quad \mathrm{p} = \sec \mathrm{x}$$

If $p = \cos x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow dy = \cos x dx \Rightarrow y = \sin x + c \Rightarrow y - \sin x - c = 0$

$\therefore p = \sec x \Rightarrow \frac{dy}{dx} = \sec x \Rightarrow dy = \sec x dx \Rightarrow y = \log(\sec x + \tan x) + c$

$$\Rightarrow \mathrm{y - \log(secx + tanx) - c = 0}$$

Therefore the solution is $(y - \sin x - c)(y - \log(\sec x + \tan x) - c) = 0$

7) Solve $p^2 + 2yp\cot x = y^2$

Solution: This is quadratic in p .Solving this for p we get,

$$p=\frac{-2y\cot x\pm\sqrt{4y^2\cot^2 x-4.1(-y^2)}}{2.1}=\frac{-2y\cot x\pm\sqrt{4y^2(\cot^2 x+1)}}{2}$$

$$=\frac{-2y\cot x\pm 2y\operatorname{cosec}x}{2}=-y\cot x\pm y\operatorname{cosec}x$$

$$\therefore\ p=-y\cot x+y\operatorname{cosec}x \quad\text{or}\quad p=-y\cot x-y\operatorname{cosec}x$$

$$\therefore\ p=-y\cot x+y\operatorname{cosec}x \text{ then } p=\frac{dy}{dx}=y(-\cot x+\operatorname{cosec}x)$$

Which $\Rightarrow\frac{dy}{y}=-\cot x+\operatorname{cosec}x\Rightarrow\log y=-\log(\sin x)-\log(\operatorname{cosec}x+\cot x)+\log c$

$$\Rightarrow\log y+\log(\sin x)+\log(\operatorname{cosec}x+\cot x)=\log c\Rightarrow\log\left(y\sin x(\operatorname{cosec}x+\cot x)\right)=\log c$$

$$\Rightarrow y\sin x(\operatorname{cosec}x+\cot x)=c\Rightarrow y\sin x\left(\frac{1+\cos x}{\sin x}\right)=c\Rightarrow y(1+\cos x)=c$$

If $p=-y\cot x-y\operatorname{cosec}x$ then $\frac{dy}{dx}=-y\cot x-y\operatorname{cosec}x$

$$\frac{dy}{dx}=-\cot x-\operatorname{cosec}x\Rightarrow\log y=-\log(\sin x)+\log(\operatorname{cosec}x+\cot x)+\log c$$

$$\Rightarrow\log y+\log(\sin x)-\log(\operatorname{cosec}x+\cot x)=\log c\Rightarrow\log\left(\frac{y\sin x}{\operatorname{cosec}x+\cot x}\right)=\log c$$

$$\Rightarrow\frac{y\sin x}{\operatorname{cosec}x+\cot x}=c\Rightarrow\frac{y\sin^2 x}{1+\cos x}=c\Rightarrow y(1-\cos x)=c$$

Therefore the solution is $\{y(1+\cos x)-c\}\{y(1-\cos x)-c\}=0$

8) Solve $xyp^2+p(3x^2-2y^2)-6xy=0$

Solution: Consider $xyp^2+p(3x^2-2y^2)-6xy=0$,which is quadratic in p ,solving for p ,we get

$$p=\frac{-(3x^2-2y^2)\pm\sqrt{(3x^2-2y^2)^2+24x^2y^2}}{2xy}=\frac{-3x^2+2y^2\pm\sqrt{9x^4+4y^4-12x^2y^2+24x^2y^2}}{2xy}$$

$$=\frac{-3x^2+2y^2\pm\sqrt{(3x^2+2y^2)^2}}{2xy}=\frac{-3x^2+2y^2\pm(3x^2+2y^2)}{2xy}$$

$$\Rightarrow p=\frac{-3x^2+2y^2+3x^2+2y^2}{2xy}\quad\text{or}\quad p=\frac{-3x^2+2y^2-3x^2-2y^2}{2xy}$$

$$\Rightarrow p=\frac{2y}{x}\quad\text{or}\quad p=\frac{-3x}{y}$$

$$\therefore\ p=\frac{2y}{x}\Rightarrow\frac{dy}{dx}=\frac{2y}{x}\Rightarrow\frac{dy}{y}=\frac{2dx}{x}\Rightarrow\log y=2\log x+\log c$$

$$\Rightarrow\log y=\log cx^2\Rightarrow y=cx^2\Rightarrow y-cx^2=0$$

$$\therefore\ \mathrm{p}=\frac{\mathrm{-3x}}{\mathrm{y}}\Rightarrow\frac{dy}{dx}=\frac{-3x}{y}\Rightarrow ydy=-3xdx$$

$$\Rightarrow\frac{y^2}{2}=\frac{-3x^2}{2}+c\Rightarrow y^2+3x^2-c=0$$

Therefore the solution is $(y-cx^2)(y^2+3x^2-c)=0$

9) Solve $xyp^2+(y^2-x^2)p-xy=0$

Solution: The given equation is quadratic in p ,solving for p ,we get,

$$p=\frac{-(y^2-x^2)\pm\sqrt{\left(y^2-x^2\right)^2+4x^2y^2}}{2xy}=\frac{-y^2+x^2\pm\sqrt{y^4+x^4-2x^2y^2+4x^2y^2}}{2xy}$$

$$=\frac{-y^2+x^2\pm\sqrt{(x^2+y^2)^2}}{2xy}=\frac{-y^2+x^2\pm(x^2+y^2)}{2xy}$$

$$\Rightarrow p=\frac{-y^2+x^2+x^2+y^2}{2xy}\quad\text{or}\quad p=\frac{-y^2+x^2-x^2-y^2}{2xy}$$

$$\Rightarrow p=\frac{x}{y}\quad\text{or}\quad p=\frac{-y}{x}$$

If $p=\frac{x}{y}\Rightarrow\frac{dy}{dx}=\frac{x}{y}\Rightarrow ydy=xdx\Rightarrow y^2-x^2-2c=0$

If $p=\frac{-y}{x}\Rightarrow\frac{dy}{dx}=\frac{-y}{x}\Rightarrow\frac{dy}{y}=-\frac{dx}{x}\Rightarrow\log y=-\log x+\log c\Rightarrow xy-c=0$

The solution is $(y^2-x^2-2c)(xy-c)=0$

10) Solve $xy(p^2+1)=(x+y)p$

Solution: The equation can be written as $xyp^2-(x+y)p+xy=0$

Which is quadratic in p and hence

$$p=\frac{(x+y)\pm\sqrt{(x+y)^2-4x^2y^2}}{2xy}=\frac{(x+y)\pm\sqrt{(x-y)^2}}{2xy}=\frac{(x+y)\pm(x-y)}{2xy}$$

$$p=\frac{x+y+x-y}{2xy}=\frac{2x}{2xy}=\frac{1}{y}\quad\text{or}\quad p=\frac{x+y-x+y}{2xy}=\frac{2y}{2xy}=\frac{1}{x}$$

If $p=\frac{1}{y}\Rightarrow\frac{dy}{dx}=\frac{1}{y}\Rightarrow ydy=dx\Rightarrow y^2-2x-c=0$

If $p=\frac{1}{x}\Rightarrow\frac{dy}{dx}=\frac{1}{x}\Rightarrow dy=\frac{dx}{x}\Rightarrow y=\log x+\log c\Rightarrow e^{y}-cx=0$

The solution is $(y^2-2x-2c)(e^y-cx)=0$

TYPE:B Equations solvable for y or x

Let the differential equation

$$p^n+P_1p^{n-1}+P_2p^{n-2}+.....+P_n=0$$

can be put in the form $f(x,y,p)=0$ .When it cannot be resolved into rational factors as in above ,it may be either solved for y or x.

Equations solvable for y

The equation $f(x,y,p)=0$ can be put in the form

$$y=F(x,p) \qquad (1)$$

Differentiating with respect to x, we get, $p=\phi\left(x,p,\frac{dp}{dx}\right)$

This being an equation in two variables p and x, can be integrated by any of the method like variable separable Homogeneous ,linear equations etc.,

Let the solution be $\psi(x,p,c)=0$ (2)

Eliminating p between (1) and (2),the solution is got.

Problems

1) Solve $xp^2-2yp+x=0$

Solution: Consider the equation $xp^2-2yp+x=0$

Solving for y , we get $y=\frac{x(p^2+1)}{2p}$ (1)

Differentiating with respect to x

$$\frac{dy}{dx}=\frac{1}{2}\left[\frac{p\left[x.2p\frac{dp}{dx}+(p^2+1).1\right]-x(p^2+1)\frac{dp}{dx}}{p^2}\right]$$

$$p\,2p^2=2p^2x\frac{dp}{dx}+p(p^2+1)-xp^2\frac{dp}{dx}-x\frac{dp}{dx}$$

$$2p^3-p^3-p=(p^2x-x)\frac{dp}{dx}\Rightarrow p(p^2-1)=x(p^2-1)\frac{dp}{dx}$$

$$\Rightarrow p=x\frac{dp}{dx}\Rightarrow\frac{dp}{p}=\frac{dx}{x}\Rightarrow\log p=\log x+\log c$$

$$\Rightarrow\log p=\log xc\Rightarrow p=cx$$

Hence we have $p = cx$ (2)

Eliminating p from (1) and (2),we get

$$y = \frac{x(c^2x^2+1)}{2cx}$$

$\therefore\ 2cy = c^2x^2 + 1$ is the solution.

2) Solve $y = xp + x(1+p^2)^{1/2}$

Solution: Let $y = xp + x(1+p^2)^{1/2}$ (1)

Differentiating with respect to x,we get

$$\frac{dy}{dx} = x\left[\frac{dp}{dx} + \frac{1}{2\sqrt{1+p^2}}2p\frac{dp}{dx}\right] + \{p + \sqrt{p^2+1}\}.1$$

$$p - p - \sqrt{1+p^2} = \frac{dp}{dx}\left[\frac{xp}{\sqrt{1+p^2}} + x\right] \Rightarrow -\sqrt{1+p^2} = \frac{x[p+\sqrt{1+p^2}]}{\sqrt{1+p^2}}\frac{dp}{dx}$$

$$\frac{dx}{x} = -\left[\frac{p+\sqrt{1+p^2}}{1+p^2}\right]dp \Rightarrow \frac{dx}{x} = -\frac{1}{2}\left[\frac{2pdp}{1+p^2}\right] - \frac{dp}{\sqrt{1+p^2}}$$

Integrating on both sides

$$\int\frac{dx}{x} = -\frac{1}{2}\int\frac{2p}{1+p^2}dp - \int\frac{dp}{\sqrt{1+p^2}} \Rightarrow \log x = -\frac{1}{2}\log(1+p^2) - \log[p+\sqrt{1+p^2}]$$

$$\log x + \log[p+\sqrt{1+p^2}] + \log\sqrt{1+p^2} = \log c$$

$$\log\left[\left(x(p+\sqrt{1+p^2}\right)\sqrt{1+p^2}\right] = \log c \Rightarrow x\left[p\sqrt{1+p^2} + 1 + p^2\right] = c \quad (2)$$

Eliminating p between (1) and (2) we get the solution.

3) Solve $y^2 = 1 + p^2$

Solution: Consider $y^2 = 1 + p^2 \Rightarrow y = \sqrt{1+p^2}$ (1)

Differentiating with respect to x,we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+p^2}}2p\frac{dp}{dx} \Rightarrow p = \frac{p}{\sqrt{1+p^2}}\frac{dp}{dx}$$

$$\therefore\ dx = \frac{dp}{\sqrt{1+p^2}}$$

Integrating on both sides

$$\int \frac{dp}{\sqrt{1+p^2}} = \int dx \Rightarrow \sinh^{-1} p = x + c$$

$$\Rightarrow p = \sinh(x+c) \qquad (2)$$

From (1), $y = \pm\sqrt{1+p^2} = \pm\sqrt{1+\sinh^2(x+c)} = \pm\sqrt{\cosh^2(x+c)}$

$$\therefore \text{ y} = \pm\cosh(x+c) \Rightarrow \cosh^{-1} y = \pm(x+c)$$

$$\log(y+\sqrt{y^2-1}) = \pm(x+c) \Rightarrow x+c = \pm\log(y+\sqrt{y^2-1})$$

Therefore, the solution is

$$\left[x+c+\log\left(y+\sqrt{y^2-1}\right)\right]\left[x+c-\log\left(y+\sqrt{y^2-1}\right)\right] = 0$$

4) Solve $y = 2p + 3p^2$

Solution: Consider $y = 2p + 3p^2$ (1)

Differentiating with respect to x we get

$$\frac{dy}{dx} = 2\frac{dp}{dx} + 6p\frac{dp}{dx} \Rightarrow p = (2+6p)\frac{dp}{dx}$$

$$dx = \frac{2+6p}{p}dp \Rightarrow dx = \left[\frac{2}{p}+6\right]dp$$

Integrating on both sides we get $\int dx = \int \frac{2}{p}dp + 6\int dp \Rightarrow x = 2\log p + 6p + c$

$$x = 2\log p + 6p + c \qquad (2)$$

Eliminating p between (1) & (2) we get the solution.

Equations solvable for x

The equation $f(x,y,p) = 0$ can be put in the form

$$x = F(y,p) \qquad (1)$$

Differentiating with respect to y, we get, $\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right)$

This being an equation in two variables p and y, can be integrated by any of the method like variable separable Homogeneous ,linear equations etc.,

Let the solution be $\psi(y,p,c) = 0$ (2)

Eliminating p between (1) and (2),the solution of (1) is got.

1) Solve $x^2 = (1+p^2)$

Solution: Consider $x^2 = (1+p^2) \Rightarrow x = \pm\sqrt{1+p^2}$ (1)

Differentiating with respect to y,

$$\frac{dx}{dy}=\frac{1}{2\sqrt{1+p^2}}2p\frac{dp}{dy}\Rightarrow dy=\frac{p^2}{\sqrt{1+p^2}}dp$$

Integrating on both sides $\int dy=\int\frac{p^2}{\sqrt{1+p^2}}dp\Rightarrow y=\int\frac{p^2+1-1}{\sqrt{1+p^2}}dp$

$$y=\int\sqrt{1+p^2}dp-\int\frac{dp}{\sqrt{1+p^2}}$$

$$y=\frac{p\sqrt{1+p^2}}{2}+\frac{1}{2}\sinh^{-1}p-\sinh^{-1}p+c$$

$$\therefore\ y=\frac{p\sqrt{1+p^2}}{2}-\frac{1}{2}\sinh^{-1}p+c \qquad (2)$$

Eliminating p between (1) & (2) we get the solution.

