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(NATIONALLY ACCREDIATED AT 'A' GRADE (3rd cycle) BY NAAC)

PG AND RESEARCH DEPARTMENT OF MATHEMATICS

STUDY MATERIAL
FOR UG
LAPLACE TRANSFORMS

NOVEMBER 2016

LAPLACE TRANSFORMS

Definition: If a function $f(t)$ is defined for all positive values of the variable t and if $\int_0^\infty e^{-st} f(t)dt$ exists and is equal to F(S) ,then F(S) is called the Laplace Transform of $f(t)$ and is denoted by the symbol $L\{f(t)\}$.

Hence $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$.The operator L that transforms $f(t)$ to F(S) is called the Laplace transform operator.

Note: $\lim_{s\to\infty} F(s) = 0$

Definition: A function $f(t)$ is said to be piecewise continuous in a closed interval $[a,b]$ if it is defined on that interval and is such that the interval can be broken up into a finite number of sub-intervals in each of which $f(t)$ is continuous. $f(t)$ can have only ordinary finite discontinuous in the interval.

Definition: A function $f(t)$ is said to be of exponential order if $\lim_{s\to\infty} e^{-st} f(t) = 0$,or if for some number s_0,the product $|f(t)| < M$ for $t < T$,that is $e^{-s_0 t}|f(t)|$ is bounded for large value of t say for $t < T$.

Sufficient conditions for the existence of the Laplace transform.

i) $f(t)$ is continuous or piecewise continuous in the closed interval $[a,b]$, where $a > 0$

ii) It is of exponential order

iii) $t^n f(t)$ is bounded near $t = 0$ for some number $n > 1$

RESULTS:

(i) $L\{f(t) + \phi(t)\} = L\{f(t)\} + L\{\phi(t)\}$

Proof: We have

$$L\{f(t) + \phi(t)\} = \int_0^\infty e^{-st}[f(t) + \phi(t)]dt = \int_0^\infty e^{-st} f(t)dt + \int_0^\infty e^{-st}\phi(t)dt = L\{f(t)\} + L\{\phi(t)\}$$

(ii) $L\{cf(t)\} = cL\{f(t)\}$,where c is a constant**.**

Proof: Consider

$$L\{cf(t)\} = \int_0^\infty e^{-st} cf(t)dt = c\int_0^\infty e^{-st} f(t)dt = cL\{f(t)\}$$

iii) $L\{f^1(t)\} = sL\{f(t)\} - f(0)$

Proof: Consider $L\{f^1(t)\} = \int_0^\infty e^{-st} f^1(t)dt$

Take $u = e^{-st}$ and $\int f^1(t)dt = dv$

then $du = -e^{-st}sdt$ and $f(t) = v$

using integration by parts ,we have,

$$L\{f^1(t)\} = \left[f(t)e^{-st}\right]_0^\infty - \int_0^\infty f(t)(-s)e^{-st}dt$$

$$= -f(0) + s\int_0^\infty f(t)e^{-st}dt$$

$$= sL\{f(t)\} - f(0)$$

iv) $L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - sf'(0)$

Proof: Take $f'(t) = F(t)$

Then $L\{f''(t)\} = L\{F'(t)\}$

$$= sL\{F(t)\} - F(0)$$

$$= sL\{f'(t)\} - f'(0)$$

$$= s\{sL\{f(t)\} - f(0)\} - f'(0)$$

$$= s^2L\{f(t)\} - sf(0) - f'(0)$$

v) If $L\{f(t)\} = F(s)$,then

a) $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

b) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Proof: We have $L\{f^1(t)\} = sL\{f(t)\} - f(0)$

$$= sF(s) - f(0)$$

Taking limits as $s \to \infty$ on both sides ,we get,

$$\lim_{s\to\infty}\{sF(s) - f(0)\} = \lim_{s\to\infty} L\{f^{'}(t)\} = \lim_{s\to\infty}\int_0^\infty e^{-st} f^{'}(t)dt = 0$$

$$\lim_{s\to\infty} sF(s) = f(0) = \lim_{t\to 0} f(t)$$

This result is known as **Initial value theorem.**

Taking limits as $s\to 0$ on both sides of $L\{f^{'}(t)\}$, we get

$$\lim_{s\to 0}[sF(s) - f(0)] = \lim_{s\to 0}\int_0^\infty e^{-st} f^{'}(t)dt$$

$$= \int_0^\infty f^{'}(t)dt = [f(t)]_0^\infty$$

$$= \lim_{t\to\infty} f(t) - f(0)$$

Therefore , $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

This result is known as **Final value theorem.**

vi) $L(e^{-at}) = \dfrac{1}{s+a}$ provided $(s+a) > 0$

Proof: Consider $L(e^{-at}) = \int_0^\infty e^{-st}e^{-at}dt$

$$= \int_0^\infty e^{-(s+a)t}dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a}$$

Similarly ,

$$L(e^{at}) = \frac{1}{s-a} \text{ provided } (s-a) > 0$$

Cor: $L(\cosh at) = L\left[\dfrac{e^{at} + e^{-at}}{2}\right] = \dfrac{1}{2}L(e^{at)}) + \dfrac{1}{2}L(e^{-at})$

$$= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} = \frac{s}{s^2-a^2}$$

Similarly ,

$$L(\sinh\ at) = \frac{a}{s^2-a^2}$$

vii) $L(\cos at) = \frac{s}{s^2+a^2}$

Proof: We have $L(\cos at) = \int_0^\infty e^{-at}\cos at dt$

$$= \left[\frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2}\right]_0^\infty = \frac{s}{s^2+a^2}$$

viii) $L(\sin\ at) = \frac{a}{s^2+a^2}$

Proof: We have $L(\sin\ at) = \int_0^\infty e^{-at}\sin\ at dt$

$$= \left[\frac{e^{-st}(-s\sin\ at + a\cos at)}{s^2+a^2}\right]_0^\infty = \frac{a}{s^2+a^2}$$

ix) $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$

Proof: We have $L(t^n) = \int_0^\infty e^{-st}t^n dt$

Put $st = x$ then $dt = \frac{1}{s}dx$

Therefore

$$L(t^n) = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x}\frac{1}{s}dx = \frac{1}{s^{n+1}}\int_0^\infty x^n e^{-x}dx = \frac{\Gamma(n+1)}{s^{n+1}}$$

When n is a positive integer $\Gamma(n+1) = n!$

Hence $L(t^n) = \frac{n!}{s^{n+1}}$,when n is a positive integer.

Cor: $L(1) = \frac{1}{s}$

$$L(t) = \frac{1}{s^2}$$

$$L(t^2) = \frac{2}{s^3}$$

$$L(t^{\frac{1}{2}}) = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$L(t^{-\frac{1}{2}}) = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{2s^{1/2}}$$

Problems :

1) Find $L\left(t^2 + 2t + 3\right)$

Solution: Consider

$$L\left(t^2 + 2t + 3\right) = L(t^2) + 2L(t) + 3L(1)$$

$$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}$$

2) Find $L\left(t^3 - 3t^2 + 2\right)$

Solution: Consider

$$L\left(t^3 - 3t^2 + 2\right) = L(t^3) - 3L(t^2) + 2L(1) = \frac{6}{s^4} - 3\left(\frac{2}{s^3}\right) + \frac{2}{s} = \frac{6}{s^4} - \frac{6}{s^3} + \frac{2}{s}$$

3) Find $L\left(at^2 + bt + c\right)$

Solution: Consider

$$L\left(at^2+bt+c\right)=L(at^2)+L(bt)+L(c)=aL(t^2)+bL(t)+cL(1)$$

$$=a\left(\frac{2}{s^3}\right)+b\left(\frac{1}{s^2}\right)+\frac{c}{s}=\frac{2a}{s^3}+\frac{b}{s^2}+\frac{c}{s}$$

4) Find $L(\sin^2 2t)$

Solution: We know that $\sin^2 2t=\left(\frac{1-\cos 4t}{2}\right)$

$$L(\sin^2 2t)=L\left(\frac{1-\cos 4t}{2}\right)=\frac{1}{2}L(1)-\frac{1}{2}L(\cos 4t)$$

$$=\frac{1}{2}\left(\frac{1}{s}\right)-\frac{1}{2}\left(\frac{s}{s^2+4^2}\right)=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+16}\right)=\frac{8}{s(s^2+16)}$$

5) Find $L(\sin^2 t)$

Solution: We know that $\sin^2 t=\left(\frac{1-\cos 2t}{2}\right)$

$$L(\sin^2 t)=L\left(\frac{1-\cos 2t}{2}\right)=\frac{1}{2}L(1)-\frac{1}{2}L(\cos 2t)$$

$$=\frac{1}{2}\left(\frac{1}{s}\right)-\frac{1}{2}\left(\frac{s}{s^2+2^2}\right)=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)=\frac{2}{s(s^2+4)}$$

6) Find $L(\cos^2 3t)$

Solution: We know that $\cos^2 3t=\left(\frac{1+\cos 6t}{2}\right)$

$$L(\cos^2 3t)=L\left(\frac{1+\cos 6t}{2}\right)=\frac{1}{2}L(1)+\frac{1}{2}L(\cos 6t)$$

$$=\frac{1}{2}\left(\frac{1}{s}\right)+\frac{1}{2}\left(\frac{s}{s^2+6^2}\right)=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+36}\right)=\frac{s^2+18}{s(s^2+36)}$$

7) Find $L\left(\sinh 3t\right)$

Solution: We know that $\sinh\, at = \dfrac{e^{at} - e^{-at}}{2}$

Consider

$$L(\sinh\, 3t) = L\left(\frac{e^{3t} - e^{-3t}}{2}\right) = \frac{1}{2}\left[L(e^{3t}) - L(e^{-3t})\right] = \frac{1}{2}\left(\frac{1}{s-3} - \frac{1}{s+3}\right) = \frac{3}{s^2 - 9}$$

8) Find $L(\sin^3 2t)$

Solution: Since $\sin 6t = 3\sin 2t - 4\sin^3 2t$**,** we have

$$\sin^3 2t = \left(\frac{3\sin 2t - \sin 6t}{4}\right)$$

Hence $L(\sin^3 2t) = L\left(\dfrac{3\sin 2t - \sin 6t}{4}\right) = \dfrac{1}{4}\left(L(3\sin 2t) - L(\sin 6t)\right) = \dfrac{3}{4}L(\sin 3t) - \dfrac{1}{4}L(\sin 6t)$

$$= \frac{3}{4}\left(\frac{2}{s^2 + 2^2}\right) - \frac{1}{4}\left(\frac{6}{s^2 + 6^2}\right) = \frac{48}{(s^2+4)(s^2+36)}$$

