CHAPTER - I

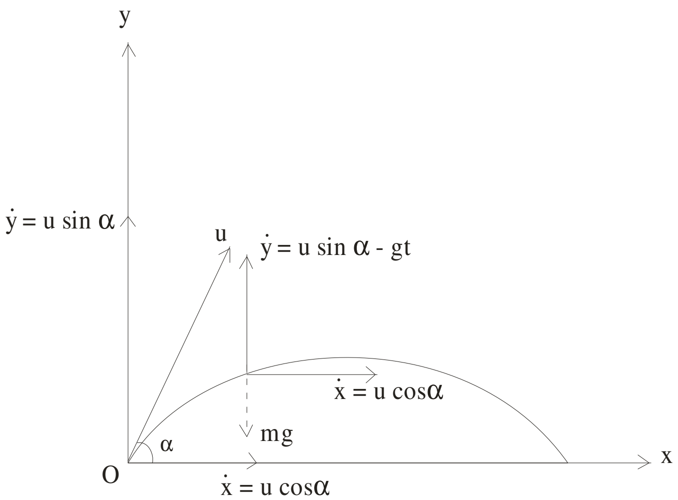
LAGRANGE'S FORMULATION

Unit 1:

In mechanics we study particle in motion under the action of a force. Equation of motion describes how particle moves under the action of a force. However, every motion of a particle is not free motion, but rather it is restricted by putting some conditions on the motion of a particle or system of particles. Hence the basic concepts like equations of motion, constraints and type of constraints on the motion of a particle, generalized coordinates, conservative force, conservation theorems, D'Alembert's principle, etc. on which the edifice of mechanics is built are illustrated in this unit.

- ***Introduction :***

Mechanics is a branch of applied mathematics deals with the motion of a particle or a system of particle with the forces Suppose a bullet is fired from a fixed point with initial velocity u, not exactly vertically upward but making an angle α with the horizontal. Then what instruments do mathematicians need to find the position of the bullet at some instant latter, its velocity at that instant, the distance covered by the bullet at that instant and also the path followed by the bullet at the end of its journey?



Well, to answer such questions, mathematicians do not need any meter stick to measure the distance covered by the bullet at the instant, they don't need any speedometer to find its speed at any instant t, nor they need any clock to see the time required to cover the definite distance. In fact, they need not have to do any such experiment. What they need to describe the motion of the bullet are simply the co-ordinates. Hence the single most important notion in mechanics is the concept of co-ordinates. But the co-ordinates however, just play a role of markers or codes and will no way influence or affect the motion of the bullet. These are just mathematical tools in the hands of a mathematician. Thus the instruments in the hands of a mathematician are the co-ordinates. With the help of these co-ordinates, the motion of a particle or system of particle can completely be described.

For instance, to discuss the motion of the bullet, take P(x, y) be any point on the path of the bullet. The only force acting on the bullet is the gravitational force in the downward direction. Resolving this force horizontally and vertically, we write from Newton's second law of motion the equations of motion as

$$\begin{aligned} \ddot{x} &= 0, \\ \ddot{y} &= -g. \end{aligned} \qquad \ldots (1)$$

Integrating the above two equations and using the initial conditions we readily obtain

$$\begin{aligned} \dot{x} &= u\cos\alpha, \\ \dot{y} &= u\sin\alpha - gt, \end{aligned} \qquad \ldots (2)$$

where u is the initial velocity of the bullet when t = 0. Integrating equations (2) once again and using the initial conditions we obtain

$$x = u\cos\alpha \cdot t, \qquad \ldots (3)$$

$$y = u\sin\alpha t - \frac{1}{2}gt^2. \qquad \ldots (4)$$

Equations (2) determine the velocity of the bullet at any time t, while equations (3) and (4) determine the position of the bullet at that instant. Further, eliminating t from equations (3) and (4), we get

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}. \qquad \text{. . . (5)}$$

This equation gives the path of the bullet and the path is a parabola.

However, the co-ordinates used to describe the motion of a particle or system of particles must be linearly independent. If not then the number of equations describing the motion of the system will be less than the number of variables and in this case the solution can not be uniquely determined. For example if the particle moves freely in space, then three independent co-ordinates are used to describe its motion. These are either the Cartesian co-ordinates (x, y, z) or the spherical polar co-ordinates (r, θ, ϕ). If however, the particle is moving along one of the co-ordinate axes in space, then all the three co-ordinates are not independent, hence these three co-ordinates can not be used for its description. Along a co-ordinate axis only one co-ordinate varies and other two are constants and only the varying co-ordinate can be used to describe the motion of the particle.