2) Solve $x(1+p^2)=1$

Solution: Consider $x(1+p^2)=1\Rightarrow x=\dfrac{1}{1+p^2}$ (1)

Differentiating with respect to y

$$dy=\frac{-2p^2}{(1+p^2)^2}dp$$

Integrating both sides

$$\int dy=-\int\frac{2p^2dp}{(1+p^2)^2}$$

Put

$$p=\tan\theta\Rightarrow dp=\sec^2\theta d\theta$$

$$\frac{2p^2}{(1+p^2)^2}=\frac{2\tan^2\theta}{(1+\tan^2\theta)^2}=\frac{2\tan^2\theta}{\sec^4\theta}$$

$$\int dy=-\int\frac{2p^2dp}{(1+p^2)^2}=-\int\frac{2\tan^2\theta\sec^2\theta d\theta}{\sec^4\theta}$$

$$=-\int\frac{2\tan^2\theta d\theta}{\sec^2\theta}=-\int\frac{2\sin^2\theta}{\cos^2\theta}.\frac{\cos^2\theta}{1}d\theta$$

$$=-\int 2\sin^2\theta d\theta=-\int(1-\cos2\theta)d\theta=-(\theta-\frac{\sin2\theta}{2})=-\tan^{-1}p+\frac{2\tan\theta}{2(1+\tan^2\theta)}$$

$$y=\tan^{-1}p+\frac{p}{1+p^2}+c$$

(2)

Eliminating p between (1) & (2), we get the solution

Clairaut's form: The equation known as Clairaut's is of the form

$$y = px + f(p) \qquad (1)$$

Differentiating with respect to x

$$p = p + \{x + f(p)\}\frac{dp}{dx} \Rightarrow \{x + f(p)\}\frac{dp}{dx} = 0$$

Either $\frac{dp}{dx} = 0$ or $x + f(p) = 0$

$\frac{dp}{dx} = 0 \Rightarrow p = c$,a constant.

Therefore the solution of (1) is $y = cx + f(c)$

We have to replace p in Clairaut's equation by c.The other factor $\{x + f(p)\} = 0$ taken along with (1) give, on eliminating of p,a solution of (1).But this solution is not included in the general solution and this is called a singular solution.

Problems:

1) Solve $y = (x-a)p - p^2$

Solution: This is in Clairaut's form hence the solution is

$$y = (x-a)c - c^2$$

2) Solve $y = px + \frac{ap}{(1+p^2)^{1/2}}$

Solution: This is in Clairaut's form hence the solution is

$$y = cx + \frac{ac}{(1+c^2)^{1/2}}$$

3) Solve $(y - px)(p-1) = p$

Solution: Consider $(y - px)(p-1) = p \Rightarrow y - px = \frac{p}{p-1} \Rightarrow y = px + \frac{p}{p-1}$

Which is in Clairaut's form, hence replace p by c we get the solution as

$$y = cx + \frac{c}{c-1}$$

4) Solve $yp = xp^2 + 2$

Solution : Consider $yp = xp^2 + 2 \Rightarrow y = xp + \frac{2}{p}$

Which is in Clairaut's form, hence the solution is $y = cx + \frac{2}{c}$

5) Solve $p = \sin(y - px)$

Solution: $p = \sin(y - px) \Rightarrow y - px = \sin^{-1} p \Rightarrow y = px + \sin^{-1} p$

Which is in Clairaut's form ,hence the solution is $y = cx + \sin^{-1} c$

6) Solve $y = (x-1)p + \tan^{-1} p$

Solution: Consider $y = (x-1)p + \tan^{-1} p \Rightarrow y = xp - p + \tan^{-1} p$

Which is in Clairaut's form ,hence the solution is $y = cx - c + \tan^{-1} c$

7) Solve $p = \log(px - y)$

Solution: Consider $p = \log(px - y) \Rightarrow px - y = e^p \Rightarrow y = px - e^p$

Which is in Clairaut's form, hence replace p by c we get the solution as

$$y = pc - e^c$$

8) Solve $y = px + \frac{a}{p}$

Solution: Consider $y = px + \frac{a}{p}$,

which is in Clairaut's form, hence the solution is $y = cx + \frac{a}{c}$

9) Solve $y = 2px + y^2 p^2$

Solution: Putting $X = 2x$ and $Y = y^2$ $dX = 2dx \,\&\, dY = 2ydy$

$$\therefore \ \mathrm{P} = \frac{\mathrm{dY}}{\mathrm{dX}} = yp$$

The equation transformed into $Y = XP + P^2$,which is in Clairaut's form .

Hence the solution is

$$Y = XC + C^2 \Rightarrow y^2 = 2xc + c^2$$

10) Solve $x^2(y - px) = yp^2$

Solution: Put $X = x^2 \,\&\, Y = y^2$ then $P = \frac{dY}{dX} = \frac{2ydy}{2xdx} = \frac{y}{x}p \Rightarrow p = \frac{x}{y}P = \frac{\sqrt{X}}{\sqrt{Y}}P$

$$\therefore \ x^2(y - px) = yp^2 \Rightarrow X(\sqrt{Y} - \frac{\sqrt{X}}{\sqrt{Y}}\sqrt{X}P) = \sqrt{Y}\,\frac{X}{Y}P^2$$

$$\Rightarrow X(Y - XP) = XP^2 \Rightarrow Y - XP = P^2 \Rightarrow Y = XP + P^2$$

which is in Clairaut's form ,the solution is

$$Y = CX + C^2 \Rightarrow y^2 = Cx^2 + C^2 \Rightarrow y^2 = Cx^2 + C^2 \ .$$

11) Solve $(px - y)(py + x) = 2p$

Solution: Put $X = x^2 \,\&\, Y = y^2$

Then $P = \frac{dY}{dX} = \frac{2ydy}{2xdx} = \frac{y}{x}p \Rightarrow p = \frac{x}{y}P = \frac{\sqrt{X}}{\sqrt{Y}}P$

$$\therefore \ (px - y)(py + x) = 2p \Rightarrow \left(\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{X} - \sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}\sqrt{Y} + \sqrt{X}\right) = 2\frac{\sqrt{X}}{\sqrt{Y}}P$$

$$\Rightarrow \left(\frac{XP - Y}{\sqrt{Y}}\right)\left(\frac{\sqrt{X}\,[P+1]}{1}\right) = 2\frac{\sqrt{X}}{\sqrt{Y}}P \Rightarrow (XP - Y)(P+!) = 2P \Rightarrow Y = XP - \frac{2P}{P+1}$$

Which is in Clairaut's form,the solution is

$$Y = XC - \frac{2C}{C+1} \Rightarrow y^2 = x^2C - \frac{2C}{C+1}$$

12) Solve $x^2(y - px) = yp^2$

Solution: Put $X = x^2$ & $Y = y^2$

Then $P = \frac{dY}{dX} = \frac{2ydy}{2xdx} = \frac{y}{x}p \Rightarrow p = \frac{x}{y}P = \frac{\sqrt{X}}{\sqrt{Y}}P$

$$\therefore \; x^2(y - px) = yp^3 \Rightarrow X\left(\sqrt{Y} - \frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{X}\right) = \sqrt{Y}\,\frac{X}{Y}P^2 \Rightarrow Y = XP + P^2$$

Which is in Clairaut's form ,the solution is

$$Y = XC + C^2 \Rightarrow y^2 = x^2C + C^2$$

UNIT.II

LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENT

Consider the linear differential equation with variable coefficient

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0 \qquad (1)$$

putting $\log x = z$ then the equation (1) reduces to

$$a\frac{d^2y}{dz^2} + b\frac{dy}{dz} + cy = 0 \qquad (2)$$

Which is a linear equation with constant coefficient which can be solved by the above methods.

Problems:

1) Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

Solution: Putting $\log x = z$ then the given equation reduces to

$$[D(D-1) - D + 1]y = 0 \text{ where } D = \frac{d}{dz}$$

The auxiliary equation is $m^2-2m+1=0 \Rightarrow (m-1)^2=0 \Rightarrow m=1,1$

Hence the solution is $y=(Az+B)e^z$

Since $\log x = z, x=e^z$, the solution will become

$$y=[A\log x+B]x$$

2) Solve $x^2\dfrac{d^2y}{dx^2}-7x\dfrac{dy}{dx}+12y=0$

Solution: Putting $\log x = z$ then the given equation reduces to

$$[D(D-1)-7D+12]y=0 \text{ where } D=\frac{d}{dz}$$

The auxiliary equation is $m^2-8m+12=0 \Rightarrow (m-6)(m-2)=0 \Rightarrow m=6,2$

Hence the solution is $y=Ae^{6z}+Be^{2z} \Rightarrow y=A(e^z)^6+B(e^z)^2$

Since $\log x = z, x=e^z$, the solution will become

$$y=Ax^6+Bx^2$$

3) Solve $x^2\dfrac{d^2y}{dx^2}+8x\dfrac{dy}{dx}+12y=x^4$

Solution: Putting $\log x = z, x=e^z$ then the given equation reduces to

$$[D(D-1)+8D+12]y=e^{4z} \text{ where } D=\frac{d}{dz}$$

$$(D^2+7D+12)y=e^{4z}$$

The auxiliary equation is $m^2+7m+12=0 \Rightarrow (m+3)(m+4)=0 \Rightarrow m=-3,-4$

the complementary function is $Ae^{-3z}+Be^{-4z} \Rightarrow A(e^z)^{-3}+B(e^z)^{-4} \Rightarrow Ax^{-3}+Bx^{-4}$

$$\text{P.I} = \frac{1}{D^2+7D+12}e^{4z} = \frac{1}{4^2+7(4)+12}e^{4z} = \frac{1}{56}x^4$$

Therefore the general solution is y = C.F+P.I

$$y=Ax^{-3}+Bx^{-4}+\frac{1}{56}x^4$$

4) Solve $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-3y=x^2$

Solution: Putting $\log x = z, x=e^z$ then the given equation reduces to

$$[D(D-1)+D-3]y=e^{2z} \text{ where } D=\frac{d}{dz}$$

$$(D^2-3)y=e^{2z}$$

The auxiliary equation is $m^2-3=0 \Rightarrow m=\pm\sqrt{3}$

the complementary function is $Ae^{+\sqrt{3}z}+Be^{-\sqrt{3}z} \Rightarrow A(e^z)^{\sqrt{3}}+B(e^z)^{-\sqrt{3}}$

$$\Rightarrow Ax^{\sqrt{3}}+Bx^{-\sqrt{3}}$$

$$\text{P.I} = \frac{1}{D^2-3}e^{2z} = \frac{1}{2^2-3}e^{2z} = e^{2z} = (e^z)^2 = x^2$$

Therefore the general solution is y = C.F+P.I

$$y = Ax^{\sqrt{3}}+Bx^{-\sqrt{3}}+x^2$$

5) Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+2y=x^2$

Solution: Putting $\log x = z, x=e^z$ then the given equation reduces to

$$[D(D-1)+D+2]y=e^{2z} \text{ where } D=\frac{d}{dz}$$

$$(D^2+2)y=e^{2z}$$

The auxiliary equation is $m^2+2=0 \Rightarrow m=\pm i\sqrt{2}$

the complementary function is $A\cos\sqrt{2}z+B\sin\sqrt{2}z = A\cos\sqrt{2}(\log x)+B\sin\sqrt{2}(\log x)$

$$\text{P.I} = \frac{1}{D^2+2}e^{2z} = \frac{1}{2^2+2}e^{2z} = \frac{1}{6}e^{2z} = \frac{1}{6}(e^z)^2 = \frac{1}{6}x^2$$

Therefore the general solution is y = C.F+P.I

$$y = A\cos\sqrt{2}(\log x)+B\sin\sqrt{2}(\log x)+\frac{1}{6}x^2$$

6) Solve $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=x^2$

Solution: Putting $\log x = z, x=e^z$ then the given equation reduces to

$$[D(D-1)-3D+4]y=e^{2z} \text{ where } D=\frac{d}{dz}$$

$$(D^2-4D+4)y=e^{2z}$$

The auxiliary equation is $m^2-4m+4=0 \Rightarrow (m-2)^2=0 \Rightarrow m=2,2$

the complementary function is $(A+Bz)e^{2z} = (A+B\log x)(e^z)^2 = (A+B\log x)x^2$

$$\text{P.I} = \frac{1}{D^2-4D+4}e^{2z} = \frac{1}{(D-2)^2}e^{2z} = \frac{z^2}{2}e^{2z} = \frac{\log x)^2}{2}(e^z)^2 = \frac{(\log x)^2}{2}x^2$$

Therefore the general solution is y = C.F+P.I

$$y=(A+B\log x)x^2+\frac{(\log x)^2}{2}x^2$$

7) Solve $(x^2D^2+4xD+2)y=x\log x$

Solution: Putting $\log x=z, x=e^z$ then the given equation reduces to

$$[D(D-1)+4D+2]y=ze^z \text{ where } D=\frac{d}{dz}$$

$$(D^2+3D+2)y=ze^z$$

The auxiliary equation is $m^2+3m+2=0\Rightarrow(m+1)(m+2)=0\Rightarrow m=-1-2$

the complementary function is $Ae^{-z}+Be^{-2z}=Ax^{-1}+Bx^{-2}=\frac{A}{x}+\frac{B}{x^2}$

P.I

$$=\frac{1}{D^2+3D+2}ze^z=e^z\frac{1}{(D+1)^2+3(D+1)+2}z=e^z\frac{1}{D^2+5D+6}z=e^z\frac{1}{5D+6}z=e^z\frac{1}{6(1+5D/6)}z$$

$$=e^z\frac{1}{6}\left(1+\frac{5D}{6}\right)^{-1}z=e^z\frac{1}{6}\left(1-\frac{5D}{6}\right)z=\frac{e^z}{6}\left(z-\frac{5}{6}\right)=\frac{x}{6}\left(\log x-\frac{5}{6}\right)$$

Therefore the general solution is $y=$ C.F+P.I

$$y=\frac{A}{x}+\frac{B}{x^2}+\frac{x}{6}\left(\log x-\frac{5}{6}\right)$$

8) Solve $(D^2+\frac{1}{x}D)y=\frac{12\log x}{x^2}$

Solution: Multiplying the equation by x^2,we get

$$(x^2D^2+xD)y=12\log x$$

Putting $\log x=z, x=e^z$ then the equation reduces to

$$\{D(D-1)+D\}y=12z \text{ where } D=\frac{d}{dz}$$

$$D^2y=12z\Rightarrow\frac{d^2y}{dz^2}=12z\Rightarrow\frac{dy}{dz}=12\frac{z^2}{2}+A\Rightarrow y=6\frac{z^3}{3}+Az+B\Rightarrow y=2z^3+Az+B$$

$y=2(\log x)^3+A\log x+B$ is the solution.