9) Find $L(\cos t \cos 2t)$

Solution: Since $\cos A \cos B = \dfrac{\cos(A+B) + \cos(A-B)}{2}$

We have $\cos t \cos 2t = \dfrac{\cos 3t + \cos t}{2}$

Hence

$$L(\cos t \cos 2t) = L(\cos t \cos 2t) = L\left(\frac{\cos 3t + \cos t}{2}\right) = \frac{1}{2}[L(\cos 3t) + L(\cos t)]$$

$$= \frac{1}{2}\left(\frac{s}{s^2 + 3^2}\right) + \frac{1}{2}\left(\frac{s}{s^2 + 1^2}\right) = \frac{s(s^2+5)}{(s^2+9)(s^2+1)}$$

10) Find $L\{f(t)\}$, where

$$f(t) = 0 \quad when \quad 0 < \text{t} \leq 2$$
$$= 3 \quad \text{when t} > 2$$

Solution: Consider $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

$$= \int_0^2 e^{-st} f(t)dt + \int_2^\infty e^{-st} f(t)dt$$

$$= \int_0^2 e^{-st}(0)dt + \int_2^\infty e^{-st}(3)dt$$

$$= \int_2^\infty e^{-st}(3)dt = 3\int_2^\infty e^{-st}dt = \frac{3}{s}e^{-2s}$$

11) Find $L\{f(t)\}$ where

$$f(t) \quad = (\text{t} - 1)^2 \quad when \quad \text{t} > 1$$
$$= 0 \quad \text{when} \quad \text{t} < 1$$

Solution: Consider $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

$$= \int_0^1 e^{-st} f(t)dt + \int_1^\infty e^{-st} f(t)dt$$

$$= \int_0^1 e^{-st} 0dt + \int_1^\infty e^{-st}(t-1)^2 dt$$

$$= \int_1^\infty e^{-st}(t-1)^2 dt$$

Using Bernoulli's formula

$$\int udu = uv - u^{'}v_1 + u^{''}v_2 - ......$$

Take $u = (t-1)^2, \;\; dv = \int e^{-st}dt$

Then $u^{'} = 2(t-1), \;\; v = -\frac{e^{-st}}{s}$

$$u'' = 2, \quad v_1 = \frac{e^{-st}}{s^2}$$

$$u''' = 0, \quad v_2 = -\frac{e^{-st}}{s^3}$$

Hence $\int_1^{\infty} e^{-st}(t-1)^2 dt = \{(t-1)^2[-\frac{e^{-st}}{s}] - 2(t-1)[\frac{e^{-st}}{s^2}] + 2[-\frac{e^{-st}}{s^3}]\}_1^{\infty} = 2\left\{\frac{e^{-s}}{s^3}\right\}$

12) Find $L\{f(t)\}$ where

$$f(t) = e^{-t} \text{ when } 0 < t < 4$$
$$= 0 \text{ when } t > 4$$

Solution: Consider $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$

$$= \int_0^4 e^{-st} f(t)dt + \int_4^{\infty} e^{-st} f(t)dt$$

$$= \int_0^4 e^{-st}e^{-t}dt + \int_4^{\infty} e^{-st}(0)dt$$

$$= \int_0^4 e^{-(s+1)t}dt = \left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^4 = \frac{1-e^{-4(s+1)}}{s+1}$$

13) Find $L\{f(t)\}$ where

$$f(t) = \sin t \text{ when } 0 < t < \pi$$
$$= 0 \text{ when } t > \pi$$

Solution: Consider $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$

$$= \int_0^{\pi} e^{-st} f(t)dt + \int_{\pi}^{\infty} e^{-st} f(t)dt$$

$$= \int_0^{\pi} e^{-st} \sin t dt + \int_{\pi}^{\infty} e^{-st}(0)dt$$

$$= \left[\frac{e^{-st}[-s\sin t - \cos t]}{(s^2+1)} \right]_0^{\pi} = \frac{1+e^{-s\pi}}{(s^2+1)}$$

Laplace transform of periodical functions

Let $f(t)$ be a periodic function wit period a.

Then $f(t) = f(a+t) = f(2a+t) = ... = f(na+t)$

Consider $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$

$$= \int_0^a e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)dt + ... + \int_{(n-1)a}^{na} e^{-st} f(t)dt + ...$$

In the second integral , put $t = T + a$

In the third integral ,put $t = T + 2a$

In the fourth integral , put $t = T + 3a$

In the nth integral , put $t = T + (n-1)a$ and so on.

Hence

$$L\{f(t)\} = \int_0^a e^{-st} f(t)dt + \int_a^{2a} e^{-s(T+a)} f(T+a)dT + ... + \int_{(n-1)a}^{na} e^{-s(T+(n-1)a)} f(T+(n-1)a)dt + ...$$

$$= \int_0^a e^{-st} f(t)dt + \int_a^{2a} e^{-s(T+a)} f(t)dT + ... + \int_{(n-1)a}^{na} e^{-s(T+(n-1)a)} f(t)dt + ...$$

$$= \int_0^a e^{-st} f(t)dt + e^{-sa}\int_0^a e^{-st} f(t)dt + ... + e^{-(n-1)a}\int_0^a e^{-st} f(t)dt + ...$$

$$= \sum_{n=0}^{\infty} \int_{na}^{(n+1)a} e^{-st} f(t)dt = \sum_{n=0}^{\infty} \int_0^{na} e^{-s(u+na)} f(u+na)du = \sum_{n=0}^{\infty} \int_a^{na} e^{-su} e^{-nsa} f(u)du$$

Where $t = u + na$ so that $dt = du$ and

$t = na \Rightarrow u = 0$

$\mathrm{t = (n+1)a \Rightarrow u = a}$

$$L\{f(t)\} = \sum_{n=0}^{\infty} e^{-nsa} \int_0^a e^{-su} f(u)du = \sum_{n=0}^{\infty} e^{-nsa} \int_0^a e^{-st} ft)dt$$

$$= (1+e^{-sa}+e^{-2sa}+.......)\int_0^a e^{-st} f(t)dt = \frac{1}{1-e^{-sa}}\int_0^a e^{-st} f(t)dt$$

Hence $L\{f(t)\} = \dfrac{1}{1-e^{-sa}}\displaystyle\int_0^a e^{-st} f(t)dt$

Problems:

1) Find the Laplace transform of $f(t)$ where

$f(t) = 1$ (0 < t < b)

= -1 (b < t < 2b)

The function is periodic in the interval $(0,,2b)$

Solution: Consider $L\{f(t)\} = \dfrac{1}{1-e^{-2bs}}\displaystyle\int_0^{2b} e^{-st} f(t)dt$

$$= \frac{1}{1-e^{-2bs}}\left\{\int_0^b e^{-st} f(t)dt + \int_b^{2b} e^{-st} f(t)dt\right\}$$

$$= \frac{1}{1-e^{-2bs}}\left\{\int_0^b e^{-st}dt + \int_b^{2b} e^{-st}(-1)dt\right\}$$

$$= \frac{1}{1-e^{-2bs}}\left\{\left(\frac{e^{-st}}{-s}\right)_0^b + \left(\frac{e^{-st}}{s}\right)_b^{2b}\right\}$$

$$= \frac{1}{s}\left\{\frac{1-2be^{-sb}+e^{-2sb}}{1-e^{2sb}}\right\}$$

$$= \frac{1}{s}\left(\frac{1-e^{-sb}}{1+e^{-sb}}\right) = \frac{1}{s}\tanh\left(\frac{bs}{2}\right)$$

2) Find the Laplace transform of $f(t)$ where

$f(t)$ = t (0 < t < b)

= 2b - t (b < t < 2b)

The function is periodic in the interval $(0,,2b)$

Solution: Consider $L\{f(t)\} = \frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st} f(t)dt$

$$= \frac{1}{1-e^{-2bs}}\left\{\int_0^b e^{-st} f(t)dt + \int_b^{2b} e^{-st} f(t)dt\right\}$$

$$= \frac{1}{1-e^{-2bs}}\left\{\int_0^b e^{-st} t dt + \int_b^{2b} e^{-st}(2b-t)dt\right\} = \frac{1}{s^2}\tanh\left(\frac{bs}{2}\right)$$

SOME GENERAL THEOREMS

Theorem :1 If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

Proof: Consider $L\{f(at)\} = \int_0^\infty e^{-st} f(at)dt$

Put a t = y, then t $= \frac{y}{a}$ and dt $= \frac{1}{\text{a}}dy$

$t = 0 \Rightarrow$ y = 0 and t = $\infty \Rightarrow$ y = ∞

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)dt = \frac{1}{a}\int_0^\infty e^{-s\left(\frac{y}{a}\right)} f(y)dy = \frac{1}{a}F\left(\frac{s}{a}\right)$$

1) Find $L(Cosat)$

Solution: Consider $L(Cosat) = \frac{1}{a}F\left(\frac{s}{a}\right)$ where $L(Cost) = F(s) = \frac{s}{s^2+1}$

Therefore, $L(Cosat) = \frac{1}{a}\left(\frac{\frac{s}{a}}{\frac{s^2}{a^2}+1}\right) = \frac{s}{s^2+a^2}$

2) Find $L(Sin\,hat)$

Solution: Consider $L(Sin\,hat) = \frac{1}{a}F\left(\frac{s}{a}\right)$ where $F(s) = L(Sinht) == \frac{1}{s^2-1}$

Therefore, $L(Sin\ hat) = \frac{1}{a}\left(\frac{1}{\frac{s^2}{a^2}-1}\right) = \frac{a}{s^2-a^2}$

3) Find $L(\text{Cos}\ hat)$

Solution: Consider $L(\text{Cos}\ hat) = \frac{1}{a}F\left(\frac{s}{a}\right)$ where $F(s) = L(Cosht) == \frac{s}{s^2-1}$

Therefore, $L(Cos\ hat) = \frac{1}{a}\left(\frac{\frac{s}{a}}{\frac{s^2}{a^2}-1}\right) = \frac{s}{s^2-a^2}$

4) Find $L(Sinat)$

Solution: Consider $L(Sin\ at) = \frac{1}{a}F\left(\frac{s}{a}\right)$ where $F(s) = L(S\ \text{int}) = \frac{1}{s^2+1}$