- ***Basic concepts***:

1. **Velocity**: Let a particle be moving along any path with respect to the fixed point O. If $\overline{r}$ is its position vector, then the velocity of the particle is defined as the time rate of change of position vector. i.e.,

$$v = \dot{\overline{r}},$$

where dot denotes the derivative with respect to time. If further $\overline{r} = xi + yj + zk$ is the position vector, then velocity of the particle is $v = \dot{\overline{r}} = \dot{x}i + \dot{y}j + \dot{z}k$, where $\dot{x}, \dot{y}, \dot{z}$ are called the components of the velocity along the coordinates axes.

2. **Linear momentum**: The linear momentum of a particle is defined as the product of mass of the particle and its velocity. It is a vector quantity and is denoted by $\overline{p}$. Thus we have $\overline{p} = m\overline{v}$. The direction of momentum is along the same direction of the velocity. In terms of the linear momentum of the particle the equation of motion is given by $\overline{F} = \dot{\overline{p}}$.

3. **Angular momentum:** The angular momentum of a particle about any fixed point O as origin is defined as $\overline{r} \times \overline{p}$. It is a vector quantity and denoted by $\overline{L}$. Thus $\overline{L} = \overline{r} \times \overline{p}$. Angular momentum is perpendicular to both the position vector and the linear momentum of the particle.

4. **Torque (Moment of a Force)**: The time rate of change of angular momentum $\overline{L}$ is defined as torque, It is denoted by $\overline{N}$. Thus

$$\overline{N} = \frac{d\overline{L}}{dt} = \frac{d}{dt}(\overline{r} \times \overline{p})$$

$$\overline{N} = \frac{d}{dt}(\overline{r} \times m\dot{\overline{r}}),$$

$$= \dot{\overline{r}} \times m\dot{\overline{r}} + \overline{r} \times \overline{F},$$

$$\overline{N} = \frac{d\overline{L}}{dt} = \overline{r} \times \overline{F}.$$

- ***Equation of Motion and Conservation Theorems :***

1. For a Particle :

Consider a particle of mass m whose position vector with respect to some fixed point is $\overline{r}$. If $\overline{F}$ is a force applied on the particle then the equation of motion of the particle is given by Newton's second law of motion

$$\overline{F} = \frac{d\overline{p}}{dt}, \qquad \ldots (1)$$

where

$$\overline{p} = m\frac{d\overline{r}}{dt} = m\dot{\overline{r}}$$

is the linear momentum of the particle. The force is defined to be

$$\bar{F} = mass \cdot accel.$$
$$\bar{F} = m\bar{a}.$$

Hence equation (1) becomes

$$\frac{d^2\bar{r}}{dt^2} = a . \qquad \text{. . . (2)}$$

Integrating this equation we get

$$\frac{d\bar{r}}{dt} = at + c , \qquad \text{. . . (3)}$$

where c is the constant of integration and is to be determined. Now applying the initial conditions, we have when $t = 0,\ \frac{d\bar{r}}{dt} = \bar{u}$ initial velocity.

$$\Rightarrow \quad c = \bar{u}$$

Hence equation (3) becomes

$$v = u + at . \qquad \text{. . . (4)}$$

This equation determines the velocity of the particle at any instant t. Integrating (4) again we get

$$r = ut + \frac{1}{2}at^2 + c_1 ,$$

where c_1 is the constant of integration. At $t = 0, r = 0 \Rightarrow c_1 = 0$. Hence we have

$$r = ut + \frac{1}{2}at^2 . \qquad \text{. . . (5)}$$

This equation gives the distance covered by the particle at any time t. One can combine equations (4) and (5) and write

$$v^2 = u^2 + 2ar . \qquad \text{. . . (6)}$$

This equation determines the velocity of the particle at a given distance. Equations (4), (5) and (6) are the algebraic equations of motion and are derived from the equation (1) namely

$$\overline{F} = \dot{\overline{p}} . \qquad \text{. . . (7)}$$

This is the differential equation of motion. It follows from equation (7) that if the applied force is zero then the linear momentum of the particle is conserved.

- ***Equation of motion of a system of particles:***

Consider a system of n particles of masses $m_1, m_2, ..., m_n$ having position vectors $\overline{r}_i, i = 1, 2, ..., n$ relative to an arbitrary fixed origin. Any particle of this system will experience two types of forces.

i) External forces on the system $\overline{F}_i^{(e)}$ $, i = 1, 2, ..., n$.

ii) Internal forces $\overline{F}_{ji}^{(\text{int})}$.