9) Solve $(x^3D^3+3x^2D^2+xD+1)y=\sin(\log x)$

Solution: Putting $\log x=z, x=e^z$

$$\{D(D-1)(D-2)+3D(D-1)+D+1\}y=\sin z\Rightarrow(D^3+1)y=\sin z$$

Auxiliary equation is $m^3+1=0\Rightarrow(m+1)(m^2-m+1)=0\Rightarrow m=1,\frac{1\pm i\sqrt{3}}{2}$

Complementary function is

$$Ae^{-z}+e^{z/2}\{B\cos\left(\frac{\sqrt{3}}{2}\right)z+C\sin\left(\frac{\sqrt{3}}{2}\right)z\}=Ax^{-1}+x^{1/2}\{B\cos\left(\frac{\sqrt{3}}{2}\right)\log x+C\sin\left(\frac{\sqrt{3}}{2}\right)\log x\}$$

$$\text{P.I}=\frac{1}{D^3+1}(\sin z)=\frac{1}{D(D^2)+1}(\sin z)=\frac{1}{D(-1)+1}(\sin z)=\frac{1+D}{1-D^2}(\sin z)=\frac{1+D}{2}(\sin z)$$

$$=\frac{\sin z+\cos z}{2}=\frac{\sin(\log x)+\cos(\log x)}{2}$$

The solution is $y=C.F+P.I$

$$y=Ax^{-1}+x^{1/2}\{B\cos\left(\frac{\sqrt{3}}{2}\log x\right)+C\sin\left(\frac{\sqrt{3}}{2}\log x\right)\}+\frac{\sin(\log x)+\cos(\log x)}{2}$$

10) Solve $x^2y''+3xy'+y=\dfrac{1}{(1-x)^2}$

Solution: Putting $\log x=z, x=e^z$ then the equation becomes

$$\{(DD-1)+3D+1\}y=\frac{1}{(1-e^z)^2} \quad \text{where} \quad D=\frac{d}{dz}$$

Auxiliary equation is $m^2+2m+1=0\Rightarrow(m+1)^2=0\Rightarrow m=-1,-1$

Complementary function is $(A+Bz)e^{-z}=(A+B\log x)x^{-1}$

$$\text{P.I}=\frac{1}{D^2+2D+1}\left(\frac{1}{(1-e^z)^2}\right)=\frac{1}{(D+1)^2}\left(\frac{e^{-z}e^z}{(1-e^z)^2}\right)=e^{-z}\left(\frac{1}{(D-1+1)^2}\right)\left(\frac{e^z}{(1-e^z)^2}\right)$$

$$=e^{-z}\left(\frac{1}{D^2}\right)\left(\frac{e^z}{(1-e^z)^2}\right)=e^{-z}\left(\frac{1}{D}\right)\int\left(\frac{e^z}{(1-e^z)^2}\right)dz=e^{-z}\left(\frac{1}{D}\right)\int t^{-2}(-dt)=e^{-z}\left(\frac{1}{D}\right)\frac{1}{t}$$

where $1-e^z=t\Rightarrow -e^zdz=dt$

P.I

$$=e^{-z}\left(\frac{1}{D}\right)\left(\frac{1}{1-e^z}\right)=e^{-z}\int\frac{dz}{1-e^z}=e^{-z}\int\frac{1-e^z+e^z}{1-e^z}dz=e^{-z}\int dz+e^{-z}\int\frac{e^zdz}{1-e^z}$$

$$=e^{-z}z-e^{-z}\log(1-e^z)=\left(\frac{1}{x}\right)(\log x-\log 1-x))=\left(\frac{1}{x}\right)\left(\log\left(\frac{x}{1-x}\right)\right)$$

Hence the general solution is $y=C.F+P.I$

$$y=(A+B\log x)x^{-1}+\left(\frac{1}{x}\right)\left(\log\left(\frac{x}{1-x}\right)\right)=\left(\frac{1}{x}\right)\left[(A+B\cos x)+\left(\log\left(\frac{x}{1-x}\right)\right)\right]$$

METHOD OF VARIATION OF PARAMETER

Consider the linear differential equation

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \qquad (1)$$

Where P,Q & R are constants.

Consider the equation

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+Qy=0 \qquad (2)$$

Suppose $u(x)$ and $v(x)$ are two linear independent solutions of (2), then the complementary function of (1) is

$$y=c_1u+c_2v$$

Where c_1, c_2 are arbitrary constants.

Then the general solution of (1) is $y=C.F+P.I$ where $C.F=c_1u+c_2v$ where c_1,c_2 are constants and $P.I=uf(x)=vg(x)$ where $f(x)=-\int\frac{vR}{w}dx$ and $g(x)=\int\frac{uR}{w}dx$ where w is wronskian of u and v .

$$w=\begin{vmatrix}u & v\\ u_1 & v_1\end{vmatrix}=uv_1-u_1v\ .$$

Problems

1.Solve $y_2+n^2y=\sec nx$ using the method of variation of parameter.

Solution: Given equation is $y_2+n^2y=\sec nx$

The auxiliary equation is $m^2+n^2=0\Rightarrow \mathrm{m}^2=-n^2\Rightarrow \mathrm{m}=\pm \mathrm{ni}$

The roots are imaginary. Hence the complementary function is $c_1\cos nx+c_2\sin nx$

Let $u=\cos nx$, $\mathrm{v}=\sin nx$.

To find w

$$w=\begin{vmatrix}u & v\\ u_1 & v_1\end{vmatrix}=\begin{vmatrix}\cos nx & \sin nx\\ -n\sin nx & n\cos nx\end{vmatrix}=n\cos^2 nx+n\sin^2 nx=n\neq 0$$

To find $f(x)$ and g(x)

$$f(x) = -\int \frac{vR}{w} dx = -\int \frac{\sin nx \sec nx}{n} dx = -\frac{1}{n}\int \sin nx \sec nx dx = -\frac{1}{n}\int \tan nx dx = -\frac{1}{n^2}\log(\sec nx)$$

$$f(x) = \frac{1}{n^2}\log(\cos nx)$$

And $g(x) = \int \frac{uR}{w} dx = \int \frac{\cos nx \sec nx}{n} dx = \frac{1}{n}\int dx = \frac{x}{n}$

Hence $g(x) = \frac{x}{n}$

Particular integral

$$P.I = uf(x) + vg(x) = \cos nx\left(\frac{\log(\cos nx)}{n}\right) + \sin nx.\left(\frac{x}{n}\right)$$

The general solution is $y = C.F + P.I$

$$y = [c_1 \cos nx + c_2 \sin nx] + \cos nx\left(\frac{\log(\cos nx)}{n^2}\right) + \sin nx.\left(\frac{x}{n}\right)$$

$$y(x) = [c_1 \cos nx + c_2 \sin nx] + \cos nx\left(\frac{\log(\cos nx)}{n^2}\right) + \left(\frac{x}{n}\right)\sin nx$$ is the general solution.

2. .Solve $y_2 + 4y = 4\tan 2x$ using the method of variation of parameter.

Solution: Given equation is $y_2 + 4y = 4\tan 2x$

The auxillary equation is $m^2 + 4 = 0 \Rightarrow \mathrm{m}^2 = -4 \Rightarrow \mathrm{m} = \pm 2\mathrm{i}$

The roots are imaginary. Hence the complementary function is $c_1 \cos 2x + c_2 \sin 2x$

Let $u = \cos 2x$, $\mathrm{v} = \sin 2\mathrm{x}$.

To find w

$$w = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2 \neq 0$$

To find $f(x)$ and g(x)

$$f(x) = -\int \frac{vR}{w} dx = -\int \frac{\sin 2x 4\tan 2x}{2} dx = -2\int \sin x \tan 2x dx = -2\int \frac{\sin^2 2x}{\cos 2x} dx = -2\frac{\log(\sec 2x + \tan 2x)}{2} + \frac{2}{2}\sin 2x$$
$$f(x) = -2\log(\sec 2x + \tan 2x) + \sin 2x$$

And $g(x) = \int \frac{uR}{w} dx = \int \frac{\cos 2x 4\tan 2x}{2} dx = \frac{4}{2}\int \cos 2x \frac{\sin 2x}{\cos 2x} dx = 2\int \sin 2x dx = -2\frac{\cos 2x}{2} = \cos 2x$

Hence $g(x) = \cos 2x$

Particular integral

$$P.I = uf(x) + vg(x) = \cos 2x[\sin 2x - \log(\sec 2x + \tan 2x)] + \sin 2x(-\cos 2x)$$
$$= -\log(\sec x + \tan 2x)\cos 2x$$

The general solution is

$$y = C.F + P.I$$

$y = [c_1 \cos 2x + c_2 \sin 2x] - \log(\sec x + \tan 2x)\cos 2x$ is the general solution.

3. Solve $(D^2 - 2D)y = e^x \sin x$

Solution: Given equation is $(D^2 - 2D)y = e^x \sin x$

Auxillary equation is $m^2 - 2m = 0 \Rightarrow m(m-2) = 0 \Rightarrow m = 0 \;\&\; m = 2$

The roots are real and unequal. Hence the complementary function is $c_1 e^{0x} + c_2 e^{2x} = c_1 + c_2 e^{2x}$

Let $u = 1$, $v = e^{2x}$ and $R = e^x \sin x$

To find w

$$w = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x} \neq 0$$

To find $f(x) \;\&\; g(x)$

$$f(x) = -\int \frac{vR}{w} dx = -\int \frac{e^{2x} e^x \sin x}{2e^{2x}} dx = -\frac{1}{2}\int e^x \sin x dx = -\frac{1}{2}\frac{e^x(\sin x - \cos x)}{2} = -\frac{e^x(\sin x - \cos x)}{4}$$

And

$$g(x)=\int\frac{uR}{w}dx=-\int\frac{e^x\sin x}{2e^{2x}}dx=-\frac{1}{2}\int e^{-x}\sin x dx=-\frac{1}{2}\frac{e^{-x}(-\sin x-\cos x)}{2}=-\frac{e^{-x}(\sin x+\cos x)}{4}$$

P.I= $uf(x)+vg(x)$

$$=-\frac{e^x(\sin x-\cos x)}{4}+e^{2x}\frac{[-e^{-x}(\sin x+\cos x)]}{4}$$

$$=-\frac{e^x}{4}\{\sin x-\cos x+\sin x+\cos x\}=-\frac{2\sin x e^x}{4}=-\frac{e^x\sin x}{2}$$

$$\text{P.I}=-\frac{e^x\sin x}{2}$$

The general solution is $y=C.F+P.I$

$$y=c_1+c_2e^{2x}-\frac{e^x\sin x}{2}$$

UNIT.III

PARTIAL DIFFERENTIAL EQUATION

Definition: A Partial differential Equation is an equation which contains one or more partial derivatives. The order of the partial differential equation is that of the derivative of highest order in the equation. Partial differential equations may be formed by

1) Eliminating arbitrary constants
2) Eliminating arbitrary functions

Eliminating arbitrary constants

Problems:

1) Form the partial equation by eliminating the arbitrary constants from $z=(x^2+a)(y^2+b)$

Solution: Given that $z=(x^2+a)(y^2+b)$

Differentiating partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = p = 2x(y^2 + b) \qquad (1)$$

$$\frac{\partial z}{\partial y} = q = 2y(x^2 + a) \qquad (2)$$

Multiplying (1) and (2) we get

$$pq = 4xy(x^2 + a)(y^2 + b) = 4xyz$$

2) Form the partial equation by eliminating the arbitrary constants from $z = (x + a)(y + b)$

Solution: Given that $z = (x^2 + a)(y^2 + b)$

Differentiating partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = p = (y + b) \qquad (1)$$

$$\frac{\partial z}{\partial y} = q = (x + a) \qquad (2)$$

Multiplying (1) and (2) we get

$$pq = (x + a)(y + b) = z$$

3) Eliminate a and b from $z = ax + by + a$

Solution: Given that $z = ax + by + a$ (1)

Differentiating partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = p = a \qquad (2)$$

$$\frac{\partial z}{\partial y} = q = b \qquad (3)$$

substituting (2) and (3) in (1) we have $z = px + qy + p$

4) Eliminate a and b from $\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1$

Solution: Given that $\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1$ (1)

Differentiating partially with respect to x and y, we get

$$\frac{2x}{a^2}+\frac{2z}{b^2}\frac{\partial z}{\partial x}=0 \Rightarrow \frac{2x}{a^2}+\frac{2z}{b^2}p=0 \Rightarrow \frac{2x}{a^2}=-\frac{2zp}{b^2} \Rightarrow \frac{2xy}{a^2}=-\frac{2yzp}{b^2} \quad (2)$$

$$\frac{2y}{a^2}+\frac{2z}{b^2}\frac{\partial z}{\partial y}=0 \Rightarrow \frac{2y}{a^2}+\frac{2z}{b^2}q=0 \Rightarrow \frac{2y}{a^2}=-\frac{2zq}{b^2} \Rightarrow \frac{2xy}{a^2}=-\frac{2xzq}{b^2} \quad (3)$$

Equating (2) and (3) we get $\frac{2xy}{a^2}=-\frac{2yzp}{b^2}=-\frac{2xzq}{b^2} \Rightarrow py=qx$

Eliminating arbitrary functions:

1) Eliminate the arbitrary function from $z=f(x^2+y^2)$

Solution: Given that $z=f(x^2+y^2)$ (1)