Therefore, $L(Sin\ at) = \frac{1}{a}\left(\frac{1}{\frac{s^2}{a^2}+1}\right) = \frac{a}{s^2+a^2}$

THEOREM:2 $L\{e^{-at}f(t)\} = F(s+a)$ where $F(s) = L\{f(t)\}$

Proof: We have $L\{e^{-at}f(t)\} = \int_0^\infty e^{-at}e^{-st}f(t)dt = \int_0^\infty e^{-(s+a)t}f(t)dt$

This is in structure exactly the Laplace transform of f(t) itself exactly that $(s+a)$ takes the place of s. Hence $L\{e^{-at}f(t)\} = F(s+a)$ where $F(s) = L\{f(t)\}$

1) Find $L(e^{-st})$

Solution: Consider $L(e^{-at}) = F(s+a)$ where $F(s) = L(1) = \frac{1}{s}$

Hence $L(e^{-at}) = F(s+a) = \frac{1}{s+a}$

2) Find $L\{e^{-at}\cos bt\}$

Solution: Consider $L\{e^{-at}\cos bt\} = F(s+a)$ where $F(s) = L(\cos bt) = \frac{s}{s^2+b^2}$

Then $L\{e^{-at}\cos bt\} = F(s+a) = \frac{s+a}{(s+a)^2+b^2}$

3) Find $L\{e^{-at}\sin bt\}$

Solution: Consider $L\{e^{-at}\sin bt\} = F(s+a)$ where $F(s) = L(\sin bt) = \frac{b}{s^2+b^2}$

Then $L\{e^{-at}\sin bt\} = F(s+a) = \frac{b}{(s+a)^2+b^2}$

4) Find $L\{e^{at}\sin bt\}$

Solution: Consider $L\{e^{at}\sin bt\} = F(s-a)$ where $F(s) = L(\sin bt) = \frac{b}{s^2+b^2}$

Then $L\{e^{at}\sin bt\} = F(s-a) = \frac{b}{(s-a)^2+b^2}$

5) Find $L(e^{-at}t^n)$

Solution: Consider $L(e^{-at}t^n) = F(s+a)$ where $F(s) = L(t^n) = \frac{n\,!}{s^{n+1}}$

Then $L(e^{-at}t^n) = F(s+a) = \frac{n!}{(s+a)^{n+1}}$

Similarly $L(e^{at}t^n) = F(s-a) = \frac{n!}{(s-a)^{n+1}}$

6) Find $L(e^{-5t}t)$

Solution: Consider $L(e^{-5t}t) = F(s+5)$ where $F(s) = L(t) = \frac{1}{s^2}$

7) Find $L(e^{3t}t^2)$

Solution: Consider $L(e^{3t}t^2) = F(s-3)$ where $F(s) = L(t^2) = \frac{2}{s^3}$

Then $L(e^{3t}t^2) = F(s-3) = \frac{2}{(s-3)^3}$

8) Find $L(e^{-2t}\sin 2t)$

Solution: Consider $L(e^{-2t}\sin 2t) = F(s+2)$ where $F(s) = L(\sin 2t) = \frac{2}{s^2+4}$

Hence $L(e^{-2t}\sin 2t) = F(s+2) = \frac{2}{(s+2)^2+4}$

9) Find $L(e^{-t}\cos 2t)$

Solution: Consider $L(e^{-t}\cos 2t) = F(s+1)$ where $F(s) = L(\cos 2t) = \frac{s}{s^2+4}$

Hence $L(e^{-t}\cos 2t) = F(s+1) = \frac{s+1}{(s+1)^2+4}$

10) Find $L(e^{t}\sin t\cos 2t)$

Solution: Consider $L(e^{t}\sin t\cos 2t) = F(s-1)$ where

$$F(s) = L(\sin t\cos 2t) = L\left\{\frac{\sin 3t - \sin t}{2}\right\} = \frac{1}{2}\{L(\sin 3t) - L\sin t)\} = \frac{1}{2}\left\{\frac{3}{s^2+9} - \frac{1}{s^2+1}\right\}$$

$$L(e^{t}\sin t\cos 2t) = F(s-1) = \frac{1}{2}\left\{\frac{3}{(s-1)^2+9} - \frac{1}{(s-1)^2+1}\right\}$$

THEOREM:3 If $L\{f(t)\} = F(s)$ then $\mathrm{L}\{t\ f(t)\} = -\frac{d}{ds}[F(s)]$

Proof: Consider $F(s) = \int_0^\infty e^{-st} f(t)dt$

$$\frac{d}{ds}\big(F(s)\big) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{\partial}{\partial s}\{e^{-st} f(t)\}dt = \int_0^\infty -t\,e^{-st} f(t)dt \quad = -\int_0^\infty t\,e^{-st} f(t)dt = -L\{t\ f(t)\}$$

Therefore, $\mathrm{L}\{t\ f(t)\} = -\frac{d}{ds}[F(s)]$

COR: $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[L\{f(t)\}]$

PROOF: We have $L\{t\ f(t)\} = -\frac{d}{ds}[F(s)]$

Then $L\{t^2 f(t)\} = L\{t.t.f(t)\} = -\frac{d}{ds}[L\{t\ f(t)\} = -\frac{d}{ds}\left[-\frac{d}{ds}L\{f(t)\}\right] = (-1)^2\frac{d^2}{ds^2}L\{f(t)\}$

Continuing the process , we get $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[L\{f(t)\}]$

Problems

1) Find $L(te^{-at})$

Solution: Consider $L(te^{-at}) = -\frac{d}{ds}L(e^{-at}) = -\frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{1}{(s+a)^2}$

2) Find $L(t\sin at)$

Solution: Consider

$$L(t\sin at) = -\frac{d}{ds}L(\sin at)$$
$$= -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

3) Find $L(t^2e^{-3t})$

Solution: Consider

$$L(t^2e^{-3t}) = (-1)\frac{d^2}{ds^2}[L(e^{-3t})] = \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right)$$
$$= \frac{d}{ds}\left(\frac{-1}{(s+3)^2}\right) = \frac{2}{(s+3)^3}$$

4) Find $L(t\cos 2t)$

Solution: Consider $L(t\cos 2t) = -\frac{d}{ds}L(\cos 2t) = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$

$$= -\left(\frac{(s^2+4).1 - s(2s)}{s^2+4}\right) = \frac{s^2-4}{(s^2+4)^2}$$

5) Find $L(t\cos^2 t)$

Solution: Consider $L(t\cos^2 t) = -\frac{d}{ds}L\left(\cos^2 t\right) = -\frac{d}{ds}L\left(\frac{1+\cos 2t}{2}\right)$

$$= -\frac{1}{2}\frac{d}{ds}[L(1) + L(\cos 2t)]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s}\right) - \frac{1}{2}\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left(-\frac{1}{s^2}\right) - \frac{1}{2}\left(\frac{(s^2+4).1 - s(2s)}{(s^2+4)^2}\right)$$

$$= \frac{1}{2s^2} + \frac{(s^2-4)}{2(s^2+4)^2}$$

6) Find $L(t^2 \sin 2t)$

Solution: Consider $L(t^2 \sin 2t) = (-1)^2 \frac{d^2}{ds^2}\left(L(\sin 2t) = \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)\right)$

$$= \frac{d}{ds}\left(\frac{-4s}{(s^2+4)^2}\right) = -4\left(\frac{(s^2+4)^2.1 - s[2(s^2+4)2s]}{(s^2+4)^4}\right) = \frac{4(3s^2-4)}{(s^2+4)^3} = \frac{12s^2-16}{(s^2+4)^3}$$

7) Find $L(t\sinh at)$

Solution: Consider

$$L(t\sinh at) = -\frac{d}{ds}\left(L(\sinh at)\right) = -\frac{d}{ds}\left(\frac{a}{s^2-a^2}\right) - a(-1)(s^2-a^2)^{-2}(2s) = \frac{2as}{(s^2-a^2)^2}$$

8) Find $L(t^2 \cosh at)$

Solution: Consider $L(t^2\cosh at) = (-1)^2 \frac{d^2}{ds^2}\left(L(\cosh at)\right) = \frac{d^2}{ds^2}\left(\frac{s}{s^2-a^2}\right)$

$$= \frac{d}{ds}\left(\frac{(s^2-a^2) - s[2s]}{(s^2-a^2)^2}\right) = \frac{d}{ds}\left(\frac{-(s^2+a^2)}{(s^2-a^2)^2}\right)$$

$$= (-1)\frac{d}{ds}\left(\frac{(s^2+a^2)}{(s^2-a^2)^2}\right) = \frac{2s(s^2+3a^2)}{(s^2-a^2)^3}$$

9) Find $L(\sin at - at\cos at)$

Solution: Consider $L(\sin at - at\cos at) = L(\sin at) - aL(t\cos at)$

$$= \frac{a}{s^2+a^2} + a\frac{d}{ds}\left(L(\cos at)\right) = \frac{a}{s^2+a^2} + a\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{a}{s^2+a^2} + a\left(\frac{(s^2+a^2).1 - s[2s]}{(s^2+a^2)^2}\right)$$

$$= \frac{2a^3}{(s^2+a^2)^2}$$

10) Find $L(te^{-t}\sin t)$

Solution: Consider $L(te^{-t}\sin t) = -\frac{d}{ds}\left(L(e^{-t}\sin t)\right) = -\frac{d}{ds}F(s+1)$

where $F(s) = L(\sin t) = \frac{1}{s^2+1}$ so that $F(s+1) = \frac{1}{(s+1)^2+1}$

Hence $L(te^{-t}\sin t) = -\frac{d}{ds}F(s+1) = -\frac{d}{ds}\left(\frac{1}{(s+1)^2+1}\right) = \frac{2(s+1)}{(s^2+2s+2)^2}$

11) Find $L(te^{-t}\cos t)$

Solution: Consider $L(te^{-t}\cos t) = -\frac{d}{ds}\left(L(e^{-t}\cos t)\right) = -\frac{d}{ds}F(s+1)$

where $F(s) = L(\cos t) = \frac{s}{s^2+1}$ so that $F(s+1) = \frac{s+1}{(s+1)^2+1}$

Hence $L(te^{-t}\cos t) = -\frac{d}{ds}F(s+1) = -\frac{d}{ds}\left(\frac{s+1}{(s+1)^2+1}\right) = \frac{s(s+2)}{(s^2+2s+2)^2}$

THEOREM: 4 If $L\{f(t)\} = F(s)$ and if $\frac{f(t)}{t}$ has a limit as $t \to 0$,then