Thus the total force acting on the i^{th} particle of the system is given by

$$\overline{F}_i = \overline{F}_i^{(e)} + \sum_j \overline{F}_{ji}^{(\text{int})} ,$$

where $\sum_j \overline{F}_{ji}^{(\text{int})}$ is the total internal force acting on the i^{th} particle due to the interaction of all other (n-1) particles of the system. Thus the equation of motion of the i^{th} particle is given by

$$\overline{F}_i^{(e)} + \sum_j \overline{F}_{ji}^{(\text{int})} = \dot{p}_i . \qquad \text{. . . (1)}$$

The equation of motion of the whole system is obtained by summing over i the equation (1) we get

$$\sum_i \overline{F}_i^{(e)} + \sum_i \left(\sum_j \overline{F}_{ji}^{(\text{int})} \right) = \sum_i \dot{p}_i .$$

We write this equation as

$$\sum_i \overline{F}_i^{(e)} + \sum_{i,j,i \neq j} \overline{F}_{ji}^{(\text{int})} = \sum_i \dot{p}_i . \qquad \text{. . . (2)}$$

The term $\sum_{i,j,i\neq j} \overline{F}_{ji}^{(\text{int})}$ represents the vector sum of all the interaction forces due to the presence of remaining n-1 particles. However, there is no self interacting force, hence $\overline{F}_{ji} = 0, \quad \forall i = j$. Also the internal forces obey the Newton's third law of motion. That is the action of one particle on the other is equal but opposite to the action of second on the first. This implies that the mutual interaction between the i^{th} and j^{th} particles are equal and opposite. i.e.

$$\overline{F}_{ji}^{(\text{int})} = -\overline{F}_{ij}^{(\text{int})}.$$

This gives

$$\sum_{i,j,i\neq j} \overline{F}_{ji}^{(\text{int})} = 0.$$

Thus equation of motion (2) of the system becomes

$$\sum_i \overline{F}_i^{(e)} = \sum_i \dot{p}_i,$$

$$F^e = \dot{P}, \qquad \ldots (3)$$

where P is the total momentum of the system and $F^e = \dot{P}$ is the total external force acting on the system.

- ***Conservation Theorem of Linear momentum of the system of particles :***

Theorem 1 : If the sum of external forces acting on the particles is zero, the total linear momentum of the system is conserved.

Proof : Proof follows immediately from equation (3). i.e., if

$$F^e = 0 \Rightarrow \quad P = const.$$

- ***Angular Momentum of the system of particles** :*

Consider a system of n particles of masses $m_1, m_2, ..., m_n$ having position vectors $\overline{r}_i, i = 1, 2, ..., n$ relative to an arbitrary fixed origin. The angular momentum of the i^{th} particle of the system about the origin is given by

$$\overline{L}_i = \overline{r}_i \times \overline{p}_i.$$

Thus the total angular momentum of the system about any point is equal to the vector sum of the angular momentum of individual particles. Hence we have

$$\sum_i \bar{L}_i = \sum_i \bar{r}_i \times \bar{p}_i \,. \qquad \ldots (1)$$

If N is the total torque acting on the system, then equation of motion of the system is given by

$$\bar{N} = \frac{d\bar{L}}{dt} = \frac{d}{dt}\left(\sum_i \bar{r}_i \times \bar{p}_i\right).$$

$$\bar{N} = \frac{d\bar{L}}{dt} = \sum_i \dot{\bar{r}}_i \times \bar{p}_i + \sum_i \bar{r}_i \times \dot{\bar{p}}_i \,. \qquad \ldots (2)$$

But we have

$$\sum_i \dot{\bar{r}}_i \times \bar{p}_i = \sum_i \dot{\bar{r}}_i \times m_i \dot{r}_i = 0 \,. \qquad \ldots (3)$$

Now consider

$$\sum_i \bar{r}_i \times \dot{\bar{p}}_i = \sum_i \bar{r}_i \times \left(\bar{F}_i^{(e)} + \sum_j \bar{F}_{ji}^{(\text{int})} \right)$$

$$\sum_i \bar{r}_i \times \dot{\bar{p}}_i = \sum_i \bar{r}_i \times \bar{F}_i^{(e)} + \sum_i \bar{r}_i \times \sum_j \bar{F}_{ji}^{(\text{int})}$$

$$\sum_i \bar{r}_i \times \dot{\bar{p}}_i = \sum_i \bar{r}_i \times \bar{F}_i^{(e)} + \sum_{i,j} \bar{r}_i \times \bar{F}_{ji}^{(\text{int})} \qquad \ldots (4)$$

However, the term $\sum_{i,j} \bar{r}_i \times \bar{F}_{ji}^{(\text{int})}$ can be expanded for $i \neq j$ as

$$\sum_{i,j} \bar{r}_i \times \bar{F}_{ji}^{(\text{int})} = (r_2 - r_1) \times F_{12}^{(\text{int})} + (r_3 - r_1) \times F_{13}^{(\text{int})} + (r_3 - r_2) \times F_{23}^{(\text{int})} + \ldots$$

$$\sum_{i,j} \bar{r}_i \times \bar{F}_{ji}^{(\text{int})} = \left| r_i - r_j \right| \times F_{ji}^{(\text{int})},$$

$$\sum_{i,j} \bar{r}_i \times \bar{F}_{ji}^{(\text{int})} = r_{ij} \times F_{ji}^{(\text{int})}, \quad for \quad r_{ij} = \left| r_i - r_j \right| \qquad \ldots (5)$$