Differentiating partially with respect to x and y, we get

$$\frac{\partial z}{\partial x}=f^{'}(x^2+y^2).2x \Rightarrow p=f^{'}(x^2+y^2)2x \Rightarrow f^{'}(x^2+y^2)=\frac{p}{2x} \quad (2)$$

$$\frac{\partial z}{\partial y}=f^{'}(x^2+y^2).2y \Rightarrow q=f^{'}(x^2+y^2)2y \Rightarrow f^{'}(x^2+y^2)=\frac{q}{2y} \quad (3)$$

Eliminating $f^{'}(x^2+y^2)$ between (2) and (3) we get $py=qx$

2) Eliminate the arbitrary function from $z=e^y f(x+y)$

Solution: Given that $z=e^y f(x+y)$ (1)

Differentiating partially with respect to x and y, we get

$$\frac{\partial z}{\partial x}=e^y f^{'}(x+y) \Rightarrow p=e^y f^{'}(x+y) \quad (2)$$

$$\frac{\partial z}{\partial y}=e^y f^{'}(x+y)+f(x+y)e^y \Rightarrow q=e^y f^{'}(x+y)+f(x+y)e^y \quad (3)$$

from (1), (2) and (3) we have $q=p+z$

3) Eliminate the arbitrary function from $z=(x+y)f(x^2-y^2)$

Solution: Given that $z=(x+y)f(x^2-y^2)$ (1)

Differentiating partially with respect to x and y, we get

$$\frac{\partial z}{\partial x}=(x+y)f^{'}(x^2-y^2).2x+f(x^2-y^2) \quad (2)$$

$$\frac{\partial z}{\partial y}=(x+y)f^{'}(x^2-y^2).(-2y)+f(x^2-y^2) \quad (3)$$

Multiply (2) by y and (3) by x and adding we get

$$yp+xq=(x+y)f(x^2-y^2)=z$$

The solution is $z=py+qx$

4) Eliminate the arbitrary function f from $f(x^2+y^2+z^2,z^2-2xy)=0$

Solution: Solving $x^2+y^2+z^2=F(z^2-2xy)$

Differentiating partially with respect to x and y

$$2x+2zp=F^{'}(z^2-2xy)(2zp-2y) \quad (1)$$

$$2y+2zq=F^{'}(z^2-2xy)(2zq-2x) \quad (2)$$

Dividing (1) by (2) to eliminate $F^{'}$

$$\frac{x+zp}{y+zq}=\frac{zp-y}{zq-x} \quad \text{or} \quad z(p-q)=y-x$$

5) Eliminate the arbitrary function f from $f(x^2+y^2,z^2-xy)=0$

Solution: Solving $x^2+y^2=F(z^2-xy)$

Differentiating partially with respect to x and y

$$2x=F^{'}(z^2-xy)(2zp-y) \quad (1)$$

$$2y=F^{'}(z^2-xy)(2zq-x) \quad (2)$$

Dividing (1) by (2) to eliminate $F^{'}$

$$\frac{2x}{2zp-y}=\frac{2y}{2zq-x}\Rightarrow 4xzq-2x^2=4yzp-2y^2\Rightarrow py-qx=y^2-x^2$$

6) Eliminate the functions f and ϕ from the relation $z=f(x+ay)+\phi(x-ay)$

Solution: Given that $z=f(x+ay)+\phi(x-ay)$

Differentiating partially with respect to x and y

$$p=f^{'}(x+ay)+\phi^{'}(x-ay)$$

$$q=f^{'}(x+ay)a-\phi^{'}(x-ay)a$$

Differentiating again partially with respect to x and y we get

$$\frac{\partial^2 z}{\partial x^2}=f^{''}(x+ay)+\phi^{''}(x-ay)$$

$$\frac{\partial^2 z}{\partial y^2}=a^2 f^{''}(x+ay)+a^2\phi^{''}(x-ay)=a^2\frac{\partial^2 z}{\partial x^2}$$

Hence the resulting equation is $r-a^2t=0$,where $r=\frac{\partial^2 z}{\partial x^2}$ and $t=\frac{\partial^2 z}{\partial y^2}$

7) Eliminate the functions f and ϕ from the relation $z=f(y+ax)+x\phi(y+ax)$

Solution: Given that $z=f(y+ax)+x\phi(y+ax)$

Differentiating partially with respect to x and y

$$p=f^{'}(y+ax)a+x\phi^{'}(y+ax)a+\phi(y+ax) \quad (1)$$

$$q=f^{'}(y+ax)+\phi^{'}(y+ax) \quad (2)$$

Differentiating again partially with respect to x and y we get

$$\frac{\partial^2 z}{\partial x^2}=r=f^{''}(y+ax)a^2+x\phi^{''}(y+ax)a^2+2\phi^{'}(y+ax)a \quad (3)$$

$$\frac{\partial^2 z}{\partial y^2} = t = f''(y+ax) + \phi''(y+a) \qquad (4)$$

Differentiating (2) partially with respect to x we get

$$\frac{\partial^2 z}{\partial x \partial y} = s = f''(y+ax).a + \phi''(y+ax).a \qquad (5)$$

From (3),(4) and (5) we get

$$r - 2as + a^2 t = 0$$

8) Form the partial differential equation by eliminating the arbitrary function form $z = f\left(x^2 + y^2 + z^2\right)$

Solution: Given that $z = f\left(x^2 + y^2 + z^2\right)$ (1)

Differentiating (1) partially with respect to x and y we get

$$\frac{\partial z}{\partial x} = p = f'(x^2 + y^2 + z^2)(2x + 2zp) \qquad (2)$$

$$\frac{\partial z}{\partial y} = q = f'(x^2 + y^2 + z^2)(2y + 2zq) \qquad (3)$$

Dividing (2) by (3) ,

$$\frac{p}{q} = \frac{x+zp}{y+zq} \Rightarrow py + zpq = qx + zpq \Rightarrow py = qx$$

9) Form the partial differential equation by eliminating the arbitrary function form $xyz = \phi\left(x^2 + y^2 - z^2\right)$

Solution: Given that $xyz = \phi\left(x^2 + y^2 - z^2\right)$ (1)

Differentiating (1) partially with respect to x and y we get

$$xyp + yz = \phi'(x^2 + y^2 - z^2)(2x - 2zp) \qquad (2)$$

$$xyq + xz = \phi'(x^2 + y^2 - z^2)(2y - 2zq) \qquad (3)$$

Dividing (2) by (3) ,

$$\frac{yz+xyp}{xz+xyq}=\frac{x-zp}{y-zq}\Rightarrow (yz+xyp)(y-zq)=(x-zp)(xz+xyq)$$

$$zy^2+xy^2p-yz^2q-xyzpq=x^2z+x^2yq-xz^2p-xyzpq$$

$$px(y^2+z^2)-qy(z^2+x^2)=z(x^2-y^2)$$

Solution of partial differential equations by direct integration

A partial differential equation can be solved by successive integration in all cases where the dependent variables occurs only in the partial derivatives.

Problems:

1) Solve $\frac{\partial z}{\partial x}=0$

Solution: Given that $\frac{\partial z}{\partial x}=0$

Integrating with respect to x, we get $z=f(y)$, where f is arbitrary function.

2) Solve $\frac{\partial^2 z}{\partial x^2}=xy$

Solution: Given that $\frac{\partial^2 z}{\partial x^2}=xy$

Integrating with respect to x, we get

$\frac{\partial z}{\partial x}=\frac{x^2}{2}y+f(y)$, where f is arbitrary function.

Again integrating with respect to x

$$z=\frac{x^3}{6}y+xf(y)+\phi(y)$$

3) Solve $\frac{\partial^2 z}{\partial y^2}=0$

Solution: Given that $\frac{\partial^2 z}{\partial y^2}=0$

Integrating with respect to y, we get

$\frac{\partial z}{\partial y}=f(x)$, where f is arbitrary function.

Again integrating with respect to y

$$z = yf(x) + \phi(x)$$

4) Solve $\dfrac{\partial^2 z}{\partial x^2} = \cos x$

Solution: Given that $\dfrac{\partial^2 z}{\partial x^2} = \cos x$

Integrating with respect to x, we get

$$\frac{\partial z}{\partial x} = -\sin x + f(y)$$, where f is arbitrary function.

Again integrating with respect to x

$$z = -\cos x + xf(y) + \phi(y)$$

5) Solve $\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{x}{y} = 6$

Solution: Given that $\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{x}{y} = 6 \Rightarrow \dfrac{\partial^2 z}{\partial x \partial y} = 6 - \dfrac{x}{y}$

Integrating with respect to x,we get

$$\frac{\partial z}{\partial y} = 6x - \frac{x^2}{2y} + f(y)$$

Integrating with respect to y,we get

$$z = 6xy - \frac{x^2}{2}\log y + F(y) + \phi(x)$$

6) Solve $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x$

Solution: Given that $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x$

Integrating with respect to x, we get

$$\frac{\partial z}{\partial y} = \cos x + f(y)$$, where f is arbitrary function.

Again integrating with respect to y

$$z = y\cos x + F(y) + \phi(x)$$

7) Solve $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$,given that $u = 0$ when $t = 0$ and $\dfrac{\partial u}{\partial t} = 0$ when $x = 0$

Solution: Given that $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$

Integrating with respect to x, we get

$$\frac{\partial u}{\partial t} = e^{-t}\sin x + f(t)$$, where f is arbitrary function.

Given that $\frac{\partial u}{\partial t} = 0$ when $x = 0$, $\frac{\partial u}{\partial t} = 0 \Rightarrow e^{-t}\sin 0 + f(t) = 0 \Rightarrow f(t) = 0$

$$\therefore \frac{\partial u}{\partial t} = e^{-t}\sin x$$

Again integrating with respect to t

$$u = -e^{-t}\sin x + \phi(x)$$

When $t = o, u = 0$

$$0 = -\sin x + \phi(x) \Rightarrow \phi(x) = \sin x$$

$$u(x,t) = -e^{-t}\sin x + \sin x$$

FIRST ORDER PARTIAL DIFFERENTIAL EQUATION

Definition : A solution of a partial differential equation is a. relation among the independent and dependent variables, which satisfies the partial differential equation.

There are three types of solutions :i) Complete integral

ii) General integral

iii) Singular integral

Definitions:

Complete integral: The solution containing as many arbitrary constants as the number of variables is called the complete integral.

General integral: The solution which contains the maximum possible number of arbitrary functions is called the general integral

Singular integral: Differentiating the complete integral

$$\phi(x, y, z, a, c) = 0 \qquad (1)$$

partially with respect to a and c we get

$$\frac{\partial \phi}{\partial a} = 0, \frac{\partial \phi}{\partial c} = 0 \qquad (2)$$

The eliminant of a and c from the three equations (1) and (2) ,if it exists is called the singular integral.

STANDARD TYPES

STANDARD :1

The partial differential equation is of the form $f(p,q)=0$

In this case the complete integral is

$$z=ax+by+c \tag{3}$$

where a,b,c are constants. From (3) we get $p=\frac{\partial z}{\partial x}=a$ and $q=\frac{\partial z}{\partial y}=b$ .Replacing p by a and q by b in $f(p,q)=0$,we get $f(a,b)=0$ .Solving this for b we get $b=\phi(a)$,say .Thus

$$z=ax+\phi(a)y+c$$

is the complete integral, where a and c are constants

Working rule for Standard .1

Step 1: Set $p=a$ and $q=b$ in the given partial differential equation and solve for b in terms of a ,say $b=\phi(a)$

Step 2: From the result $dz=pdx+qdy$,we get $dz=adx+\phi(a)dy$

Step 3: Integrate this and get the complete integral as $z=ax+\phi(a)y$ where a and c are arbitrary constants.

PROBLEMS:

(1) Solve $p^2+q^2=npq$

Solutions: This is of the form $f(p,q)=0$.Hence the complete integral is

$z=ax+by+c$ where $a^2+b^2=nab$ or $a^2+b^2-nab=0$

Solving this for b we get

$$b=\frac{na\pm\sqrt{n^2a^2-4a^2}}{2}=a\left(\frac{n\pm\sqrt{n^2-4}}{2}\right)$$

Now from

$dz=pdx+qdy$,we get

$$dz=adx+bdy=adx+\frac{a}{2}\left(n\pm\sqrt{n^2-4}\right)dy$$

/integrating, we get the complete integral is

$$z=ax+\frac{a}{2}\left(n\pm\sqrt{n^2-4}\right)y+c$$

Where a and c are arbitrary constants.

General integral: Putting $c = g(a)$ in the complete integral, we get

$$z = ax + \frac{a}{2}\left(n \pm \sqrt{n^2 - 4}\right)y + g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get $0 = 1$,which is an absurd result. Hence there is no singular integral

(2) Solve $pq + p + q = 0$

Solution: This is of the form $f(p,q) = 0$.Hence the complete integral is $z = ax + by + c$

where $ab + a + b = 0$ or $b(a+1) + a = 0 \Rightarrow b = -\frac{a}{a+1}$

Now from $dz = pdx + qdy$,we get $dz = adx + bdy = adx - \frac{a}{a+1}dy$

Integrating we get the **complete integral** as

$$z = ax - \frac{a}{a+1}y + c$$

Where a and c are arbitrary constants

General integral: Putting $c = g(a)$ in the complete integral, we get

$$z = ax - \frac{a}{a+1}y + g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get $0 = 1$, which is an absurd result. Hence there is no singular integral.

(3) **Solve** $\sqrt{p} + \sqrt{q} = 1$

Solution: This is of the form $f(p,q) = 0$.Hence the complete integral is $z = ax + by + c$

where $\sqrt{a} + \sqrt{b} = 1$,that is $b = (1 - \sqrt{a})^2$

Therefore the complete integral is $z = ax + (1 - \sqrt{a})^2 y + c$ where a and c are arbitrary constants.

General integral: Putting $c = g(a)$ in the complete integral, we get

$$z = ax + (1 - \sqrt{a})^2 y + g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

(4) **Solve** $p+q=pq$

Solution: This is of the form $f(p,q)=0$.Hence the complete integral is $z=ax+by+c$ where $a+b=ab$,that is $b=-\frac{a}{(1-a)}$

Therefore the complete integral is $z=ax-\frac{a}{(1-a)}y+c$ where a and c are arbitrary constants.