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)ds \quad \text{where} \quad F(s) = L\{f(t)\}$$

Proof: Consider $F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

then $\int_0^\infty F(s)ds = \int_0^\infty \int_s^\infty e^{-st} f(t)dtds$

By changing the order of integration ,we have

$$\int_0^\infty F(s)ds = \int_0^\infty \int_s^\infty e^{-st} f(t)dsdt = \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt = \int_0^\infty \frac{f(t)}{t}e^{-st}dt = L\left\{\frac{f(t)}{t}\right\}$$

Problems:

1) Find $L\left\{\frac{1-e^t}{t}\right\}$

Solution: Consider $L\left\{\frac{1-e^t}{t}\right\}=\int_s^\infty L(1-e^t)ds$ since $\lim_{t\to 0}\frac{1-e^t}{t}=-1$

$$L\left\{\frac{1-e^t}{t}\right\}=\int_s^\infty L(1-e^t)ds=\int_s^\infty [L(1)-L(e^t)]ds$$

$$=\int_s^\infty \left\{\frac{1}{s}-\frac{1}{s-1}\right\}ds=\left\{\log\left(\frac{s}{s-1}\right)\right\}_s^\infty=\log\left\{\frac{s-1}{s}\right\}$$

2) Find $L\left\{\frac{\sin at}{t}\right\}$

Solution: Consider $L\left\{\frac{\sin at}{t}\right\}=\int_s^\infty L(\sin at)ds$ since $\lim_{t\to 0}\frac{\sin at}{t}=a$

$$L\left\{\frac{\sin at}{t}\right\}=\int_s^\infty L(\sin at)ds=\int_s^\infty \frac{a}{s^2+a^2}ds=\tan^{-1}\left(\frac{s}{a}\right)_s^\infty$$

$$=\tan^{-1}\infty-\tan^{-1}\left(\frac{s}{a}\right)=\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{a}\right)=\cot^{-1}\left(\frac{s}{a}\right)$$

3) Find $L\left\{\frac{1-\cos t}{t}\right\}$

Solution: $L\left\{\frac{1-\cos t}{t}\right\}=\int_s^\infty L[1-\cos t]ds$ since

$$\lim_{t\to 0}\frac{1-\cos t}{t}=\frac{0}{0}=\lim_{t\to 0}\frac{\sin t}{1}=0$$

$$L\left\{\frac{1-\cos t}{t}\right\}=\int_s^\infty L[1-\cos t]ds$$

$$=\int_s^\infty [L(1)-L(\cos t)]ds$$

$$=\int_s^\infty \left\{\frac{1}{s}-\frac{s}{s^2+1}\right\}ds=\left\{\log s-\frac{1}{2}\log(s^2+1)\right\}_s^\infty$$

$$=\log\left(\frac{s}{\sqrt{s^2+1}}\right)_s^\infty=\log\left\{\frac{\sqrt{s^2+1}}{s}\right\}$$

4) Find $L\left\{\frac{\sin^2 t}{t}\right\}$

Solution: Consider $L\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty L\{\sin^2 t\}ds$

since $\lim_{t\to 0}\frac{\sin^2 t}{t} = \underset{t\to 0}{Lim}\frac{1-\cos 2t}{t} = \frac{0}{0} = \underset{t\to 0}{Lim}\frac{2\sin 2t}{1} = 0$

Then $L\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty L\{\sin^2 t\}ds = \int_s^\infty L\left(\frac{1-\cos 2t}{2}\right)ds = \frac{1}{2}\int_s^\infty [L(1) - L(\cos 2t)]ds$

$$= \frac{1}{2}\int_s^\infty \left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}ds = \frac{1}{2}\left\{\log s - \frac{1}{2}\log(s^2+4)\right\}_s^\infty$$

$$= \frac{1}{2}\log\left\{\frac{s}{\sqrt{s^2+4}}\right\}_s^\infty = \frac{1}{2}\log\left\{\frac{\sqrt{s^2+4}}{s}\right\}$$

5) Find $L\left\{\frac{e^{-t}-e^{-2t}}{t}\right\}$

Solution: Consider $L\left\{\frac{e^{-t}-e^{-2t}}{t}\right\} = \int_s^\infty L[e^{-t}-e^{-2t}]ds$

since $\underset{t\to 0}{Lim}\frac{e^{-t}-e^{-2t}}{t} = -3$

$$L\left\{\frac{e^{-t}-e^{-2t}}{t}\right\} = \int_s^\infty L[e^{-t}-e^{-2t}]ds = \int_s^\infty [L(e^{-t}) - L(e^{-2t})]ds$$

$$= \int_s^\infty \left\{\frac{1}{s+1} - \frac{1}{s+2}\right\}ds = \log\left\{\frac{s+1}{s+2}\right\}_s^\infty = \log\left\{\frac{s+2}{s+1}\right\}$$

6) Find $L\left(\frac{\cos 2t - \cos 3t}{t}\right)$

Solution: Consider $L\left(\frac{\cos 2t - \cos 3t}{t}\right) = \int_s^\infty [L(\cos 2t) - L(\cos 3t)]ds$

since $\lim_{t\to 0}\left(\frac{\cos 2t - \cos 3t}{t}\right) = 0$

$$L\left(\frac{\cos 2t - \cos 3t}{t}\right) = \int_s^\infty [L(\cos 2t) - L(\cos 3t)]ds = \int_s^\infty \left\{\frac{s}{s^2+4} - \frac{s}{s^2+9}\right\}ds$$

$$=\left\{\frac{1}{2}\log(s^2+4)-\frac{1}{2}\log(s^2+9)\right\}_s^\infty=\frac{1}{2}\log\left\{\frac{s^2+4}{s^2+9}\right\}_s^\infty=\frac{1}{2}\log\left\{\frac{s^2+9}{s^2+4}\right\}$$

Result: Using Laplace transform, we can evaluate certain integrals .

1) Evaluate $\int_0^\infty e^{-2t}\sin 3t\,dt$

Solution: By definition of Laplace transform, we have

$$\int_0^\infty e^{-st}\sin 3t\,dt = L(\sin 3t)=\frac{3}{s^2+9}$$

Putting $s=2$ .we get $\int_0^\infty e^{-2t}\sin 3t\,dt=\frac{3}{4+9}=\frac{3}{13}$

2) Evaluate $\int_0^\infty te^{-3t}\cos t\,dt$

Solution: We have $L(t\cos t)=\int_0^\infty te^{-st}\cos t\,dt$

Hence the required integral is the value of $L(t\cos t)$ when $s=3$

Consider

$$L(t\cos t)=-\frac{d}{ds}\big(L(\cos t)\big)=-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)=-\left(\frac{(s^2+1).1-s(2s)}{(s^2+1)^2}\right)=\frac{(s^2-1)}{(s^2+1)^2}$$

Hence the required integral is obtained by putting $s=3$

$$\int_0^\infty te^{-3t}\cos t\,dt=\frac{9-1}{(9+1)^2}=\frac{8}{100}=\frac{2}{25}.$$

3) Evaluate $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}dt$

Solution: Consider

$$\int_0^\infty e^{-st}\left(\frac{e^{-t}-e^{-2t}}{t}\right)dt=L\left(\frac{e^{-t}-e^{-2t}}{t}\right)=\int_s^\infty[L(e^{-t})-L(e^{-2t})]ds$$

$$=\int_s^\infty\left(\frac{1}{s+1}-\frac{1}{s+2}\right)ds=\log\left(\frac{s+2}{s+1}\right)$$

Hence the required integral is obtained by putting $s=0$

$$\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}dt=\log\left(\frac{2}{1}\right)=\log 2$$

4) Evaluate $\int_0^\infty \frac{e^{-3t}-e^{-6t}}{t}dt$

Solution: Consider

$$\int_0^\infty e^{-st}\left(\frac{e^{-3t}-e^{-6t}}{t}\right)dt = L\left(\frac{e^{-3t}-e^{-6t}}{t}\right) = \int_s^\infty [L(e^{-3t})-L(e^{-6t})]ds$$

$$= \int_s^\infty \left(\frac{1}{s+3}-\frac{1}{s+6}\right)ds = \log\left(\frac{s+3}{s+6}\right)_s^\infty = \log\left(\frac{s+6}{s+3}\right)$$

Hence the required integral is obtained by putting $s=0$

$$\int_0^\infty \frac{e^{-3t}-e^{-6t}}{t}dt = \log\left(\frac{6}{3}\right) = \log 2$$

5) Evaluate $\int_0^\infty te^{-3t}\sin t\,dt$

Solution: We have $L(t\sin t) = \int_0^\infty te^{-st}\sin t\,dt$

Hence the required integral is the value of $L(t\cos t)$ when $s=3$

Consider

$$L(t\sin t) = -\frac{d}{ds}\big(L(\sin t)\big) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\left(\frac{(s^2+1).0-1(2s)}{(s^2+1)^2}\right) = \frac{2s}{(s^2+1)^2}$$

Hence the required integral is obtained by putting $s=3$

$$\int_0^\infty te^{-3t}\sin t\,dt = \frac{2(3)}{(9+1)^2} = \frac{6}{100} = \frac{3}{50}.$$

INVERSE LAPLACE TRANSFORMS

Let the $L^{-1}\{F(s)\}$ denote a function, whose Laplace transform is F(s).Thus if $L\{ft)\} = F(s)$ then $f(t) = L^{-1}\{F(s)\}$

We can complete the table of transforms from the known results:

S.No	**f(t)**	**F(s)**
1	e^{at}	$\frac{1}{s-a}$

2	$\cosh at$	$\frac{s}{s^2-a^2}$
3	$\sinh at$	$\frac{a}{s^2-a^2}$
4	$\cos at$	$\frac{s}{s^2+a^2}$
5	$\sin at$	$\frac{a}{s^2+a^2}$
6	1	$\frac{1}{s}$
7	t	$\frac{1}{s^2}$
8	t^n	$\frac{n!}{s^{n+1}}$ (n is a +ive integer)
9	te^{at}	$\frac{1}{(s-a)^2}$
10	t^2e^{at}	$\frac{2}{(s-a)^3}$
11	t^ne^{at}	$\frac{n!}{(s-a)^{n+1}}$ (n is a positive integer)
12	$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$
S.No	**f(t)**	**F(s)**
13	$e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2+b^2}$
14	$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$

15	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$

Result: 1 If $L\{f(t)\}=F(s)$ then $L\{e^{-at}f(t)\}=F(s+a)$.