Interchanging i and j on the r. h. s. of equation (5) we get

$$\sum_{i,j} \overline{r}_i \times \overline{F}_{ji}^{(\text{int})} = r_{ji} \times F_{ij}^{(\text{int})},$$

$$\sum_{i,j} \overline{r}_i \times \overline{F}_{ji}^{(\text{int})} = -r_{ij} \times F_{ji}^{(\text{int})} \qquad \ldots (6)$$

Adding equations (5) and (6) we get

$$\sum_{i,j} \overline{r}_i \times \overline{F}_{ji}^{(\text{int})} = 0 \qquad \ldots (7)$$

Consequently, on using equations (3), (4) and (7) in equation (2) we readily obtain

$$\overline{N} = \frac{d\overline{L}}{dt} = \sum_i \overline{r}_i \times \overline{F}_i^{(e)}. \qquad \ldots (8)$$

This equation shows that the total torque on the system is equal to the vector sum of torques acting on the individual particles of the system.

- ***Conservation Theorem of Angular momentum of the system of particles:***

Theorem 2 : If the total external torque acting on the system of particles is zero, then the total angular momentum of the system is conserved.

Proof : Proof follows immediately from equation (8). i.e., if

$$\overline{N} = 0 \Rightarrow \quad \overline{L} = const.$$

- ***Some definitions****:*

Centre of Gravity (Centre of Mass): It is the point of the body at which the whole mass of the body is supposed to be concentrated. If $\overline{R}$ is the position vector of the centre of mass of the body with respect to the origin then its coordinates are given by

$$\overline{R} = (\overline{x}, \overline{y}) = \frac{\sum_i m_i \overline{r}_i}{M}, \text{ where } M = \sum_i m_i \text{ is the total mass of the body.}$$

Example 1: Show that the total angular momentum of a system of particles can be expressed as the sum of the angular momentum of the motion of the centre of mass about origin plus the total angular momentum of the system about the centre of mass.

Solution: Consider a system of n particle of masses $m_1, m_2, ..., m_n$ having position vectors $\overline{r}_i, i = 1, 2, ..., n$ relative to an arbitrary fixed origin. The angular momentum of the i^{th} particle of the system about the origin is given by

$$\overline{L}_i = \overline{r}_i \times \overline{p}_i.$$

Thus the total angular momentum of the system about any point is equal to the vector sum of the angular momentum of individual particles. Hence we have

$$\sum_i \overline{L}_i = \sum_i \overline{r}_i \times \overline{p}_i. \qquad \ldots(1)$$

Let $\overline{R}$ be the radius vector of the centre of mass with respect to the origin and $\overline{r}_i'$ be the position vector of the i^{th} particle with respect to the centre of mass. Then we have

$$\overline{r}_i = \overline{r}_i' + \overline{R}. \qquad \ldots(2)$$

Differentiating this equation with respect to t we get

$$\dot{\overline{r}}_i = \dot{\overline{r}}_i' + \dot{\overline{R}}.$$

i.e., $$v_i = v_i' + v,$$

where v_i is the velocity of the i^{th} particle with respect to O,

v_i' - velocity of the i^{th} particle with respect to centre of mass,

v - velocity of the centre of mass with respect to O.

Using the equation (2) in the equation (1) we get

$$L = \sum_i \overline{L}_i = \sum_i \left(\overline{r}_i' + \overline{R}\right) \times m_i \left(v' + v_i\right),$$

$$L = \sum_i \overline{r}_i' \times m_i v' + \sum_i m_i \overline{r}_i' \times v + \sum_i \overline{R} \times m_i v + \sum_i \overline{R} \times m_i v_i',$$

$$L = \sum_i \overline{r}_i' \times \overline{p}_i' + \sum_i m_i \overline{r}_i' \times v + \overline{R} \times \sum_i m_i v + \overline{R} \times \frac{d}{dt} \sum_i m_i \overline{r}_i',$$

Consider the term

$$\sum_i m_i \overline{r}_i' = \sum_i m_i \left(\overline{r}_i - \overline{R}\right),$$

$$\sum_i m_i \overline{r}_i' = \sum_i m_i \overline{r}_i - \sum_i m_i \overline{R},$$

$$\sum_i m_i \overline{r}_i' = M\overline{R} - M\overline{R},$$

$$\sum_i m_i \overline{r}_i' = 0.$$

Consequently we have from above equation

$$L = \sum_i \overline{r}_i' \times \overline{p}_i' + \overline{R} \times \sum_i m_i v,$$

$$L = \overline{R} \times Mv + \sum_i \overline{r}_i' \times \overline{p}_i'. \qquad \ldots (4)$$

This shows that the total angular momentum about the point O is the sum of the angular momentum of the centre of mass about the origin and the angular momentum of the system about the centre of mass.