General integral: Putting $c=g(a)$ in the complete integral, we get

$$z=ax-\frac{a}{(1-a)}y+g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a}=0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

(5) Solve $pq=1$

Solution: This is of the form $f(p,q)=0$.Hence the complete integral is $z=ax+by+c$ where $ab=1$,that is $b=\frac{1}{a}$

Therefore the complete integral is $z=ax+\frac{1}{a}y+c$ where a and c are arbitrary constants.

General integral: Putting $c=g(a)$ in the complete integral, we get

$$z=ax+\frac{1}{a}y+g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a}=0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

(6) **Solve** $p=q^2$

Solution: This is of the form $f(p,q)=0$.Hence the complete integral is $z=ax+by+c$ where $a=b^2$,that is $b=\sqrt{a}$

Therefore the complete integral is $z = ax + \sqrt{a}y + c$ where a and c are arbitrary constants.

General integral: Putting $c = g(a)$ in the complete integral, we get

$$z = ax + \sqrt{a}y + g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

(7) Solve $2p + 3q = 1$

Solution: This is of the form $f(p,q) = 0$. Hence the complete integral is $z = ax + by + c$ where $2a + 3b = 1$,that is $b = \frac{1-2a}{3}$

Therefore the complete integral is $z = ax + \left(\frac{1-2a}{3}\right)y + c$ where a and c are arbitrary constants.

General integral: Putting $c = g(a)$ in the complete integral, we get

$$z = ax + \left(\frac{1-2a}{3}\right)y + g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

(8) Solve $q^2 - p^2 = 9$

Solution: This is of the form $f(p,q) = 0$. Hence the complete integral is $z = ax + by + c$ where $b^2 - a^2 = 9$,that is $b = \pm\sqrt{9 + a^2}$

Therefore the complete integral is $z = ax \pm \left(\sqrt{9 + a^2}\right)y + c$ where a and c are arbitrary constants.

General integral: Putting $c = g(a)$ in the complete integral, we get

$$z = ax \pm \left(\sqrt{9 + a^2}\right)y + g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

STANDARD:II The partial differential equations of the form

i) $f(x,p,q)=0$

ii) $f(y,p,q)=0$

iii) $f(z,p,q)=0$

Case:1 The equation of the form $f(x,p,q)=0$

Set $q=a$ in the given p.d.e and solve the resulting equation for p in terms of x and a .Let $p=g(x,a)$ .Then $dz=pdx+qdy=g(x,a)dx+ady$

Integrating this we get the complete integral as

$$z=\int g(x,a)dx+ay+c$$

Where a and c are arbitrary constants.

Problem:

1) Solve $\sqrt{p}+\sqrt{q}=x$

Solution: The p.d.e is of standard 2.So,setting $q=a^2$ in the given equation we get $\sqrt{p}+a=x$ or $p=(x-a)^2$

Now from $dz=pdx+qdy=(x-a)^2dx+a^2dy$

Integrating we get the complete integral as

$$z=\frac{(x-a)^3}{3}+a^2y+c$$

Where a and c are arbitrary constants.

General integral: Putting $c=g(a)$ in the complete integral, we get

$$z=\frac{(x-a)^3}{3}+a^2y+g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a}=0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

2) Solve $xp+p^2=q$

Solution: The p.d.e is of standard 2. So, setting $q=a$ in the given equation we get

$$xp+p^2=a \text{ or } p^2+xp-a=0 \Rightarrow p=\frac{-x\pm\sqrt{x^2+4a}}{2}$$

Now from $dz=pdx+qdy=\frac{-x\pm\sqrt{x^2+4a}}{2}dx+ady$

Integrating this, we get the complete integral as

$$z=\int\frac{-x\pm\sqrt{x^2+4a}}{2}dx+ay+c \quad \text{(or)}$$

$$z=-\frac{x^2}{4}\pm\frac{1}{4}\left[x\sqrt{x^2+4a}\right]+4a\log(x+\sqrt{4a+x^2})+ay+c$$

Where a and c are arbitrary constants.

General integral: Putting $c=g(a)$ in the complete integral, we get

$$z=-\frac{x^2}{4}\pm\frac{1}{4}\left[x\sqrt{x^2+4a}\right]+4a\log(x+\sqrt{4a+x^2})+ay+g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a}=0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

Case:2 The equation is of the form $f(y,p,q)=0$ Set $p=a$ in the given p.d.e and solve the resulting equation for p in terms of y and a .Let $q=g(y,a)$ .Then

$dz=pdx+qdy=adx+g(a,y)dy$

Integrating this we get the complete integral as

$$z=ax+\int g(y,a)dy+c$$

Where a and c are arbitrary constants.

Problems:

1) **Solve** $pq=y$

Solution: The given equation is of the form $f(y,p,q)=0$. So we put $p=a$ in the given equation .Then we get $aq=y\Rightarrow q=\frac{y}{a}$

Now from $dz=pdx+qdy=adx+\frac{y}{a}dy$

Integrating this, we get the complete integral as

$$z=ax+\frac{y^2}{2a}+c$$

Where a and c are arbitrary constants.

General integral: Putting $c=g(a)$ in the complete integral, we get

$$z=ax+\frac{y^2}{2a}+g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a}=0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

2) Solve $p=(1+q^2)y^2$

Solution: The given equation is of the form $f(y,p,q)=0$. So we put $p=a^2$ in the given equation .Then we get $a^2 = (1+q^2)y^2 \Rightarrow q = \frac{\sqrt{a^2-y^2}}{y}$

Now from $dz = pdx + qdy = a^2 dx + \frac{\sqrt{a^2-y^2}}{y} dy$

Integrating this, we get the complete integral as

$$z = a^2 x + \int \frac{\sqrt{a^2-y^2}}{y} dy + c$$

To integrate this, put $y = a\sin\theta$ then $dy = a\cos\theta\, d\theta$

$$z = a^2 x + \int \frac{a\cos\theta}{a\sin\theta} a\cos\theta d\theta + c$$

$$z = a^2 x + a\int (\operatorname{cosec}\theta - \sin\theta) d\theta + c$$

$$z = a^2 x - a\log(\operatorname{cosec}\theta + \cot\theta) + a\cos\theta + c$$

$$z = a^2 x - a\log\left(\frac{a+\sqrt{a^2-y^2}}{y}\right) + \sqrt{a^2-y^2} + c$$ is the complete integral

where a and c are arbitrary constants.

General integral: Putting $c = g(a)$ in the complete integral, we get

$$z = a^2 x - a\log\left(\frac{a+\sqrt{a^2-y^2}}{y}\right) + \sqrt{a^2-y^2} + g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

Case:3 The equation is of the form $f(z,p,q) = 0$ Set $q = ap$ in the given p.d.e solve the resulting equation for p in terms of z and a .Let $p = g(z,a)$ .Then

$$dz = pdx + qdy = g(z,a)dx + ag(z,a)dy \Rightarrow \frac{dz}{g(z,a)} = dx + ady$$

Integrating this we get the complete integral as

$$\int \frac{dz}{g(z,a)} = \int dx + a\int dy \Rightarrow \int \frac{dz}{g(z,a)} = x + ay + c$$

Where a and c are arbitrary constants.

Problems:

1) **Solve** $z^2(p^2+q^2+1) = 1$

Solution: The equation is of the form $f(z,p,q)=0$ Set $q=ap$ in the given p.d.e then the given equation becomes $z^2(p^2+a^2p^2+1)=1$ or $z^2p^2(1+a^2)=1-z^2$

$$p=\pm\frac{\sqrt{1-z^2}}{z\sqrt{1+a^2}}$$

Now using $dz=pdx+qdy$, we get

$$dz=\pm\frac{\sqrt{1-z^2}}{z\sqrt{1+a^2}}(dx+ady)$$

$$\sqrt{1+a^2}\,\frac{z}{\sqrt{1-z^2}}dz=\pm(dx+ady)$$

Integrating this, we get

$$\sqrt{1+a^2}\left(-\sqrt{1-z^2}\right)=\pm(x+ay)+c$$

is the complete integral where a and c are arbitrary constants.

General integral: Putting $c=g(a)$ in the complete integral, we get

$$\sqrt{1+a^2}\left(-\sqrt{1-z^2}\right)=\pm(x+ay)+g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a}=0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

2) **Solve** $z^4q^2-z^2p=1$

Solution: The equation is of the form $f(z,p,q)=0$ Set $q=ap$ in the given p.d.e then the given equation becomes $z^4a^2p^2-z^2p=1 \Rightarrow a^2(z^2p)^2-z^2p-1=0$

$$z^2p=\frac{1\pm\sqrt{1+4a^2}}{2a^2}\Rightarrow p=\frac{1\pm\sqrt{1+4a^2}}{2a^2z^2}$$

Now from $dz=pdx+qdy$,we get $dz=p(dx+ady)$

$$dz=\left(\frac{1\pm\sqrt{1+4a^2}}{2a^2z^2}\right)(dx+ady)\quad\text{(or)}\quad z^2dz=\left(\frac{1\pm\sqrt{1+4a^2}}{2a^2}\right)(dx+ady)$$

Integrating this we get the complete integral as

$$\frac{z^3}{3}=\left(\frac{1\pm\sqrt{1+4a^2}}{2a^2}\right)(x+ay)+c$$

where a and c are arbitrary constants.

General integral: Putting $c=g(a)$ in the complete integral, we get

$$\frac{z^3}{3} = \left(\frac{1 \pm \sqrt{1+4a^2}}{2a^2}\right)(x+ay) + g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero, we get an absurd result. Hence there is no singular integral.

3) **Solve** $pz = 1 + q^2$

Solution: The equation is of the form $f(z, p, q) = 0$ Set $q = ap$ in the given p.d.e then the given equation becomes $pz = 1 + a^2 p^2$ (or) $pz - 1 + a^2 p^2 = 0 \Rightarrow a^2 p^2 + zp - 1 = 0$

$$p = \frac{-z \pm \sqrt{z^2 + 4a^2}}{2a^2} \text{ and } q = \frac{-z \pm \sqrt{z^2 + 4a^2}}{2a}$$

Now from $dz = pdx + qdy$,we get $dz = p(dx + ady)$

$$dz = \left(\frac{-z \pm \sqrt{z^2 + 4a^2}}{2a^2}\right)(dx + ady) \quad \text{(or)} \quad \frac{2a^2 dz}{-z \pm \sqrt{z^2 + 4a^2}} = dx + ady$$

Integrating this we get the complete integral as

$$\int \frac{2a^2 dz}{-z \pm \sqrt{z^2 + 4a^2}} = x + ay + c$$

where a and c are arbitrary constants.

General integral: Putting $c = g(a)$ in the complete integral, we get

$$\int \frac{2a^2 dz}{-z \pm \sqrt{z^2 + 4a^2}} = x + ay + g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero, we get an absurd result. Hence there is no singular integral.

4) **Solve** $pq = z^2$

Solution: The equation is of the form $f(z, p, q) = 0$ Set $q = ap$ in the given p.d.e then the given equation becomes $p(ap) = z^2$ (or) $ap^2 = z^2 \Rightarrow p = \pm \frac{z}{\sqrt{a}}$

And $q = \pm \frac{az}{\sqrt{a}}$

Now from $dz = pdx + qdy$,we get $dz = p(dx + ady)$

$$dz = \left(\pm \frac{z}{\sqrt{a}} \right)(dx + ady) \quad \text{(or)} \quad \sqrt{a}\int \frac{dz}{z} = \pm \int dx + ady$$

Integrating this we get the complete integral as

$$\sqrt{a}\log z = \pm(x + ay + c) \Rightarrow a(\log z)^2 = (x + ay + c)^2$$

where a and c are arbitrary constants.

General integral: Putting $c = g(a)$ in the complete integral, we get

$$a(\log z)^2 = (x + ay + g(a))^2$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$ we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero, we get an absurd result. Hence there is no singular integral.

5) **Solve** $z^2 p^2 + q^2 = 1$

Solution: The equation is of the form $f(z, p, q) = 0$ Set $q = ap$ in the given p.d.e then the given equation becomes $z^2 p^2 + a^2 p^2 = 1$ or $p^2(z^2 + a^2) = 1 \Rightarrow p^2 = \frac{1}{z^2 + a^2}$

$$p = \pm \frac{1}{\sqrt{z^2 + a^2}}$$

Now using $dz = pdx + qdy$, we get

$$dz = \pm \frac{1}{\sqrt{z^2 + a^2}}\left(dx + ady\right)$$

$$\sqrt{z^2 + a^2}\,dz = \pm(dx + ady)$$

Integrating this, we get

$$\int \sqrt{z^2 + a^2}\,dz = \pm \int (dx + ady)$$

$$\frac{z}{2}\sqrt{z^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{z}{a}\right) = \pm(x + ay + c)$$

is the complete integral where a and c are arbitrary constants.

General integral: Putting $c = g(a)$ in the complete integral, we get

$$\frac{z}{2}\sqrt{z^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{z}{a}\right) = \pm(x + ay + g(a))$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$, we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

6) **Solve** $9(p^2 z + q^2) = 4$

Solution: The equation is of the form $f(z,p,q)=0$ Set $q=ap$ in the given p.d.e then the given equation becomes $9(p^2z+a^2p^2)=4$ or $9p^2(z+a^2)=4 \Rightarrow p^2=\frac{4}{9(z+a^2)}$

$$p=\pm\frac{2}{3\sqrt{z+a^2}}$$

Now using $dz=pdx+qdy$, we get

$$dz=\pm\frac{2}{3\sqrt{z+a^2}}(dx+ady)$$

$$\sqrt{z+a^2}\,dz=\pm(dx+ady)$$

Integrating this, we get

$$\frac{3}{2}\int\sqrt{z+a^2}\,dz=\pm\int(dx+ady)$$

$$\frac{3}{2}\frac{(z^2+a^2)^{3/2}}{3/2}=\pm(x+ay+c)\Rightarrow(z^2+a^2)^3=(x+ay+c)^2$$

is the complete integral where a and c are arbitrary constants.