Hence we get the result $L^{-1}\{F(s+a)\}=e^{-at}f(t)=e^{-at}L^{-1}(F(s))$

Problems

1) Find $L^{-1}\left\{\frac{1}{(s+a)^2}\right\}$

Solution: Consider

$$L^{-1}\left\{\frac{1}{(s+a)^2}\right\}=e^{-at}L^{-1}\left\{\frac{1}{s^2}\right\}=e^{-at}t$$

2) Find $L^{-1}\left\{\frac{1}{(s+2)^2+16}\right\}$

Solution: Consider

$$L^{-1}\left\{\frac{1}{(s+2)^2+16}\right\}=e^{-2t}L^{-1}\left\{\frac{1}{s^2+4^2}\right\}=e^{-2t}\left\{\frac{\sin 4t}{4}\right\}=\frac{e^{-2ts}\sin 4t}{4}$$

3) Find $L^{-1}\left\{\frac{s-3}{(s-3)^2+4}\right\}$

Solution: Consider

$$L^{-1}\left\{\frac{s-3}{(s-3)^2+4}\right\}=e^{3tt}L^{-1}\left\{\frac{s}{s^2+2^2}\right\}=e^{3t}\cos 2t$$

4) Find $L^{-1}\left\{\frac{s}{s^2+2s+5}\right\}$

Solution: Consider

$$L^{-1}\left\{\frac{s}{s^2+2s+5}\right\}=L^{-1}\left\{\frac{s}{(s+1)^2+4}\right\}=L^{-1}\left\{\frac{(s+1)-1}{s+1)^2+4}\right\}$$

$$=L^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\}-L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$$=e^{-t}L^{-1}\left\{\frac{s}{s^2+2^2}\right\}-e^{-t}L^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

$$=e^{-t}\cos 2t-e^{-2t}\left(\frac{\sin 2t}{2}\right)=\frac{e^{-t}}{2}(2\cos 2t-\sin 2t)$$

5) Find $L^{-1}\left\{\frac{1}{(s-3)^2}\right\}$

Solution: Consider

$$L^{-1}\left\{\frac{1}{(s-3)^2}\right\}=e^{3t}L^{-1}\left\{\frac{1}{s^2}\right\}=e^{3t}t$$

6) Find $L^{-1}\left\{\frac{s}{(s-b)^2+a^2}\right\}$

Solution: Consider

$$L^{-1}\left\{\frac{s}{(s-b)^2+a^2}\right\}=L^{-1}\left\{\frac{(s-b)+b}{(s-b)^2+a^2}\right\}$$

$$=L^{-1}\left\{\frac{s-b}{(s-b)^2+a^2}\right\}+L^{-1}\left\{\frac{b}{(s-b)^2+a^2}\right\}$$

$$=e^{bt}L^{-1}\left\{\frac{s}{s^2+a^2}\right\}+e^{bt}bL^{-1}\left(\frac{1}{s^2+a^2}\right)$$

$$=e^{bt}\cos at+e^{bt}b\left(\frac{\sin at}{a}\right)=\frac{e^{bt}}{a}\left(a\cos bt+b\sin bt\right)$$

7) Find $L^{-1}\left\{\frac{cs+d}{(s+a)^2+b^2}\right\}$

Solution: Consider

$$L^{-1}\left\{\frac{cs}{(s+a)^2+b^2}\right\} = L^{-1}\left\{\frac{c(s+a)-ca}{(s+a)^2+b^2}\right\}$$

$$= L^{-1}\left\{\frac{c(s+a)}{(s+a)^2+b^2}\right\} - cL^{-1}\left\{\frac{a}{(s+a)^2+b^2}\right\}$$

$$= e^{-at}cL^{-1}\left\{\frac{s}{s^2+b^2}\right\} - e^{-at}acL^{-1}\left(\frac{1}{s^2+b^2}\right)$$

$$= e^{-at}c\cos bt - ce^{-at}a\left(\frac{\sin bt}{b}\right)$$

$$= e^{-at}c\cos bt - e^{-at}ac\left(\frac{\sin bt}{b}\right) = e^{-at}c\left\{\frac{b\cos at - a\sin bt}{b}\right\}$$

Result:2 If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

This result can be written in the form $L^{-1}\left\{\frac{1}{a}F\left(\frac{s}{a}\right)\right\} = f(at)$ where

$f(t) = L^{-1}\{F(s)\}$

Put $\frac{1}{a} = k$, we have

$L^{-1}\{F(ks)\} = \frac{1}{k}f\left(\frac{t}{k}\right)$ where $f(t) = L^{-1}\{F(s)\}$.

Problems:

1) Find $L^{-1}\left\{\frac{s}{s^2a^2+b^2}\right\}$

Solution: Consider $\frac{s}{s^2a^2+b^2} = \frac{1}{a}\left(\frac{sa}{s^2a^2+b^2}\right) = \frac{1}{a}F(sa)$

Where $F(sa) = \frac{sa}{s^2a^2+b^2}$ and therefore $F(s) = \frac{s}{s^2+b^2}$

$$L^{-1}\left\{\frac{s}{s^2a^2+b^2}\right\} = \frac{1}{a}L^{-1}\left\{\frac{sa}{s^2a^2+b^2}\right\} = \frac{1}{a}L^{-1}\{F(sa)\} = \frac{1}{a}\frac{1}{a}f\left(\frac{t}{a}\right)$$

Where $f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt$

$\therefore$ $f\left\{\frac{t}{a}\right\} = \cos\left(\frac{bt}{a}\right)$

Hence $L^{-1}\left\{\frac{s}{s^2a^2+b^2}\right\} = \frac{1}{a^2}\cos\left(\frac{bt}{a}\right)$

Result:3 If $L\{f(t)\} = F(s)$ then $L\{t\,f(t)\} = -F'(s)$.

Hence we get the result $L^{-1}\{F'(s)\} = -t\,f(t) = -t\,L^{-1}\{F(s)\}$

Problems

1) Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Solution: Let $F'(s) = \frac{s}{(s^2+a^2)^2}$

$\therefore$ $F(s) = \int \frac{s}{(s^2+a^2)^2}\,ds = -\frac{1}{2(s^2+a^2)}$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -t\,L^{-1}\left\{-\frac{1}{2(s^2+a^2)}\right\} = \frac{t}{2}L^{-1}\left(\frac{1}{(s^2+a^2)}\right)$$

$$= \frac{t}{2a}L^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{t}{2a}\sin at$$

2) Find $L^{-1}\left\{\frac{s}{(s^2-1)^2}\right\}$

Solution: Let $F'(s) = \frac{s}{(s^2-1)^2}$

$\therefore$ $F(s) = \int \frac{s}{(s^2-1)^2}\,ds = -\frac{1}{2(s^2-1)}$

$$\therefore\ L^{-1}\left\{\frac{s}{(s^2-1)^2}\right\} = -t\,L^{-1}\left\{-\frac{1}{2(s^2-1)}\right\} = \frac{t}{2}L^{-1}\left(\frac{1}{(s^2-1)}\right)$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{s^2-1}\right) = \frac{t}{2}\sinh t$$

3) Find $L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$

Solution: Let $F'(s) = \frac{s}{(s^2+4)^2}$

$$\therefore\ F(s) = \int \frac{s}{(s^2+4)^2}\,ds = -\frac{1}{2(s^2+4)}$$

$$\therefore\ L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = -t\,L^{-1}\left\{-\frac{1}{2(s^2+4)}\right\} = \frac{t}{2}L^{-1}\left(\frac{1}{(s^2+4)}\right)$$

$$= \frac{t}{4}L^{-1}\left(\frac{2}{s^2+2^2}\right) = \frac{t}{4}\sinh\ 2t$$

4) Find $L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}$

Solution: Let $F'(s) = \frac{s+2}{(s^2+4s+5)^2}$

$$\therefore\ F(s) = \int \frac{s+2}{(s^2+4s+5)^2}\,ds = -\frac{1}{2(s^2+4s+5)}$$

$$\therefore\ L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = -t\,L^{-1}\left\{-\frac{1}{2(s^2+4s+5)}\right\} = \frac{t}{2}L^{-1}\left(\frac{1}{(s^2+4s+5)}\right)$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{(s+2)^2+1}\right) = \frac{t}{2}e^{-2t}L^{-1}\left\{\frac{1}{s^2+1^2}\right\} = \frac{te^{-2t}\sin t}{2}$$

5) Find $L^{-1}\left\{\frac{s+3}{(s^2+6s+13)^2}\right\}$

Solution: Let $F'(s) = \frac{s+3}{(s^2+6s+13)^2}$

$$\therefore \ \mathrm{F(s)} = \int \frac{\mathrm{s}+3}{(\mathrm{s}^2+6s+13)^2}\, ds = -\frac{1}{2(s^2+6s+13)}$$

$$\therefore \ \mathrm{L}^{-1}\left\{\frac{s+3}{(s^2+6s+13)^2}\right\} = -t\,\mathrm{L}^{-1}\left\{-\frac{1}{2(s^2+6s+13)}\right\} = \frac{t}{2}L^{-1}\left(\frac{1}{(s^2+6s+13)}\right)$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{(s+3)^2+4}\right) = \frac{t}{2}e^{-3t}L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{te^{-2t}\sin 2t}{4}$$

6) Find $L^{-1}\left\{\frac{2(s+1)}{(s^2+2s+2)^2}\right\}$

Solution: Let $F^{'}(s) = \frac{2(s+1)}{(s^2+2s+2)^2}$

$$\therefore \ \mathrm{F(s)} = \int \frac{2(\mathrm{s}+1)}{(\mathrm{s}^2+2s+2)^2}\, ds = -\frac{1}{(s^2+2s+2)}$$

$$\therefore \ \mathrm{L}^{-1}\left\{\frac{2(s+1)}{(s^2+2s+2)^2}\right\} = -t\,\mathrm{L}^{-1}\left\{-\frac{1}{(s^2+2s+2)}\right\} = tL^{-1}\left(\frac{1}{(s^2+2s+2)}\right)$$

$$= tL^{-1}\left(\frac{1}{(s+1)^2+1}\right) = te^{-t}L^{-1}\left\{\frac{1}{s^2+1^2}\right\} = te^{-t}\sin t$$

Result:4 If $L\{f(t)\} = F(s)$ then $L\{t\,\mathrm{f(t)}\} = -F^{'}(s)$.This theorem can be used to get inverse transform of certain functions:

Problem:

1) Find $L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}$

Solution: Let $L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\} = f(t)$

Then $L\{f(t)\} = \log\left(\frac{s+1}{s-1}\right)$

$$\therefore \ \mathrm{L}\{\mathrm{t\ f(t)}\} = -\frac{d}{ds}\log\left(\frac{s+1}{s-1}\right) = -\frac{d}{ds}\left[\log(s+1) - \log(s-1)\right]$$

$$= -\frac{1}{s+1} + \frac{1}{s-1}$$

$$\therefore \text{ t f(t) } = \text{L}^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s+1}\right) = e^{t} - e^{-t} = 2\sinh t$$

$$\therefore \text{f(t)} = \frac{\text{2sinht}}{\text{t}}$$

2) Find $L^{-1}\left\{\log\left(\frac{s+1}{s}\right)\right\}$

Solution: Let

$$L^{-1}\left\{\log\left(\frac{s+1}{s}\right)\right\} = f(t)$$

Then $L\{f(t)\} = \log\left(\frac{s+1}{s}\right)$

$$\therefore \text{ L}\{\text{t f(t)}\} = -\frac{d}{ds}\log\left(\frac{s+1}{s}\right) = -\frac{d}{ds}\left[\log(s+1) - \log s\right]$$

$$= -\frac{1}{s+1} + \frac{1}{s}$$

$$\therefore \text{ t f(t) } = \text{L}^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) = 1 - e^{-t}$$

$$\therefore \text{f(t)} = \frac{1-e^{-t}}{t}$$

3) Find $L^{-1}\left\{\log\left(\frac{s}{s^{2}+1}\right)\right\}$

Solution: Let $L^{-1}\left\{\log\left(\frac{s}{s^{2}+1}\right)\right\} = f(t)$

Then $L\{f(t)\} = \log\left(\frac{s}{s^{2}+1}\right)$

$$\therefore \; L\{t\, f(t)\} = -\frac{d}{ds}\log\left(\frac{s}{s^2+1}\right) = -\frac{d}{ds}\left[\log s - \log(s^2+1)\right]$$

$$= -\frac{1}{s} + \frac{2s}{s^2+1}$$

$$\therefore \; t\, f(t) = 2L^{-1}\left(\frac{s}{s^2+1}\right) - L^{-1}\left(\frac{1}{s}\right) = 2\cos t - 1$$

$$\therefore f(t) = \frac{2\cos t - 1}{t}$$

4) Find $L^{-1}\left\{\log\left(\frac{1-s^2}{s^2}\right)\right\}$

Solution: Let $L^{-1}\left\{\log\left(\frac{1-s^2}{s^2}\right)\right\} = f(t)$

Then $L\{f(t)\} = \log\left(\frac{1-s^2}{s^2}\right)$

$$\therefore \; L\{t\, f(t)\} = -\frac{d}{ds}\log\left(\frac{1-s^2}{s^2}\right) = -\frac{d}{ds}\left[\log(1-s^2) - \log(s^2)\right]$$

$$= -\left(\frac{-2s}{1-s^2}\right) + \frac{2s}{s^2} = \left(\frac{2s}{1-s^2}\right) + \frac{2}{s} = -2\left(\frac{s}{s^2-1}\right) + \frac{2}{s}$$

$$\therefore f(t) = \frac{2(1-\cosh t)}{t}$$

Result:5 If $L\{f(t)\} = sF(s)$ and $\phi(t)$ is a function such that $\phi(t) = F(s)$ and $\phi(0) = 0$ then $f(t) = \phi'(t)$.

We have $L\{\phi'(t)\} = sL\{\phi(t)\} - \phi(0) = sF(s) = L\{f(t)\}$.Therefore , $f(t) = \phi'(t)$

This result can be used to get the inverse transforms of certain functions.

$$\therefore \; L^{-1}\{sF(s) = f(t)\} = \frac{d}{dt}(\phi(t)) = \frac{d}{dt}L^{-1}[F(s)] \text{ provided } L^{-1}[F(s)] = 0 \text{ when } t = 0$$

Problems:

1) Find $L^{-1}\left[\frac{s}{s^2+k^2}\right]$

Solution: Consider $L^{-1}\left[\frac{s}{s^2+k^2}\right]=\frac{d}{dt}L^{-1}\left(\frac{1}{s^2+k^2}\right)=\frac{d}{dt}\left(\frac{\sin kt}{k}\right)=\cos kt$

Here $\frac{\sin kt}{t}=0$ when $t=0$

2) Find $L^{-1}\left\{\frac{s}{(s+3)^2+4}\right\}$

Solution: consider

$$L^{-1}\left\{\frac{s}{(s+3)^2+4}\right\}=\frac{d}{dt}L^{-1}\left[\frac{1}{(s+3)^2+4}\right]=\frac{d}{dt}\left\{e^{-3t}L^{-1}\left[\frac{1}{s^2+4}\right]\right\}$$

$$=\frac{d}{dt}\left(e^{-3t}\left[\frac{\sin 2t}{2}\right]\right)=\frac{d}{dt}\left\{\frac{e^{-3t}\sin 2t}{2}\right\}$$

$$=e^{-3t}\cos 2t-\frac{3}{2}e^{-3t}\sin 2t$$

$$=\frac{e^{-3t}}{2}\left[2\cos 2t-3\sin 2t\right]$$

3) Find $L^{-1}\left\{\frac{s}{(s-b)^2+a^2}\right\}$

Solution: consider

$$L^{-1}\left\{\frac{s}{(s-b)^2+a^2}\right\}=\frac{d}{dt}L^{-1}\left[\frac{1}{(s-b)^2+a^2}\right]=\frac{d}{dt}\left\{e^{bt}L^{-1}\left[\frac{1}{s^2+a^2}\right]\right\}$$

$$=\frac{d}{dt}\left(e^{bt}\left[\frac{\sin at}{a}\right]\right)=\frac{d}{dt}\left\{\frac{e^{bt}\sin at}{a}\right\}$$

$$=e^{btt}\cos at+\frac{b}{a}e^{btt}\sin at$$

$$=\frac{e^{btt}}{a}\left[a\cos bt+b\sin at\right]$$

4) Find $L^{-1}\left[\frac{s-3}{s^2+4s+13}\right]$

Solution: Consider

$$L^{-1}\left[\frac{s-3}{s^2+4s+13}\right]=L^{-1}\left[\frac{s}{s^2+4s+13}\right]-L^{-1}\left[\frac{3}{s^2+4s+13}\right]$$

$$=\frac{d}{dt}L^{-1}\left[\frac{1}{s^2+4s+13}\right]-3L^{-1}\left[\frac{1}{s^2+4s+13}\right]$$

$$=\frac{d}{dt}L^{-1}\left[\frac{1}{(s+2)^2+}\right]-3L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$=\frac{d}{dt}\left[\frac{e^{-2t}\sin 3t}{3}\right]-3\left[\frac{e^{-2t}\sin 3t}{3}\right]$$

$$=\frac{e^{-2t}}{3}\left(3\cos 3t-5\sin 3t\right)$$

5) Find $L^{-1}\left(\frac{s}{(s+2)^2}\right)$

Solution: Consider

$$L^{-1}\left(\frac{s}{(s+2)^2}\right)=\frac{d}{dt}\left(\frac{1}{(s+2)^2}\right)$$

$$=\frac{d}{dt}\left(e^{-2t}L^{-1}\left[\frac{1}{s^2}\right]\right)=\frac{d}{dt}\left(e^{-2t}t\right)$$

$$=e^{-2t}(1-2t)$$

6) Find $L^{-1}\left\{\frac{s}{(s+3)^5}\right\}$

Solution: Consider

$$L^{-1}\left\{\frac{s}{(s+3)^5}\right\} = \frac{d}{dt}L^{-1}\left(\frac{1}{(s+3)^5}\right) = \frac{d}{dt}\left\{e^{-3t}L^{-1}\left(\frac{1}{s^5}\right)\right\}$$

$$= \frac{d}{dt}\left\{e^{-3t}\frac{t^4}{24}\right\} = \frac{1}{24}\frac{d}{dt}\left(e^{-3t}t^4\right) = \frac{1}{24}\left[e^{-3t}4t^3 + t^4e^{-3t}(-3)\right]$$

$$= \frac{e^{-3t}}{24}\left\{4t^3 - 3t^4\right\}$$

7) Find $L^{-1}\left[\frac{s^2}{(s-1)^3}\right]$

Solution:

$$L^{-1}\left[\frac{s^2}{(s-1)^3}\right] = \frac{d}{dt}L^{-1}\left(\frac{s}{(s-1)^3}\right) = \frac{d^2}{dt^2}L^{-1}\left(\frac{1}{(s-1)^3}\right)$$

$$= \frac{d^2}{dt^2}\left(e^tL^{-1}\left(\left[\frac{1}{s^2}\right]\right)\right) = \frac{d^2}{dt^2}\left[\frac{e^tt^2}{2}\right]$$

$$= \frac{e^t}{2}\left(t^2 + 4t + 2\right)$$

Result:6 $L\left[\int_0^t f(x)dx\right] = \frac{1}{s}L\{f(t)\}$ then $L^{-1}\left\{\frac{1}{s}F(s) = \int_0^t L^{-1}\{F(s)\}dt\right\}$

Proof: Let $\int_0^t f(x)dx = F(t)$ .Then $F^{'}(t) = f(t)$ and $F(0) = 0$

$\therefore$ $L\{F^{'}(t)\} = sL\{F(t)\} - F(0) = sL\{F(t)\}$

That is $L\{f(t)\} = sL\left\{\int_0^t f(x)dx\right\}$

Hence $L\left\{\int_0^t f(x)dx\right\} = \frac{1}{s}L\{f(t)\} \Rightarrow \int_0^t f(x)dx = L^{-1}\left[\frac{1}{s}L\{f(t)\}\right]$

This result can be used to find the inverse transforms of certain functions .