Constraint Motion :

Some times the motion of a particle or a system of particles is not free but it is limited by putting some restrictions on the position co-ordinates of the particle or system of particles. The motion under such restrictions is called constraint motion or restricted motion. The mathematical relations establishing the limitations on the position co-ordinates are called as the equations of constraints. Mathematically, the constraints are thus the relations between the co-ordinates and the time t. Consequently, all the co-ordinates are not linearly independent; constraint relations relate some of them. Thus in general the constraints on the motion of a particle or system of particles are always possible to express in the form

$$f_r\left(x_i, y_i, z_i, t\right) \leq or \geq or = 0,$$

where r =1,2,3,...,k the number of constraints and $\left(x_i, y_i, z_i\right)$ are the position co-ordinates of the i^{th} particle of the system, $i = 1, 2, ..., n$.

Examples of motion under constraints:

1. The motion of a rigid body,
2. The motion of a simple pendulum,
3. The motion of a particle on the surface of a sphere,
4. The motion of a particle along the parabola $x^2 = 4ay$,
5. The motion of a particle on an inclined plane.

Holonomic and non-holonomic Constraints:

If the constraints on a particle or system of particles are expressible as equations in the form

$$f_r(x_i, y_i, z_i, t) = 0, \quad r = 1, 2, ..., k \qquad ... (1)$$

then constraints are said to be holonomic otherwise non-holonomic constraints. A system of particles is called respectively holonomic or non-holonomic system if it involves holonomic or non-holonomic constraints.

For example:

Constraints involved on the motion of a rigid body, and simple pendulum are examples of holonomic constraints, while constraints involved in the motion of a particle on the surface of a sphere, the motion of a gas molecules inside the container are the examples of non-holonomic constraints. However, this is not the only way to describe the non-holonomic system. A system is also said to be non-holonomic, if it corresponds to non-integrable differential equations of constraints. Such constraints can not be expressed in the form of equation of the type

$$f_r(x_i, y_i, z_i, t) = 0.$$

Hence such constraints are also called non-holonomic constraints. Obviously, holonomic system has integrable differential equations of constraints expressible in the form of equations.

The constraints are further classified in to two parts viz., Scleronomic and Rheonomic Constraints.

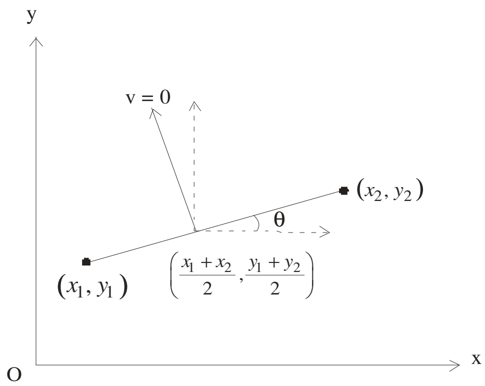
Scleronomic and Rheonomic Constraints :

When the constraint relations do not explicitly depend on time are called scleronomic constraints. While the constraints, which involve time explicitly are called rheonomic constraints. The examples cited above are all scleronomic constraints. A bead moving along a circular wire of radius r with angular velocity ω is an example of rheonomic constraint and the constraints relations are

$$x = r\cos\omega t, \quad y = r\sin\omega t .$$

Worked Examples

Example 2 : Consider a system of two particles joined by a mass less rod of fixed length l. Suppose for simplicity, the system is confined to the horizontal plane xy. Suppose further that the system is so constrained that the centre of the rod cannot have a velocity component perpendicular to the rod. Show that the constraint involved in the system is non-holonomic.



Solution: Let (x_1, y_1) and (x_2, y_2) be the positions of the two particles connected by the mass less rod of length l. The system is shown in the fig.

Since the length between the two particles is constant, clearly one of the constraint relations is

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = l^2 \qquad \ldots (1)$$

where,

$$\begin{aligned} x_2 - x_1 &= l\cos\theta, \\ y_2 - y_1 &= l\sin\theta \end{aligned} \qquad \ldots (2)$$

The constraint (1) is clearly holonomic. The other constraint is such that the centre of the rod cannot have velocity component perpendicular to the rod. Mathematically this is expressed as

$$(\dot{x}_1+\dot{x}_2)\cos(90+\theta)+(\dot{y}_1+\dot{y}_2)\cos\theta=0,$$

$$(\dot{x}_1+\dot{x}_2)\sin\theta=(\dot{y}_1+\dot{y}_2)\cos\theta. \quad \ldots (3)$$

This constraint can not be integrated and hence the constraint is non-holonomic and consequently, the system is non-holonomic.

- ***Degrees of freedom and Generalized co-ordinates :***

Consider the motion of a free particle. To describe its motion we need three independent co-ordinates, such as the Cartesian co-ordinates x, y, z or the spherical polar co-ordinates r,θ,ϕ etc. The particle is free to execute motion along any one of the axes independently with change in only one co-ordinate. In this case we say that the particle has three degrees of freedom. Thus we define

Definition : The least possible number of independent co-ordinates required to specify the motion of the system completely by taking into account the constraints is called degrees of freedom.
e.g. For a system of N particles free from constraints moving independent of each other has 3N degree of freedom.