General integral: Putting $c=g(a)$ in the complete integral, we get

$$(z^2+a^2)^3=(x+ay+g(a))^2$$

Eliminating a between z and $\frac{\partial z}{\partial a}=0$, we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

Standard:3 The partial differential equation is of the form $f_1(x,p)=f_2(y,q)$

Put $f_1(x,p)=f_2(y,q)=a$ then we get $p=\phi_1(x,a)$ and $q=\phi_2(y,a)$

Then $pdx+qdy=dz$ becomes $dz=\phi_1(x,a)dx+\phi_2(y,a)dy$

Integrating this we get $z=\int\phi_1(x,a)dx+\int\phi_2(a,y)dy+c$ is the complete integral

General integral :Putting $c=g(a)$ in the complete integral

$$z=\int\phi_1(x,a)dx+\int\phi_2(a,y)dy+g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a}=0$, we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

1) **Solve** $p+q=x+y$

Solution: The partial differential equation is of the form $f_1(x,p)=f_2(y,q)$

Put $p-x=y-q=a$ then we get $p=x+a$ and $q=y-a$

Then $pdx+qdy=dz$ becomes $dz=(x+a)dx+(y-a)dy$

Integrating this we get $z=\int(x+a)dx+\int(y-a)dy \Rightarrow z=\frac{(x+a)^2}{2}+\frac{(y-a)^2}{2}+c$ is the complete integral

General integral :Putting $c=g(a)$ in the complete integral

$$z=\frac{(x+a)^2}{2}+\frac{(y-a)^2}{2}+g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a}=0$, we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral

2) **Solve** $p^2+q^2=x+y$

Solution: The partial differential equation is of the form $f_1(x,p)=f_2(y,q)$

Put $p^2-x=y-q^2=a$ then we get $p^2=x+a$ and $q^2=y-a$ (or) $p=\sqrt{x+a}$ and $q=\sqrt{y-a}$

Then $pdx+qdy=dz$ becomes $dz=(x+a)^{1/2}dx+(y-a)^{1/2}dy$

Integrating this we get

$$z=\int(x+a)^{1/2}dx+\int(y-a)^{1/2}dy \Rightarrow z=\frac{(x+a)^{3/2}}{3/2}+\frac{(y-a)^{3/2}}{3/2}+c$$

$$z=\frac{2}{3}(x+a)^{3/2}+\frac{2}{3}(y-a)^{3/2}+c$$

is the complete integral

General integral :Putting $c=g(a)$ in the complete integral

$$z=\frac{2}{3}(x+a)^{3/2}+\frac{2}{3}(y-a)^{3/2}+g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a}=0$, we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral

3) **Solve** $pq=xy$

Solution: The partial differential equation is of the form $f_1(x,p)=f_2(y,q)$

Putting $\frac{p}{x}=\frac{y}{q}=a$ we get $p=ax$ and $q=\frac{y}{a}$

Then $dz=pdx+qdy$ will become $dz=axdx+\frac{y}{a}dy$

Integrating this we get $z = a\left(\frac{x^2}{2}\right) + \frac{1}{a}\left(\frac{y^2}{2}\right) + c$ is the complete integral .

General integral: Putting $c = g(a)$ in the complete integral

$$z = a\left(\frac{x^2}{2}\right) + \frac{1}{a}\left(\frac{y^2}{2}\right) + g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$, we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

4) **Solve** $q(p - \sin x) = \cos y$

Solution: The partial differential equation is of the form $f_1(x, p) = f_2(y, q)$

Let $p - \sin x = \frac{\cos y}{q} = a$

Then $p = (a + \sin x), q = \frac{\cos y}{a}$

Therefore $dz = pdx + qdy$ will become $dz = (a + \sin x)dx + \frac{\cos y}{a}dy$

Integrating this we get $z = ax - \cos x + \frac{\sin y}{a} + c$ is the complete integral

General integral: Putting $c = g(a)$ in the complete integral

$$z = ax - \cos x + \frac{\sin y}{a} + g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$, we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

5) **Solve** $p + q = \sin x + \sin y$

Solution: The partial differential equation is of the form $f_1(x, p) = f_2(y, q)$

Let $p - \sin x = \sin y - q = a$

Then $p = (a + \sin x), q = \sin y - a$

Therefore $dz = pdx + qdy$ will become $dz = (a + \sin x)dx + (\sin y - a)dy$

Integrating this we get $z = ax - \cos x - \cos y - ay + c$ is the complete integral

General integral: Putting $c = g(a)$ in the complete integral

$$z = a(x - y) - (\cos x + \cos y) + g(a)$$

Eliminating a between z and $\frac{\partial z}{\partial a} = 0$, we get the general integral.

Singular integral: Differentiating z partially with respect to c and equating to zero we get an absurd result. Hence there is no singular integral.

STANDARD .IV CLAIRAUT'S FORM

This is of the form $z = px + qy + f(p,q)$

The solution of the equation is $z = ax + by + f(a,b)$ for $p = a$ & $q = b$

Problems :

1) Solve $z = px + qy + \sqrt{1 + p^2 + q^2}$

Solution: The complete integral is obviously $z = ax + by + \sqrt{1 + a^2 + b^2}$ (1)

General integral: Assume $b = f(a)$ where f is arbitrary.

Differentiating partially with respect to a and eliminating a between the two equations we get the general integral.

Singular integral: Differentiating partially with respect to a and b

$$x + \frac{a}{\sqrt{1 + a^2 + b^2}} = 0 \text{ and } y + \frac{b}{\sqrt{1 + a^2 + b^2}} = 0$$

Eliminating a and b the singular integral is $x^2 + y^2 + z^2 = 1$

2) Solve $z = px + qy + pq$

Solution: The complete integral is obviously $z = ax + by + ab$ (1)

General integral:

Assume $b = f(a)$ where f is arbitrary. Differentiating partially with respect to a and eliminating a between the two equations we get the general integral.

Singular integral:

Differentiating (1) partially with respect to a and b

$$x + b = 0 \text{ and } y + a = 0$$

Eliminating a and b the singular integral is $z = -xy - xy + xy = 0 \Rightarrow z + xy = 0.$

3) Solve $z = px + qy + p^2q^2$

Solution: This equation is in clairauts form .

The complete integral is $z = ax + by + a^2b^2$ (1)

Where a and b are arbitrary constants

Singular integral:

Differentiating (1) partially with respect to a and b we get

$0 = x + 2ab^2$ (2)

$0 = y + 2a^2b$ (3)

From (2) $x = -2ab^2 = \frac{-2ay^2}{4a^4} = \frac{-y^2}{2a^3} \Rightarrow a^3 = \frac{-y^2}{2x} \Rightarrow a = \left[\frac{-y^2}{2x}\right]^{\frac{1}{3}}$

$$y = -2a^2b = \frac{-2bx^2}{4b^4} = -\frac{x^2}{2b^3} \Rightarrow b^3 = \left[\frac{-x^2}{2y}\right] \Rightarrow b = \left[\frac{-x^2}{2y}\right]^{\frac{1}{3}}$$

Thus the singular integral is given by

$$z = x\sqrt[3]{\frac{y^2}{2x}} - y\sqrt[3]{\frac{x^2}{2y}} + \left(\sqrt[3]{\frac{y^2}{2x}}\right)^2\left(\sqrt[3]{\frac{x^2}{2y}}\right)^2 = -\left(\frac{1}{2}\right)^{\frac{1}{3}} x^{\frac{2}{3}} y^{\frac{2}{3}} - \left(\frac{1}{2}\right)^{\frac{1}{3}} x^{\frac{2}{3}} y^{\frac{2}{3}} + \frac{(xy)^{\frac{4}{3}}}{(4xy)^{\frac{2}{3}}}$$

$z = -2^{\frac{2}{3}} x^{\frac{2}{3}} y^{\frac{2}{3}}$ is the singular integral

General integral:

Assume $b = f(a)$ where f is arbitrary . Differentiating partially with respect to a and eliminating a between the two equations we get the general integral.

LAGRANGES LINEAR EQUATION

A linear partial differential equation of the first order known as Lagrange's linear equation is of the form

$$Pp + Qq = R \qquad (1)$$

Where P , Q and R are functions of x, y, z .

Working rule:

To solve the equation $Pp + Qq = R$,

(i) Form the auxiliary simultaneous equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

(ii) Solve these auxiliary equations giving two independent solutions $u = a$ and $v = b$

(iii) Then write down the solutions as $\phi(u,v) = 0$ or $u = f(v)$ or $v = F(u)$ where the function is arbitrary

Problems:

1) Find the general integral of $px + qy = z$

Solution: Comparing the equation with $Pp + Qq = R$,we get

$P = x, Q = y$ and $R = z$

The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

In these the variables are separated.

From $\frac{dx}{x} = \frac{dy}{y}$, we get $\log x = \log y + \log c_1 \Rightarrow \log x = \log yc_1 \Rightarrow \frac{x}{y} = c_1$

Similarly from $\frac{dy}{y} = \frac{dz}{z}$,we get $\frac{y}{z} = c_2$

Hence the general solution is $\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$

2) Solve $(y+z)p + (z+x)q = x+y$

Solution: Comparing the equation with $Pp + Qq = R$,we get

$$P = y+z, Q = z+x \text{ and } R = x+y$$

The subsidiary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

Consider $\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z} = \frac{dz-dx}{z-x} = \frac{dx+dy+dz}{2(x+y+z)}$

Comparing the first two members and integrating we get

$$\frac{x-y}{y-z} = c_1$$

Comparing first and last members and integrating we get $(x-y)^2(x+y+z) = c_2$

The solution is $\phi\left[\frac{x-y}{y-z}, (x-y)^2(x+y+z)\right] = 0$

3) Solve $y^2 p + x^2 q = x^2 y^2 z^2$

Solution: Comparing the equation with $Pp + Qq = R$,we get

$$P = y^2, Q = x^2 \text{ and } R = x^2 y^2 z^2$$

The subsidiary equations are

$$\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2 y^2 z^2}$$

Comparing the first two members and integrating we get

$$x^2 dx = y^2 dy \Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + c_1 \Rightarrow x^3 - y^3 = c_1$$

Comparing second and last members and integrating we get $y^2 dy = \frac{dz}{z^2} \Rightarrow \frac{y^3}{3} = \frac{-1}{z} + c_2$

The solution is $\phi\left[x^3 - y^3, \frac{y^3}{3} + \frac{1}{z}\right] = 0$

4) Solve $y^2 zp + x^2 zq = xy^2$

Solution: Comparing the equation with $Pp+Qq=R$,we get

$P=y^2z, Q=x^2z$ and $R=xy^2$

The subsidiary equations are

$$\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{xy^2}$$

Comparing the first two members and integrating we get

$$\frac{dx}{y^2z}=\frac{dy}{x^2z}\Rightarrow x^2dx=y^2dy\Rightarrow\frac{x^3}{3}=\frac{y^3}{3}+c_1\Rightarrow x^3-y^3=3C_1=c_1$$

Comparing first and last members and integrating we get

$$\frac{dx}{y^2z}=\frac{dz}{xy^2}\Rightarrow xdx-zdz=0\Rightarrow x^2-z^2=c_2$$

The solution is $\phi\left[x^3-y^3, x^2-z^2\right]=0$

5) Solve $(mz-ny)p+(nx-lz)q=ly-mx$

Solution: Comparing the equation with $Pp+Qq=R$,we get

$P=mz-ny, Q=(nx-lz)$ and $R=ly-mx$

The subsidiary equations are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Using multipliers x,y,z we get each ratio equal to

$$\frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}=\frac{xdx+ydy+zdz}{0}\Rightarrow xdx+ydy+zdz=0\Rightarrow x^2+y^2+z^2=c_1$$

Using multipliers l,m,n we get each ratio equal to

$$\frac{ldx+mdy+ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}=\frac{ldx+mdy+ndz}{0}\Rightarrow ldx+mdy+ndz=0\Rightarrow lx+my+nz=c_2$$

The solution is $\phi\left[x^2+y^2+z^2, lx+my+nz\right]=0$

6) Solve $(3z-4y)p+(4x-2z)q=2y-3x$

Solution: Comparing the equation with $Pp+Qq=R$,we get

$P=3z-4y, Q=(4x-2z)$ and $R=2y-3x$

The subsidiary equations are

$$\frac{dx}{3z-4y}=\frac{dy}{4x-2z}=\frac{dz}{2y-3x}$$

Using multipliers x,y,z we get each ratio equal to

$$\frac{xdx+ydy+zdz}{x(3z-4y)+y(4x-2z)+z(2y-3x)}=\frac{xdx+ydy+zdz}{0}\Rightarrow xdx+ydy+zdz=0\Rightarrow x^2+y^2+z^2=c_1$$

Using multipliers $2,3\,\&\,4$ we get each ratio equal to

$$\frac{2dx+3dy+4dz}{2(3z-4y)+3(4x-2z)+4(2y-3x)}=\frac{2dx+3dy+4dz}{0}\Rightarrow 2dx+3dy+4dz=0\Rightarrow 2x+3y+4z=c_2$$

The solution is $\phi\left[x^2+y^2+z^2, 2x+3y+4z\right]=0$

7) Solve $x(y^2+z)p-y(x^2+z)q=z(x^2-y^2)$

Solution: Comparing the equation with $Pp+Qq=R$, we get

$P=x(y^2+z), Q=-y(x^2+z)$ and $R=z(x^2-y^2)$

The subsidiary equations are

$$\frac{dx}{x(y^2+z)}=\frac{dy}{-y(x^2+z)}=\frac{dz}{z(x^2-y^2)}=\frac{xdx+ydy}{z(x^2-y^2)}$$

Comparing the last two members and integrating we get

$$\frac{dz}{z(x^2-y^2)}=\frac{xdx+ydy}{z(x^2-y^2)}\Rightarrow dz=xdx+ydy$$

$$\Rightarrow z+c_1=\frac{x^2}{2}+\frac{y^2}{2}\Rightarrow x^2+y^2=2z+c_1\Rightarrow x^2+y^2-2z=c_1$$

The subsidiary equations can also be written as

$$\frac{\frac{dx}{x}}{(y^2+z)}=\frac{\frac{dy}{y}}{-(x^2+z)}=\frac{dz}{z(x^2-y^2)}=\frac{\frac{dx}{x}+\frac{dy}{y}}{y^2-x^2}$$

Taking the last two ratios $\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$

Integrating we get $\log x+\log y+\log z=\log c_2\Rightarrow \log xyz=\log c_2\Rightarrow xyz=c_2$