If $L\{f(t)\} = F(s)$ then $L^{-1}\left[\frac{1}{s}L\{f(t)\}\right] = \int_0^t f(x)dx$

Where $f(t) = L^{-1}(F(s))$,

$\therefore \; L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}\{F(s)\}dt$

Problem:

1) Find $L^{-1}\left\{\frac{1}{s(s+a)}\right\}$

Solution: Consider

$$L^{-1}\left\{\frac{1}{s(s+a)}\right\} = \int_0^t L^{-1}\left(\frac{1}{s+a}\right)dt$$

$$= \int_0^t e^{-at}dt = \left(\frac{e^{-at}}{-a}\right)_0^t = \frac{1-e^{-at}}{a}$$

2) Find $L^{-1}\left\{\frac{1}{s(s-a)}\right\}$

Solution: Consider

$$L^{-1}\left\{\frac{1}{s(s-a)}\right\} = \int_0^t L^{-1}\left(\frac{1}{s-a}\right)dt$$

$$= \int_0^t e^{at}dt = \left(\frac{e^{at}}{a}\right)_0^t = \frac{e^{at}-1}{a}$$

3) Find $L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\}$

Solution: Consider

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t L^{-1}\left(\frac{1}{s^2+a^2}\right)dt$$

$$= \int_0^t \frac{\sin at}{a}dt = \frac{1}{a}\left(\frac{-\cos at}{a}\right)_0^t = \frac{1-\cos at}{a^2}$$

4) Find $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$

Solution: Consider

$$L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t L^{-1}\left(\frac{1}{s^2+4}\right)dt$$

$$= \int_0^t \frac{\sin\ 2t}{2}\, dt = \frac{1}{2}\left(\frac{-\cos 2t}{2}\right)_0^t = \frac{1-\cos 2t}{4}$$

5) Find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

Solution: Consider

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s}\,\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t L^{-1}\left(\frac{s}{(s^2+a^2)}\right)dt$$

$$= \int_0^t \frac{t\sin\ at}{2a}\, dt = \frac{1}{2a}\left(\frac{-t\cos at}{a} + \frac{\sin\ at}{a^2}\right)_0^t$$

$$= \frac{\sin\ at - at\cos at}{2a^3}$$

6) Find $L^{-1}\left(\frac{1}{s(s+2)^3}\right)$

Solution: Consider $L^{-1}\left(\frac{1}{s(s+2)^3}\right) = \int_0^t L^{-1}\left(\frac{1}{(s+2)^3}\right)dt$

$$= \int_0^t e^{-2t} L^{-1}\left(\frac{1}{s^5}\right)dt = \int_0^t e^{-2t}\left(\frac{t^4}{24}\right)dt$$

$$= \frac{1}{24}\int_0^t e^{-2t} t^4 dt$$

Using Bernoulli's formula

$$\int u du = uv - u^{'} v_1 + u^{''} v_2 - \ldots\ldots$$

Take $u = t^4, \quad dv = \int e^{-2t} dt$

Then $u' = 4t^3, \; v = -\dfrac{e^{-2t}}{2}$

$$u'' = 12t^2, \quad v_1 = \frac{e^{-2t}}{2^2}$$

$$u''' = 24t, \quad v_2 = -\frac{e^{-2t}}{2^3}$$

$$u'''' = 24, \quad v_3 = \frac{e^{-2t}}{2^4}$$

$$u''''' = 0, \quad v_4 = -\frac{e^{-2t}}{2^5}$$

$$\frac{1}{24}\int_0^t t^4 e^{-2t} dt = \frac{1}{24}\left\{-t^4\frac{e^{-2t}}{2} - 4t^3\frac{e^{-2}}{2^2} - 12t^2\frac{e^{-2t}}{8} - 24t\frac{e^{-2t}}{16} - 24\frac{e^{-2t}}{32}\right\}_0^t$$

$$= \frac{1}{24}\left\{-t^4\frac{e^{-2t}}{2} - 4t^3\frac{e^{-2}}{2^2} - 12t^2\frac{e^{-2t}}{8} - 24t\frac{e^{-2t}}{16} - 24\frac{e^{-2t}}{32}\right\} - \frac{1}{24}\left(\frac{-24}{32}\right)$$

$$= \frac{1}{24}\left\{-\frac{t^4e^{-2t}}{2} - t^3e^{-2t} - \frac{3t^2e^{-2t}}{4} - \frac{3te^{-2t}}{2} - \frac{3e^{-2t}}{4}\right\} + \left(\frac{1}{32}\right)$$

Result: 7 The method of partial fraction can be used to find the inverse transform of certain functions.

Problems:

1) Find $L^{-1}\left\{\dfrac{1}{s(s+1)(s+2)}\right\}$

Solution: Consider

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$= \frac{A(s+1)(s+2) + Bs(s+2) + Css+1)}{s(s+1)(s+2)}$$

So that $1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$

Putting $s = 0$ we get $1 = 2A \Rightarrow A = \frac{1}{2}$

Putting $s = -1$,we get $1 = -B \Rightarrow B = -1$

Putting $s = -2$,we get $1 = 2C \Rightarrow C = \frac{1}{2}$

Hence $\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = L^{-1}\left(\frac{1}{2s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left[\frac{1}{2(s+2)}\right]$$

$$= \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}L^{-1}\left[\frac{1}{(s+2)}\right]$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

2) Find $L^{-1}\left\{\frac{s^2 - s + 2}{s(s-3)(s+2)}\right\}$

Solution: Consider

$$\frac{s^2 - s + 2}{s(s-3)(s+2)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s+2}$$

$$= \frac{A(s-3)(s+2) + Bs(s+2) + Cs(s-3)}{s(s-3)(s+2)}$$

So that $s^2 - s + 2 = A(s-3)(s+2) + Bs(s+2) + Cs(s-3)$

Putting $s = 0$ we get $2 = -6A \Rightarrow A = -\frac{1}{3}$

Putting $s = 3$,we get $8 = 15B \Rightarrow B = \frac{8}{15}$

Putting $s=-2$, we get $8=10C \Rightarrow C=\frac{8}{10}-=\frac{4}{5}$

Hence $\frac{s^2-s+2}{s(s-3)(s+2)}=-\frac{1}{3s}+\frac{8}{15}\frac{1}{s-3}+\frac{4}{5(s+2)}$

$$L^{-1}\left[\frac{s^2-s+2}{s(s-3)(s+2)}\right]=-L^{-1}\left(\frac{1}{3s}\right)+\frac{8}{15}L^{-1}\left(\frac{1}{s-3}\right)+\frac{4}{5}L^{-1}\left[\frac{1}{(s+2)}\right]$$

$$=-\frac{1}{3}L^{-1}\left(\frac{1}{s}\right)+\frac{8}{15}L^{-1}\left(\frac{1}{s-3}\right)+\frac{4}{5}L^{-1}\left[\frac{1}{(s+2)}\right]$$

$$=-\frac{1}{3}+\frac{8}{15}e^{3t}+\frac{4}{5}e^{-2t}$$

3) Find $L^{-1}\left\{\frac{1}{(s+1)(s^2+2s+2)}\right\}$

Solution: Consider $\frac{1}{(s+1)(s^2+2s+2)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+2s+2}$

By putting suitable values to s, we can find the values of A, B and C as

$A=1, B=-1 \,\&\, C=-1$

so that $\frac{1}{(s+1)(s^2+2s+2)}=\frac{1}{s+1}-\frac{s+1}{s^2+2s+2}$

Hence $L^{-1}\left\{\frac{1}{(s+1)(s^2+2s+2)}\right\}=L^{-1}\left(\frac{1}{s+1}\right)-L^{-1}\left(\frac{s+1}{s^2+2s+2}\right)$

$$=e^{-t}-L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right)$$

$$=e^{-t}-e^{-t}L^{-1}\left(\frac{s}{s^2+1}\right)$$

$$=e^{-t}-e^{-t}\cos t=e^{-t}(1-\cos t)$$

4) Find $L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

Solution: Consider $\dfrac{5s+3}{(s-1)(s^2+2s+5)}=\dfrac{A}{s-1}+\dfrac{Bs+C}{s^2+2s+5}$

$$5s+3=A(s^2+2s+5)+(Bs+C)(s-1)$$

Putting $s=1$,we get $A=1$

Equating the coefficient of s^2,we get $A+B=0\Rightarrow B=-A\Rightarrow B=-1$

Equating the coefficient of s,we get $2A-B+C=5\Rightarrow C=5-2A+B\Rightarrow C=2$

so that $\dfrac{5s+3}{(s-1)(s^2+2s+5)}=\dfrac{1}{s-1}+\dfrac{-s+2}{s^2+2s+5}$

Hence $L^{-1}\left\{\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right\}=L^{-1}\left(\dfrac{1}{s-1}\right)+L^{-1}\left(\dfrac{-s+2}{s^2+2s+2}\right)$

$$=e^t-L^{-1}\left(\frac{s-2}{(s+1)^2+4}\right)=e^t-L^{-1}\left(\frac{(s+1)-3}{(s+1)^2+4}\right)$$

$$=e^t-e^{-t}L^{-1}\left(\frac{s}{s^2+4}\right)+3e^{-t}L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$=e^t-e^{-t}\cos 2t+3e^{-t}\left(\frac{\sin 2t}{2}\right)$$

$$=e^t-e^{-t}\cos 2t+\frac{3e^{-t}\sin 2t}{2}$$

5) Find $L^{-1}\left\{\dfrac{1+2s}{(s+2)^2(s-1)^2}\right\}$

Solution: Consider

$$(s+2)^2-(s-1)^2=s^2+4s+4-s^2+2s-1=6s+3=3(1+2s)$$

$$\Rightarrow 1+2s=\frac{1}{3}\left\{(s+2)^2-(s-1)^2\right\}$$

Consider

$$\frac{1+2s}{(s+2)^2(s-1)^2}=\frac{1}{3}\left\{\frac{(s+2)^2-(s-1)^2}{(s+2)^2(s-1)^2}\right\}$$

$$= \frac{1}{3}\left\{\frac{1}{(s-1)^2} - \frac{1}{(s+2)^2}\right\}$$

Hence $L^{-1}\left\{\frac{1+2s}{(s+2)^2(s-1)^2}\right\} = \frac{1}{3}\left\{L^{-1}\left(\frac{1}{(s-1)^2}\right) - L^{-1}\left(\frac{1}{(s+2)^2}\right)\right\}$

$$= \frac{1}{3}\left\{e^t L^{-1}\left(\frac{1}{s^2}\right) - e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)\right\}$$

$$= \frac{1}{3}\left\{e^t t - e^{-2t}t\right\} = \frac{t}{3}\left[e^t - e^{-2t}\right]$$

6) Find $L^{-1}\left\{\frac{2s-1}{s^2(s-1)^2}\right\}$

Solution: Consider

$$(s-1)^2 - s^2 = s^2 - 2s + 1 - s^2 = -2s + 1 = -(2s-1)$$

$$\Rightarrow 2s - 1 = -\left\{(s-1)^2 - s^2\right\}$$

Consider

$$\frac{2s-1}{s^2(s-1)^2} = -\left\{\frac{(s-1)^2 - s^2}{s^2(s-1)^2}\right\}$$

$$= -\left\{\frac{1}{s^2} - \frac{1}{(s-1)^2}\right\} = -\frac{1}{s^2} + \frac{1}{(s-1)^2}$$

$$= -t + e^t L^{-1}\left(\frac{1}{s^2}\right) = -t + e^t t = t(e^t - 1)$$

Result: Laplace transformation can be used to solve ordinary differential equations with constant coefficients.