Generalized co-ordinates:

A system of N particles free from constraints has 3N degrees of freedom. If however, there exists k holonomic constraints expressed in k equations

$$f_i(r_1,r_2,\ldots,r_n,t)=0,\quad i=1,2,\ldots,k, \quad \ldots (1)$$

then 3N co-ordinates are not all independent but related by k equations given in (1). We may use these k equations to eliminate k of the 3N co-ordinates, and we are left with $3N-k=n$ (say) independent co-ordinates. These are generally denoted by $q_j,\quad j=1,2,\ldots,n$ called the generalized co-ordinates and the system has 3N-k degrees of freedom.

Definition: A set of linearly independent variables $q_1, q_2, q_3, ...q_n$ that are used to describe the configuration of the system completely by taking into account the constraints forces acting on it is called generalized co-ordinates.

Thus in general we have

No. of degrees of freedom – No. of constraints= No. of generalized co-ordinates.

Note: The generalized co-ordinates need not be the position co-ordinates, which have the dimensions of length, breath and height, but they can be angles, charges or momentum of the particle.

- ***Transformation Relations:***

It is always possible to express the position co-ordinates of a particle or a system of particles in terms of generalized co-ordinates and vice-versa. This expression is called the transformation relation.

e.g., If $r_i, i = 1, 2, 3, ...n$ are the position vectors of the n particles of the system and q_j, $j = 1, 2, ..., n$ are the generalized co-ordinates, then there exists a relation

$$r_i = r_i(q_1, q_2, q_3, ...q_n, t), \qquad \text{. . . (1)}$$

called the transformation relation.

Work: Let a force $\bar{F}$ be acted on a particle whose position vector is $\bar{r}$. Suppose the particle is displaced through an infinitesimal distance dr due to the application of force F. Then the work done by the force F is given by

$$dW = \bar{F}\, d\bar{r}$$

If the particle is finitely displaced from point $P(r_1)$ to $P(r_2)$ along any path, then the work done by F is given by

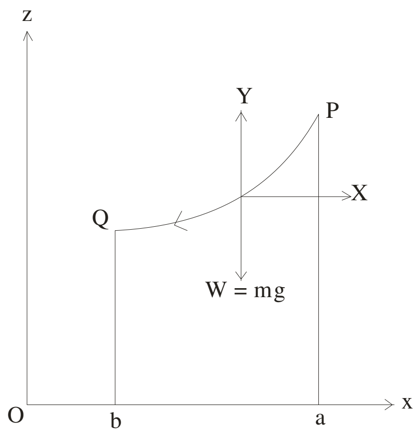
$$W = \int_{r_1}^{r_2} F\, dr \qquad \text{. . . (1)}$$

Conservative Force : The work given in expression (1) is in general depends on the extreme positions of the particle and also the path along which it travels. If a force is such that the work depends only upon the positions P_1 , P_2 and not on the path followed by the particle, then the force F is called conservative force, otherwise non-conservative.

Worked Examples

Example 3 : Show that the gravitational force is conservative.

Solution : Let a particle of mass m move along a curve PQ under gravity. Thus the only force acting on the particle is its own weight in the down ward direction. Therefore, work done by the force is given by



$$W = \int_P^Q F\, dr ,$$

If $F = Xi + Yj$ and $\overline{r} = xi + yj$, where X and Y are the components of the force along the co-ordinate axes. We see that $X = 0$, and $Y = -w$.

Hence $$W = \int_a^b -w dy ,$$

where a and b are the ordinates at points P and Q respectively.

This implies

$$W = w(a-b).$$

This shows that the work does not depend upon the path but depends on the extreme points. Hence the gravitational force is conservative. Alternately, we say that, the force F is conservative if the work done by it around the closed path is zero.

i. e. F is conservative iff $\oint F\, dr = 0$ . . . (1)

However, by Stokes theorem, we have

$$\oint F\, dr = \int_s \nabla \times F.ds, \quad . . . (2)$$

where ds is an arbitrary surface element. Thus from equations (1) and (2) we have

$$\text{F is conservative iff} \quad \nabla \times F = 0 \qquad \ldots (3)$$

However, $\nabla \times F = 0 \Rightarrow F$ is a gradient of some potential V.

$$\Rightarrow \quad F = -\nabla V \quad or \quad F = -\frac{\partial V}{\partial r},$$

where V is a potential called potential energy of the particle and is a function of position only. Thus the force F is conservative if

$$F = -\nabla V \qquad \ldots (4)$$

and conversely. The negative sign indicates that F is in the direction of decreasing V.

Example 4: Show that the inverse square law of attractive force (central force) is conservative.