The solution is $\phi\left[x^2+y^2-2z, xyz\right]=0$

8) Solve $(y-z)p+(z-x)q=x-y$

Solution: Comparing the equation with $Pp+Qq=R$, we get

$P=y-z, Q=z-x$ and $R=x-y$

The subsidiary equations are

$$\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}$$

Consider $\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}=\frac{dx+dy+dz}{0}$

Integrating we get

$$x+y+z=c_1$$

Also each ratio= $\dfrac{xdx+ydy+zdz}{x(y-z)+y(z-x)+z(x-y)}=\dfrac{xdx+ydy+zdz}{0}$

Integrating we get $x^2+y^2+z^2=c_2$

The solution is $\phi\left[x+y+z,x^2+y^2+z^2\right]=0$

9) Solve $x(z^2-y^2)p+y(x^2-z^2)q=z(y^2-x^2)$

Solution: Comparing the equation with $Pp+Qq=R$, we get

$P=x(z^2-y^2),Q=y(x^2-z^2)$ and $R=z(y^2-x^2)$

The subsidiary equations are

$$\frac{dx}{x(z^2-y^2)}=\frac{dy}{y(x^2-z^2)}=\frac{dz}{z(y^2-x^2)}$$

Using multipliers x,y,z we get each ratio equal to

$$\frac{xdx+ydy+zdz}{x^2z^2-x^2y^2+y^2x^2-y^2z^2+z^2y^2-z^2x^2}=\frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow xdx+ydy+zdz=0\Rightarrow x^2+y^2+z^2=c_1$$

Using multipliers $\frac{1}{x},\frac{1}{y}\&\frac{1}{z}$ we get each ratio equal to

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{z^2-y^2+x^2-z^2+y^2-x^2}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}\Rightarrow\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$$

$$\Rightarrow\log x+\log y+\log z=\log c_2\Rightarrow\log xyz=\log c_2\Rightarrow xyz=c_2$$

The solution is $\phi\left[x^2+y^2+z^2,xyz\right]=0$

10) Solve $\dfrac{y^2z}{x}p+xzq=y^2$

Solution: Comparing the equation with $Pp+Qq=R$, we get

$P=\dfrac{y^2z}{x},Q=xz$ and $R=y^2$

The subsidiary equations are

$$\frac{dx}{\frac{y^2z}{x}}=\frac{dy}{xz}=\frac{dz}{y^2}$$

From the first two ratios $x^2dx=y^2dy\Rightarrow x^3-y^3=c_1$

From the first and ratio $xdx=ydy\Rightarrow x^2-y^2=c_2$

Hence the solution is $\phi\left(x^3-y^3,x^2-y^2\right)=0$

CHARPIT'S METHOD

A general method of solving a partial differential equation is due to charpit's.

Consider $F(x, y, z, p, q) = 0$

The auxiliary equations are

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q} = \frac{df}{0}$$

Problems:

1) Solve $p^2 + q^2 - 2px - 2qy + 1 = 0$ (1)

Solution: Let $F(x, y, z, p, q) = p^2 + q^2 - 2px - 2qy + 1$

Here $F_x = \frac{\partial F}{\partial x} = -2p$, $F_y = \frac{\partial F}{\partial y} = -2q$, $F_z = \frac{\partial F}{\partial z} = 0$

$$F_p = \frac{\partial F}{\partial p} = 2(p - x), \quad F_q = \frac{\partial F}{\partial q} = 2(q - y)$$

The auxiliary equations are

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q} = \frac{df}{0}$$

$$\frac{dp}{-2p} = \frac{dq}{-2q} = \frac{dz}{-2p(p-x) - 2q(q-y)} = \frac{dx}{-2(p-x)} = \frac{dy}{-2(q-y)} = \frac{df}{0}$$

From the first two ratios

$$\frac{dp}{-2p} = \frac{dq}{-2q} \Rightarrow \frac{dp}{p} = \frac{dq}{q} \Rightarrow \log p = \log q + \log a$$

$$\Rightarrow \log p = \log aq \Rightarrow p = aq$$

Putting $p = aq$ in (1) we get $q^2(a^2 + 1) - 2q(ax + y) + 1 = 0$ which is quadratic in q .Solving this for q ,we get

$$q = \frac{(ax + y) \pm \sqrt{(ax + y)^2 - a^2 - 1}}{a^2 + 1}$$

Taking the positive root for the radical,

$$p = a\frac{(ax + y) \pm \sqrt{(ax + y)^2 - a^2 - 1}}{a^2 + 1}$$

$$\therefore dz = pdx + qdy = \left[a\frac{(ax+y) \pm \sqrt{(ax+y)^2 - a^2 - 1}}{a^2+1} \right] dx + \left[\frac{(ax+y) \pm \sqrt{(ax+y)^2 - a^2 - 1}}{a^2+1} \right] dy$$

$$\text{dz} = \left\{ \frac{(ax+y) \pm \sqrt{(ax+y)^2 - a^2 - 1}}{a^2+1} \right\} d(ax+y)$$

Integrating

$$2(a^2+1)z + b = (ax+y)^2 + (ax+y)\sqrt{(ax+y)^2 - a^2 - 1} - (a^2+1)\cosh^{-1}\left(\frac{ax+y}{\sqrt{a^2+1}} \right)$$

There is no singular integral as differentiation with respect to b leads to an absurd result.
To get the general integral, as usual we put $b = f(a)$ where f is arbitrary and differentiating with respect to a Eliminating a ,the general integral is obtained.

2) Solve $pxy + pq + qy = yz$ (1)

Solution: Let $F = pxy + pq + qy - yz$

Here $F_x = \frac{\partial F}{\partial x} = py$, $F_y = \frac{\partial F}{\partial y} = px + q$, $F_z = \frac{\partial F}{\partial z} = -y$

$$F_p = \frac{\partial F}{\partial p} = xy + q, \quad F_q = \frac{\partial F}{\partial q} = p + y$$

The auxiliary equations are

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q} = \frac{df}{0}$$

The equation $\frac{df}{0} = \ldots \Rightarrow p = a$ (constant) (2)

substituting (2) in (1),we get $axy + aq + qy - yz = 0$

$$q = \frac{y(z-ax)}{a+y}$$

$$\therefore dz = pdx + qdy = adx + \frac{y(z-ax)}{a+y} dy$$

$$\therefore \frac{d(z-ax)}{z-ax} = \frac{ydy}{a+y}$$

Integrating this we get $\log(z-ax) = y - a\log(a+y) + b$
This is the complete integral
There is no singular integral as differentiation with respect to b leads to an absurd result.
To get the general integral, as usual we put $b = f(a)$ where f is arbitrary and differentiating with respect to a Eliminating a ,the general integral is obtained.

3) Obtain a complete integral of $xp^2 - ypq + y^3q - y^2z = 0$ (1)

Solution: Let $F = xp^2 - ypq + y^2q - y^2z$

Here $F_x = \frac{\partial F}{\partial x} = p^2, F_y = \frac{\partial F}{\partial y} = -pq + 3y^2q - 2yz, F_z = \frac{\partial F}{\partial z} = -y^2$

$F_p = 2px - yq, \quad F_q = \frac{\partial F}{\partial q} = -yp + y^3$

The auxiliary equations are

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q} = \frac{df}{0}$$

$$\frac{dp}{p(p-y^2)} = \frac{dy}{y(p-y^2)} \Rightarrow \frac{dp}{p} = \frac{dy}{y} \Rightarrow \log p = \log y + \log a \Rightarrow \log p = \log ay \Rightarrow p = ay$$

Substituting $p = ay$ in the equation (1) ,we get

$$q = \frac{z - a^2x}{y - a}$$

Then $dz = pdx + qdy$ becomes $dz = aydx + \left[\frac{z - a^2x}{y - a}\right]dy$ (2)

Treating this as a total differential equation neglect dy for the present and integrating we get

$z = axy + f(y)$

Differentiating totally and comparing with (2)

$$\frac{z - axy}{y - a} = \frac{df}{dy} \Rightarrow \frac{f}{y - a} = \frac{df}{dy} \Rightarrow \frac{df}{f} = \frac{dy}{y - a}$$

Integrating $\log f = \log(y-a) + \log b \Rightarrow \log f = \log b(y-a) \Rightarrow f = (y-a)$

Therefore the complete integral is $z = axy + b(y-a)$

4) Solve $px + qy = pq$ (1)

Solution: Let $F = px + qy - pq$

Here $F_x = \frac{\partial F}{\partial x} = p, F_y = \frac{\partial F}{\partial y} = q, F_z = \frac{\partial F}{\partial z} = 0$

$F_p = x - q, \quad F_q = \frac{\partial F}{\partial q} = y - p$

The auxiliary equations are

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q} = \frac{df}{0}$$

$$\frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(y-p) - q(x-q)} = \frac{dx}{-(x-q)} = \frac{dy}{-(y-p)} = \frac{df}{0}$$

From the first two ratios

$$\frac{dp}{p}=\frac{dq}{q}\Rightarrow \log p=\log q+\log a\Rightarrow \log p=\log qa\Rightarrow p=qa$$

Substituting $p=qa$ in (1) we get

$$qax+qy=qaq\Rightarrow ax+y=qa\Rightarrow q=\frac{ax+y}{a}$$

Then $p=qa=ax+y$

Then $dz=pdx+qdy=(ax+y)dx+\left(\frac{ax+y}{a}\right)dy=\frac{(ax+y)[adx+dy]}{a}=\frac{(ax+y)d(ax+y)}{a}$

$$adz=(ax+y)d(ax+y)$$

Integrating we get $az=\frac{(ax+y)^2}{2}+b' \Rightarrow z=\frac{(ax+y)^2}{2a}+\frac{b'}{a}=\frac{(c_1x+y)^2}{2a}+b$

Hence the complete integral is $z=\frac{(ax+y)^2}{2a}+b$

There is no singular integral as differentiation with respect to b leads to an absurd result.
To get the general integral, as usual we put $b=f(a)$ where f is arbitrary and differentiating with respect to a Eliminating a ,the general integral is obtained.

5) Obtain a complete integral of $1+p^2=qz$ (1)

Solution: Let $F=1+p^2-qz$

Here $F_x=\frac{\partial F}{\partial x}=0\,,F_y=\frac{\partial F}{\partial y}=0\,,F_z=\frac{\partial F}{\partial z}=-q$

$$F_p=2p\,,\quad F_q=\frac{\partial F}{\partial q}=-q$$

The auxiliary equations are

$$\frac{dp}{F_x+pF_z}=\frac{dq}{F_y+qF_z}=\frac{dz}{-pF_p-qF_q}=\frac{dx}{-F_p}=\frac{dy}{-F_q}=\frac{df}{0}$$

$$\frac{dp}{-pq}=\frac{dq}{-q^2}=\frac{dz}{-p(2p)-q(-q)}=\frac{dx}{-2p}=\frac{dy}{-z}=\frac{df}{0}$$

Consider the first two ratios

$$\frac{dp}{-pq}=\frac{dq}{-q^2}\Rightarrow\frac{dp}{p}=\frac{dq}{q}\Rightarrow \log p=\log q+\log a\Rightarrow \log p=\log aq\Rightarrow p=qa$$

Substituting $p=qa$ in (1) we get

$1+q^2a^2=qz\Rightarrow a^2q^2-zq+1=0$,which is quadratic in q .

$$\therefore q=\frac{z\pm\sqrt{z^2-4a^2}}{2a^2}$$

Hence $p = \left[\dfrac{z \pm \sqrt{z^2 - 4a^2}}{2a^2}\right] a = \left[\dfrac{z \pm \sqrt{z^2 - 4a^2}}{2a}\right]$

Substituting these values of $p \,\&\, q$ in (1) we get

$$1 + \left[\frac{z \pm \sqrt{z^2 - 4a^2}}{2a}\right]^2 = \left[\frac{z \pm \sqrt{z^2 - 4a^2}}{2a^2}\right] z$$

$4a^2 + \left[z \pm \sqrt{z^2 - 4a^2}\right]^2 - 2\left(z \pm \sqrt{z^2 - 4a^2}\right) = 0$ is the complete integral.

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Consider a linear equation of the second order with constant coefficient

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = X \qquad (1)$$

Where a, b and c are constants and X is any function of x.

Consider the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (2)$$

The solution of this equation (2) is called the complementary function of (1).

The equation

$$am^2 + bm + c = 0 \qquad (3)$$

is called the auxiliary equation of the equation (1).

Three cases can arise in the solution of the auxiliary equation.

Case:1 Let the auxiliary equation (3) has two real and distinct roots m_1 and m_2. Then $y = Ae^{m_1 x} + Be^{m_2 x}$ is the general solution of (2).

Case:2 Let the auxiliary equation (3) has two roots equal and real..Let $m_1 = m_2 = m$ (say)

Then $y = (Ax + B)e^{mx}$ is the general solution of (2).

Case:3 Let the auxiliary equation (3) has imaginary roots. As imaginary roots occur in pairs, let $m_1 = \alpha + i\beta$ where α and β are real; then $m_2 = \alpha - i\beta$ .Then $y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$ is the general solution of (2).

Problems:

1. Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0$

   Solution: The auxiliary equation is $m^2 - 5m + 4 = 0 \Rightarrow m = 1$ and $\mathrm{m} = 4$.
   The roots are real and distinct. Thus $y = Ae^x + Be^{4x}$ is the general solution.

2. Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$

   Solution: The auxiliary equation is $m^2 - 6m + 8 = 0 \Rightarrow m = 2$ and $\mathrm{m} = 4$.
   The roots are real and distinct. Thus $y = Ae^{2x} + Be^{4x}$ is the general solution.

3. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

   Solution: The auxiliary equation is $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = \text{-}1$ twice .
   The roots are real and equal. Thus $y = (Ax + B)e^{-x}$ is the general solution.

4. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

   Solution: The auxiliary equation is $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2$ twice .
   The roots are real and equal. Thus $y = (Ax + B)e^{2x}$ is the general solution.

5. Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 5y = 0$

   Solution: The auxiliary equation is $m^2 - 3m + 5 = 0 \Rightarrow \mathrm{m} = \frac{3 \pm \sqrt{9 - 20}}{2} \Rightarrow \mathrm{m} = \frac{3 \pm \mathrm{i}\sqrt{11}}{2}$.

   The roots are imaginary. Thus $y = e^{\frac{3x}{2}}\left[A\cos\frac{\sqrt{11}}{2}x + B\sin\frac{\sqrt{11}}{2}x\right]$ is the general solution.

6. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 4y = 0$

   Solution: The auxiliary equation is $m^2 + m + 4 = 0 \Rightarrow \mathrm{m} = \frac{-1 \pm \sqrt{1-16}}{2} \Rightarrow \mathrm{m} = \frac{-1 \pm \mathrm{i}\sqrt{15}}{2}$.

The roots are imaginary. Thus $y = e^{-\frac{x}{2}}\left[A\cos\frac{\sqrt{15}}{2}x + B\sin\frac{\sqrt{15}}{2}x\right]$ is the general solution.

To find the particular integral:

Consider the equation (1) which can be written symbolically as $(aD^2 + bD + c)y = X$ or shortly $f(D)y = X$, where $f(D) = aD^2 + bD + c$

Let $y = u$ be a particular solution of this equation .Let Y be the complementary function of (1).Then $y = Y + u$ is the general solution of (1). u is called the particular integral of (1).In symbolic form it is written as $\frac{1}{f(D)}X$

P.I= $\frac{1}{f(D)}X = \frac{1}{aD^2 + bD + c}X$

TYPE.1 Let X be of the form $e^{\alpha x}$

P.I = $\frac{1}{f(D)}e^{\alpha x}$

Case:1 If $f(\alpha) \neq 0$, $\frac{1}{f(D)}e^{\alpha x} = \frac{1}{f(\alpha)}e^{\alpha x}$

Hence the rule is, replace D by α if $f(\alpha) \neq 0$.

Case:2 If $f(\alpha) = 0$,then α satisfies the equation $f(m) = 0$

Then we proceed as follows.

(i) Let the auxiliary equation have two distinct roots m_1 and m_2 and let $\alpha = m_1$ then the

P.I= $\frac{x}{a(\alpha - m_2)}e^{\alpha x}$

(ii) Let the auxiliary equation have two equal roots, each equal to α .(i.e) $m_2 = m_1 = \alpha$

P.I= $\frac{x^2}{2}e^{\alpha x}$

Problems:

1. Solve $(D^2 + 5D + 6)y = e^x$

Solution: The auxiliary equation is

$m^2 + 5m + 6 = 0 \Rightarrow (m+2)(m+3) = 0 \Rightarrow m = -2 \text{ and } m = -3$

The equation has real and distinct roots.

The complementary function is $\text{C.F} = Ae^{-2x} + Be^{-3x}$

P.I= $\frac{1}{(D^2 + 5D + 6)}e^x = \frac{1}{1+5+6}e^x = \frac{1}{12}e^x$

Hence the general solution is $y = C.F + P.I = Ae^{-2x} + Be^{-3x} + \frac{1}{12}e^{x}$

2. Solve $(D^2 - 5D + 6)y = e^{4x}$

Solution: The auxiliary equation is

$m^2 - 5m + 6 = 0 \Rightarrow (\mathrm{m}-2)(\mathrm{m}-3) = 0 \Rightarrow \mathrm{m} = 2$ and $\mathrm{m} = 3$

The equation has real and distinct roots.

The complementary function is C.F $= Ae^{2x} + Be^{3x}$

P.I= $\frac{1}{(D^2 - 5D + 6)}e^{4x} = \frac{1}{16 - 20 + 6}e^{4x} = \frac{1}{2}e^{4x}$

Hence the general solution is $y = C.F + P.I = Ae^{2x} + Be^{3x} + \frac{1}{2}e^{4x}$

3. Solve $(D^2 - 13D + 12)y = e^{x}$

Solution: The auxiliary equation is

$m^2 - 13m + 12 = 0 \Rightarrow (\mathrm{m}-1)(\mathrm{m}-12) = 0 \Rightarrow \mathrm{m} = 1$ and $\mathrm{m} = 12$

The equation has real and distinct roots.

The complementary function is C.F $= Ae^{x} + Be^{12x}$

Here $\alpha = 1$

P.I= $\frac{1}{(D^2 - 13D + 12)}e^{x} = \frac{1}{(D-1)(D-12)}e^{x} = \frac{1}{(D-1)(-11)}e^{x} = \frac{-x}{11}e^{x}$

Hence the general solution is $y = C.F + P.I = Ae^{x} + Be^{12x} - \frac{x}{11}e^{x}$

4. Solve $(3D^2 + D - 14)y = 13e^{2x}$

Solution: The auxiliary equation is

$3m^2 + m - 14 = 0 \Rightarrow (\mathrm{m}-2)(3\mathrm{m}+7) = 0 \Rightarrow \mathrm{m} = 2$ and $\mathrm{m} = -7/3$

The equation has real and distinct roots.

The complementary function is C.F $= Ae^{2x} + Be^{(-7/3)x}$

Here $\alpha = 2$

P.I= $\frac{1}{(3D^2 + D - 14)}13e^{2x} = \frac{1}{(D-2)(3D+7)}13e^{x2} = \frac{1}{(D-2)(13)}13e^{2x} = xe^{2x}$

Hence the general solution is $y = C.F + P.I = Ae^{2x} + Be^{(-7/3)x} + xe^{2x}$

5. Solve $(D^2 - 2mD + m^2)\, y = e^{mx}$.

Solution: The auxiliary equation is $k^2 - 2mk + m^2 = 0$.

i.e., $(k - m)^2 = 0$, $\therefore$ k = m twice.

The equation has real and equal roots.

The complementary function is

C.F. = e^{mx} (A + Bx).

P.I. = $\frac{1}{(\mathrm{k-m})^2}e^{mx}$

= $\frac{x^2}{2}e^{mx}$

Hence the general solution is

$y = C.F + P.I$ = e^{mx} (A + Bx + $\frac{x^2}{2}$).

∴ y = e^{mx} (A + Bx + $\frac{x^2}{2}$).

6. Solve $(D^2+4D+4)y = e^{-2x}$

Solution: The auxiliary equation is

$m^2+4m+4=0 \Rightarrow (\mathrm{m}+2)^2 = 0 \Rightarrow \mathrm{m} = -2$ twice

The equation has real and equal roots.

The complementary function is $\mathrm{C.F} = e^{-2x}(A+Bx)$

Here $\alpha = -2$

P.I= $\frac{1}{(D^2+4D+4)}e^{-2x} = \frac{1}{(D+2)^2}e^{-2x} = \frac{x^2}{2}e^{-2x}$

Hence the general solution is $y = C.F + P.I = e^{-2x}(A+Bx) + \frac{x^2}{2}e^{-2x}$

TYPE:2 Let X be of the form $\sin\alpha x$ or $\cos\alpha x$ where α is a constant.

P.I= $\frac{1}{\phi(D^2)}\sin\alpha x = \frac{1}{\phi(-\alpha^2)}\sin\alpha x$

Hence (i) the rule is **Replace** D^2 by $-\alpha^2$ if $\phi(-\alpha^2) \neq 0$

(ii) If $\phi(-\alpha^2) = 0$ then $\mathrm{D}^2+\alpha^2$ is a factor ,

hence $\frac{1}{D^2+\alpha^2}\sin\alpha x = -\frac{\mathrm{x}\cos\alpha\mathrm{x}}{2\alpha}$

and $\frac{1}{D^2+\alpha^2}\cos\alpha x = \frac{x\sin\alpha x}{2\alpha}$

1. Problem: Solve $(D^2 - 3D + 2)\,y = \sin 3x$.

Solution:. The auxiliary equation is $m^2 - 3m + 2 = 0$.

Solving, m = 2 and 1.

C.F. = $A e^{2x} + B e^{x}$.

$$\text{P.I.} = \frac{\sin 3x}{D^2 - 3D + 2}$$

$$= \frac{\sin 3x}{-9-3D+2}\ , \text{ replace } D^2 \text{ by } -\alpha^2 = -9$$

$$= \frac{7-3D}{7-3D}\times\frac{\sin 3x}{-7-3D}$$

$$= \frac{7\sin 3x - 3D(\sin 3x)}{-49+9D^2}$$

$$= \frac{7\sin 3x - 3(3\cos 3x)}{-49+9(-9)}$$

$$= \frac{7\sin 3x - 9\cos 3x}{-49-81}$$

$$= \frac{7\sin 3x - 9\cos 3x}{-130}$$

$$\text{P.I} = -\left[\frac{7\sin 3x - 9\cos 3x}{130}\right]$$

The general solution is y = C.F. + P.I.

$$y = A e^{2x} + B e^{x} - \left[\frac{7\sin 3x - 9\cos 3x}{130}\right].$$

2. Solve $(D^2+9)y = \sin 3x$

Solution: The auxiliary equation is $m^2 + 9 = 0 \Rightarrow m = \pm 3i$

The roots are imaginary.

The complementary equation is $A\cos 3x + B\sin 3x$

P.I = $\frac{1}{D^2+9}(\sin 3x)$

Here $\alpha = 3$ and $-\alpha^2 = -9$.

Replacing D^2 by -9 we get zero.

Hence $\frac{1}{D^2+9}\sin 3x = -\frac{\text{xcos}\,3\text{x}}{2.3} = -\frac{\text{xcos3x}}{6}$

General solution is y=C.F +P.I

$$y = A\cos 3x + B\sin 3x - \frac{x\cos 3x}{6}$$

3. Solve $(D^2+4)y = \cos 2x$

Solution: The auxiliary equation is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

The roots are imaginary.

The complementary equation is $A\cos 2x + B\sin 2x$

P.I = $\frac{1}{D^2+4}(\cos 2x)$

Here $\alpha = 2$ and $-\alpha^2 = -4$.Replacing D^2 by -4 ,we get zero.

Hence $\frac{1}{D^2+4}\cos 2x = \frac{\text{x}\sin 2\text{x}}{2.2} = \frac{\text{xsin2x}}{4}$

General solution is y=C.F +P.I

$$y = A\cos 2x + B\sin 2x + \frac{x\sin 2x}{4}$$

Type:3 Let X be of the form x^m ,m being appositive integer.

To evaluate $\frac{1}{f(D)}\, x^m$,raise f(D) to power (-1) and expand in ascending powers of D as far as D^m. These terms in the expansion of $[f(D)]^{-1}$ operating on x^m give the particular integral required.

,**Problem:** Solve $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+3y=5x^2$.

Solution: To find the C.F. solve $(D^2 + 2D + 3)\, y = 0$.

The auxiliary equation is $m^2 + 2m + 3 = 0$.

$$m = \frac{-2\pm\sqrt{2^2-4.1.3}}{2.1}$$

$$= \frac{-2\pm\sqrt{4-12}}{2}$$

$$= \frac{-2\pm\sqrt{-8}}{2}$$

$$= \frac{-2\pm 2i\sqrt{2}}{2}$$

$$= -1\pm i\sqrt{2}$$

$\alpha=-1$, $\beta=\sqrt{2}$

C.F. = $e^{-x}(A\cos\sqrt{2}\,x + B\sin\sqrt{2}\,x)$

$$\text{P.I.} = \frac{5x^2}{D^2+2D+3} = \frac{5x^2}{3+2D+D^2} = \frac{5x^2}{3\left[1+\frac{2D+D^2}{3}\right]}$$

$$= \frac{5}{3}\left[1+\frac{2D+D^2}{3}\right]^{-1}x^2 \qquad = \frac{5}{3}\left[1-\left(\frac{2D+D^2}{3}\right)+\left(\frac{2D+D^2}{3}\right)^2-\text{..........}\right]x^2$$

$$= \frac{5}{3}\left[1-\left(\frac{2D+D^2}{3}\right)+\left(\frac{4D^2+4D^3+D^4}{9}\right)-\text{..........}\right]x^2$$

$$= \frac{5}{3}\left[1-\left(\frac{2D+D^2}{3}\right)+\left(\frac{4D^2}{9}\right)\right]x^2 \quad \text{(Neglecting Higher Powers)}$$

$$= \frac{5}{3}\left[x^2-\left(\frac{2D(x^2)+D^2(x^2)}{3}\right)+\left(\frac{4D^2(x^2)}{9}\right)\right] = \frac{5}{3}\left[x^2-\left(\frac{2(2x)+2}{3}\right)+\left(\frac{4(2)}{9}\right)\right]$$

$$= \frac{5}{3}\left[x^2-\left(\frac{4x+2}{3}\right)+\left(\frac{8}{9}\right)\right] = \frac{5}{3}\left[x^2-\frac{4x}{3}-\frac{2}{3}+\frac{8}{9}\right]$$

$$= \frac{5}{3}\left[x^2-\frac{4x}{3}+\frac{2}{9}\right]$$

y = C.F. + P.I. $= e^{-x}(A\cos\sqrt{2}\,x + B\sin\sqrt{2}\,x) + \frac{5}{3}\left[x^2-\frac{4x}{3}+\frac{2}{9}\right]$.

TYPE ;4 Let X be of the form $e^{ax}F(x)$ where $F(x)$ is any function of x

$$\text{P.I}= \frac{1}{f(D)}\left[e^{ax}F(x)\right]=e^{ax}\frac{1}{f(D+a)}F(x).$$

Hence the rule is Replace D by $D+a$

Problem: Solve $(D^2 + 4)\, y = e^{2x}\sin 2x$.

Solution: The auxiliary equation $m^2 + 4 = 0. \Rightarrow m^2 = -4 \Rightarrow m = \pm 2i$

The roots are imaginary.

C.F. = $e^{0x}(A\cos 2x + B\sin 2x) = A\cos 2x + B\sin 2x$

$$\text{P.I.} = \frac{e^{2x}\sin 2x}{D^2+4} = \frac{e^{2x}\sin 2x}{(D+2)^2+4}, \text{ replace D by D+2}$$

$$= \frac{e^{2x}\sin 2x}{D^2+4D+8} = \frac{e^{2x}\sin 2x}{-4+4D+8}, \quad \text{replace } D^2 \text{ by } -4$$

$$= \frac{e^{2x}(4D-4)}{(4D+4)(4D-4)}\sin 2x = \frac{e^{2x}[4D(\sin 2x)-4\sin 2x]}{16D^2-16} = \frac{e^{2x}[4D(\sin 2x)-4\sin 2x]}{16(-4)-16}$$

$$= \frac{4e^{2x}[2\cos 2x-\sin 2x]}{-80}$$

The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = A\cos 2x + B\sin 2x - \frac{4e^{2x}[2\cos 2x - \sin 2x]}{80}.$$