Problems

1) Solve the equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$ given that $y = \frac{dy}{dt} = 0$ when $t = 0$

Solution : The equation can be written in the form

$$y'' + 2y' - 3y = \sin t$$

Applying Laplace transform to both sides, we have

$$L\left(y'' + 2y' - 3y\right) = L(\sin t)$$

i.e ., $L(y'') + 2L(y') - 3L(y) = L(\sin t)$

i.e ., $s^2L(y) - sy(0) - y'(0) + 2\{sL(y) - y(0)\} - 3L(y) = \dfrac{1}{s^2+1}$

substituting the values of $y(0)$ and $y'(0)$ in the equation ,we get

$$s^2L(y) + 2sL(y) - 3L(y) = \frac{1}{s^2+1}$$

Putting $L(y) = \bar{y}$,

$$(s^2 + 2s - 3)\bar{y} = \frac{1}{s^2+1}$$

$$\bar{y} = \frac{1}{(s^2+2s-3)(s^2+1)} = \frac{1}{(s+3)(s-1)(s^2+1)}$$

$$\therefore\ y = L^{-1}\left\{\frac{1}{(s-1)(s+3)(s^2+1)}\right\}$$

Consider $\dfrac{1}{(s-1)(s+3)(s^2+1)} = \dfrac{A}{s+3} + \dfrac{B}{s-1} + \dfrac{Cs+D}{s^2+1}$

By giving suitable values to s we can find $A = -\dfrac{1}{40}, B = \dfrac{1}{8}$ and $C = -\dfrac{1}{10}$ & $D = -\dfrac{1}{5}$

Hence

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{-\frac{1}{40}}{s-1} + \frac{\frac{1}{8}}{s+3} + \frac{-\frac{1}{10}s - \frac{1}{5}}{s^2+1}$$

So that

$$y = -\frac{1}{40}L^{-1}\left(\frac{1}{s+3}\right) + \frac{1}{8}L^{-1}\left(\frac{1}{s-1}\right) - \frac{1}{10}L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{5}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$\therefore y = -\frac{1}{40}e^{-3t} + \frac{1}{8}e^{t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t$$

2) Solve the equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 4e^{-t}$ given that $y(0) = y(0) =' 0$

Solution : The equation can be written in the form

$$y'' + 2y' + 5y = 4e^{-t}$$

Applying Laplace transform to both sides, we have

$$L\left(y'' + 2y' + 5y\right) = L(4e^{-t})$$

i.e . , $L(y'') + 2L(y') + 5L(y) = 4L(e^{-t})$

i.e . , $s^2L(y) - sy(0) - y'(0) + 2\{sL(y) - y(0)\} + 5L(y) = \frac{4}{s+1}$

substituting the values of $y(0)$ and $y'(0)$ in the equation ,we get

$$s^2L(y) + 2sL(y) + 5L(y) = \frac{4}{s+1}$$

Putting $L(y) = \bar{y}$,

$$(s^2 + 2s + 5)\bar{y} = \frac{4}{s+1}$$

$$\bar{y} = \frac{4}{(s^2+2s+5)(s+1)} = \frac{4}{(s^2+2s+5)(s+1)}$$

$$\therefore \ y = L^{-1}\left\{\frac{1}{(s+1)(s^2+2s+5)}\right\}$$

Consider $\frac{1}{(s^2+2s+5)(s+1)} = \frac{As+B}{s^2+2s+5} + \frac{C}{s+1}$

$$4 = (As+B)(s+1) + C(s^2+2s+5)$$

Putting $s=-1$,we get $4=4C \Rightarrow C=1$

Equating the coefficient of s^2, $A+C=0 \Rightarrow A=-C \Rightarrow A=-1$

Equating the coefficient of s, $A+B+2C=0 \Rightarrow B=-A-2C=1-2 \Rightarrow B=-1$

Hence $\frac{1}{(s^2+2s+5)(s+1)} = \frac{-s-1}{s^2+2s+5} + \frac{1}{s+1}$

So that

$$y = L^{-1}\left(\frac{-s-1}{s^2+2s+5}\right) + L^{-1}\left(\frac{1}{s+1}\right) = -L^{-1}\left\{\frac{(s+1)}{(s+1)^2+4}\right\} + L^{-1}\left(\frac{1}{s+1}\right)$$

$$= -e^{-t}L^{-1}\left(\frac{s}{s^2+4}\right) + e^{-t} = -e^{-t}\cos 2t + e^{-t} = e^{-t}(1-\cos 2t)$$

3) Solve the equation $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 5$ given that $y(0)=0, y'(0)=2$ using Laplace transforms.

Solution : The equation can be written in the form

$$y'' + 4y' - 5y = 5$$

Applying Laplace transform to both sides, we have

$$L(y'' + 4y' - 5y) = \text{L}(5)$$

i.e . , $L(y'') + 4L(y') - 5L(y) = 5L(1)$

i.e . , $s^2L(y) - sy(0) - y'(0) + 4\{sL(y) - y(0)\} - 5L(y) = \frac{5}{s}$

substituting the values of $y(0)$ and $y'(0)$ in the equation ,we get

$$s^2L(y) + 4sL(y) - 5L(y) - 2 = \frac{5}{s}$$

Putting $L(y) = \bar{y}$,

$$(s^2+4s-5)\bar{y} = \frac{5}{s} + 2 = \frac{2s+5}{s}$$

$$\bar{y} = \frac{2s+5}{(s^2+4s-5)s} = \frac{2s+5}{s(s+5)(s-1)}$$

$$\therefore\ y = L^{-1}\left\{\frac{2s+5}{s(s+5)(s-1)}\right\}$$

Consider $\frac{2s+5}{s(s+5)(s-1)} = \frac{A}{s} + \frac{B}{s+5} + \frac{C}{s-1}$

$$2s+5 = A(s+5)(s-1) + Bs(s-1) + Cs(s+5)$$

Putting $s=0$, we get $5 = -5A \Rightarrow A = -1$

Putting $s=-5$, we get $-5 == 30B \Rightarrow B = -\frac{1}{6}$

Putting $s=1$, we get $7 = 6C \Rightarrow C = \frac{7}{6}$

Hence $\frac{2s+5}{s(s+5)(s-1)} = \frac{-1}{s} - \frac{\frac{1}{6}}{s+5} + \frac{\frac{7}{6}}{s-1} = -\frac{1}{s} - \frac{1}{6}\left(\frac{1}{s+5}\right) + \frac{7}{6}\left(\frac{1}{s-1}\right)$

$$\therefore\ y = L^{-1}\left\{\frac{2s+5}{s(s+5)(s-1)}\right\}$$

$$= -L^{-1}\left(\frac{1}{s}\right) - \frac{1}{6}L^{-1}\left(\frac{1}{s+5}\right) + \frac{7}{6}L^{-1}\left(\frac{1}{s-1}\right)$$

$$= -1 - \frac{1}{6}e^{-5t} + \frac{7}{6}e^{t}$$

4) Solve the equation $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 24y = 24x$ given that $y(0) = 0 = y'(0)$ using Laplace transforms.

Solution : The equation can be written in the form

$$y'' - 10y' + 24y = 24x$$

Applying Laplace transform to both sides, we have

$$L\left(y'' - 10y' + 24y\right) = \mathrm{L}(24\mathrm{x}) \Rightarrow L(y'') - 10L(y') + 24L(y) = 24L(x)$$

i.e . , $s^2L(y) - sy(0) - y'(0) - 10\{sL(y) - y(0)\} + 24L(y) = \frac{24}{s^2}$

substituting the values of $y(0)$ and $y'(0)$ in the equation ,we get

$$s^2L(y) - 10sL(y) + 24L(y) = \frac{24}{s^2}$$

Putting $L(y) = \bar{y}$,

$$(s^2 - 10s + 24)\bar{y} = \frac{24}{s^2}$$

$$\bar{y} = \frac{24}{s^2(s^2 - 10s + 24)} = \frac{24}{s^2(s-4)(s-6)}$$

$$\therefore \ \mathrm{y} = \mathrm{L}^{-1}\left\{\frac{24}{s^2(s-4)(s-6)}\right\}$$

Taking into partial fraction $\frac{24}{s^2(s-4)(s-6)} = \frac{As+B}{s^2} + \frac{C}{s-4} + \frac{D}{s-6}$

$$24 = (As+B)(s-4)(s-6) + Cs^2(s-6) + Ds^2(s-4)$$

Putting $s = 0$,we get $24 = 24B \Rightarrow B = 1$

Putting $s = 6$,we get $24 = 72D \Rightarrow D = \frac{1}{3}$

Putting $s = 4$,we get $24 = -32C \Rightarrow C = -\frac{3}{4}$

Equating the coefficient of s^2 we get

$$A + C + D = 0 \Rightarrow A = -C - D \Rightarrow A = \frac{5}{12}$$

Hence $\frac{24}{s^2(s-4)(s-6)} = \frac{\left(\frac{5}{12}\right)s + 1}{s^2} + \frac{\left(-\frac{3}{4}\right)}{s-4} + \frac{\frac{1}{3}}{s-6}$

$$= \frac{5}{12}\frac{1}{s} + \frac{1}{s^2} - \frac{3}{4}\left(\frac{1}{s-4}\right) + \frac{1}{3}\left(\frac{1}{s-6}\right)$$

$$\therefore \ \mathrm{y} = \mathrm{L}^{-1}\left\{\frac{24}{s^2(s-4)(s-6)}\right\} = \frac{5}{12}L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s^2}\right) - \frac{3}{4}L^{-1}\left(\frac{1}{s-4}\right) + \frac{1}{3}L^{-1}\left(\frac{1}{s-6}\right)$$

$$y = \frac{5}{12}.1 + x - \frac{3}{4}e^{4x} + \frac{1}{3}e^{6x} = \frac{5}{12} + x - \frac{3}{4}e^{4x} + \frac{1}{3}e^{6x}$$