Solution: The inverse square law of force is the force of attraction between two particles and is given by

$$\bar{F} = -G\frac{m_1 m_2}{r^2} \qquad \ldots (1)$$

where negative sign indicates that the force is directed towards the fixed point and it is called the attractive force. We write the force as

$$\bar{F} = -\frac{k}{r^3}\bar{r}, \quad for \quad k = Gm_1 m_2 \qquad \ldots (2)$$

For $\bar{r} = xi + yj + zk$

We have
$$\bar{F} = -\frac{k(xi + yj + zk)}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \qquad \ldots (3)$$

For conservative force we have $\nabla \times \bar{F} = 0$.

Consider therefore

$$\nabla\times\bar{F}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-Kx}{r^3} & \frac{-Ky}{r^3} & \frac{-Kz}{r^3} \end{vmatrix}$$

$$\nabla\times\bar{F}=-K\left[i\left(\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right)-\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right)\right)+j\left(\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right)-\frac{\partial}{\partial x}\left(\frac{z}{r^3}\right)\right)+k\left(\frac{\partial}{\partial x}\left(\frac{y}{r^3}\right)-\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right)\right].$$

$$\nabla\times\bar{F}=-K\left[i\left(\frac{-3yz}{r^{\frac{5}{2}}}+\frac{3yz}{r^{\frac{5}{2}}}\right)+j\left(\frac{-3xz}{r^{\frac{5}{2}}}+\frac{3xz}{r^{\frac{5}{2}}}\right)+k\left(\frac{-3yx}{r^{\frac{5}{2}}}+\frac{3yx}{r^{\frac{5}{2}}}\right)\right].$$

$$\nabla\times\bar{F}=0.$$

This shows that the inverse square law of attractive force $\bar{F}$ is conservative.

- ***Virtual Work :***

If the system of forces acting on a particle be in equilibrium then their resultant is zero and hence the work done is zero.

Thus in the case of a particle be in equilibrium there is no motion, hence there arises no question of displacement. In this case we assume the particle receives a small virtual displacement (the displacement of the system which causes no real motion is called as virtual or imaginary displacement) and it is denoted by δr_i. Virtual displacement δr_i is assumed to take place only in the co-ordinates and at fixed instant t, hence δ change in time t is zero.

$$\delta r_i=\left(dr_i\right)_{dt=o}.$$

Thus the work done by the system of forces in causing imaginary displacement is called virtual work. It is the amount of work that would have been done if the actual displacement had been caused. Hence the expression for the virtual work done by the forces is given by

$$\text{Virtual work } \delta W = \sum_i F_i \delta r_i . \qquad \ldots (1)$$

- ***Principle of Virtual Work :***

If the forces are in equilibrium then the resultant is zero. Hence the algebraic sum of the virtual work is zero. Conversely, if the algebraic sum of the virtual work is zero then the forces are in equilibrium.

Note that this principle is applicable in statics. However, an analogous principle in dynamics was put forward by D'Alembert.

- ***D'Alembert's Principle :***

D'Alembert started with the equation of motion of a particle $F_i = \dot{p}_i$, where p_i is the linear momentum of the i^{th} particle. This can be written as $F_i - \dot{p}_i = 0$.

Hence $$\sum_i (F_i - \dot{p}_i) = 0,$$

implying a system of particles is in equilibrium. This equation states that the dynamical system appears to be in equilibrium under the action of applied forces F_i and an equal and opposite 'effective forces' $\dot{p}_i$. In this way dynamics reduces to static. Thus

$$\sum_i (F_i - \dot{p}_i) = 0 \Leftrightarrow \text{ the system is in equilibrium (the resultant is zero).}$$

Hence the virtual work done by the forces is zero. This implies that

$$\sum_i (F_i - \dot{p}_i)\delta r_i = 0 .$$

This is known as the mathematical form of D'Alembert principle. This states that "a system of particles moves in such a way that the total virtual work done by the applied forces and reverse effective forces is zero''.

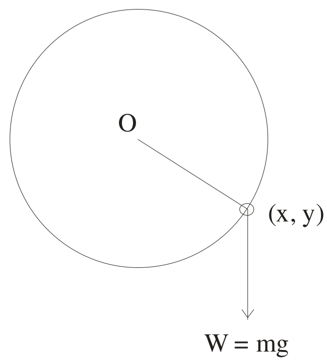
Note :

1. D'Alembert Principle describes the motion of the system by considering its equilibrium.
2. All the laws of mechanics may be derived from this single principle. Hence D'Alembert principle has been called the fundamental principle of mechanics. We will solve some examples by using this principle.

Worked Examples

Example 5: A particle is constrained to move in a circle in a vertical plane xy. Apply the D'Alembert's principle to show that for equilibrium we must have $\ddot{x}y - \ddot{y}x - gx = 0$.

Solution: Consider a particle of mass m be moving along a circle of radius r in xy plane. Let (x, y) be the position of the particle at any instant t with respect to the fixed-point 0.The constraint on the motion of the particle is that the position co-ordinates of the particle always lie on the circle. Hence the equation of the constraint is



$$x^2 + y^2 = r^2 \quad \text{. . . (1)}$$

$$\Rightarrow \quad 2x\delta x + 2y\delta y = 0$$

or $$\delta x = -\frac{y}{x}\delta y \quad \text{. . . (2)}$$

where δx and δy are displacement in x and y respectively. Now from D'Alembert's principle, we have

$$\left(F - m\ddot{r}\right)\delta r = 0.$$

In terms of components we have

$$\left(F_x - m\ddot{x}\right)\delta x + \left(F_y - m\ddot{y}\right)\delta y = 0. \quad \text{. . . (3)}$$

However, the only force acting on the particle at any instant t is its weight mg in the downward direction. Resolving the force horizontally and vertically, we have $F_x = 0$ and $F_y = -mg$. Therefore equation (3) becomes

$$-m\ddot{x}\delta x - (mg + m\ddot{y})\delta y = 0.$$

On using (2) we have

$$m(-\ddot{x}y + \ddot{y}x + gx)\delta x = 0.$$

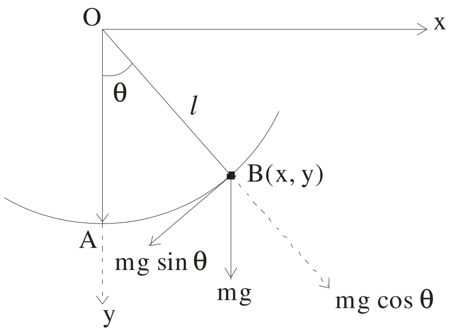
For $\delta x \neq 0, \quad and \quad m \neq 0$ we have

$$\ddot{x}y - \ddot{y}x - gx = 0, \qquad \ldots (4)$$

which is the required equation of motion.

Example 6: Use D'Alembert's principle to determine the equation of motion of a simple pendulum.

Solution : Consider a particle of mass m attached to one end of the string and other end is fastened to a fixed point 0. Let l be the length of the pendulum and θ the angular displacement of the pendulum shown in the fig.



According to the D'Alembert's principle we have

$$\sum_i (F_i - \dot{p}_i)\delta r_i = 0$$

where i is the number of particles in the system.

$$\Rightarrow \quad (F - m\ddot{r})\delta r = 0,$$

where r is the distance of the particle from the starting point along the curve. Resolving the force acting on the particle along the direction of motion and perpendicular to the direction of motion we have

$$\left(-mg\sin\theta - m\ddot{r}\right)\delta r = 0,$$

where the negative sign indicates the force is opposite to the direction of motion. Since $\delta r \neq 0$ we have

$$\ddot{r} = -g\sin\theta. \qquad \text{. . . (1)}$$

From the figure we have r = arcAB $\Rightarrow$ $r = l\theta$ $\Rightarrow$ $\ddot{r} = l\ddot{\theta}$.

Equation (1) becomes

$$\ddot{\theta} = -\frac{g}{l}\sin\theta. \qquad \text{. . . (2)}$$

For small angle, we have $\sin\theta = \theta$ $\Rightarrow$ $\ddot{\theta} = -\frac{g}{l}\theta$.

- ***Generalized Velocities :***

  From transformation equations we have

$$r_i = r_i\left(q_1, q_2, q_3, \ldots q_n, t\right), \qquad \text{. . . (1)}$$

Differentiating this with respect to t we get

$$\dot{r} = \sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} \qquad \text{. . . (2)}$$

where $\dot{q}_j$, $j = 1, 2, 3, \ldots, n$ are called generalized velocities.

- ***Virtual displacement :***

  We find δ variation (change) in the transformation equation (1) to get

$$\delta r_i = \sum \frac{\partial r_i}{\partial q_j}\delta q_j$$

Note here that δt term is absent because virtual displacement is assumed to take place at fixed instant t, hence $\delta t = 0$.

- ***Generalized force :***

If F_i are forces acting on a dynamical system with position vectors r_i then virtual work done by these forces is given by

$$
\begin{aligned}
\delta W &= \sum_i F_i \delta r_i, \\
&= \sum_i \sum_j F_i \frac{\partial r_i}{\partial q_j} \delta q_j \ , \\
&= \sum_j \left(\sum_i F_i \frac{\partial r_i}{\partial q_j} \right) \delta q_j \ , \\
&= \sum_j Q_j \delta q_j \ ,
\end{aligned}
$$

where

$$Q_j = \sum_i F_i \frac{\partial r_i}{\partial q_j} \qquad \ldots (1)$$

are called the components of generalized forces.

Note :

1. If forces are conservative then they are derived from potential V and are given by

$$F_i = -\nabla_i V = -\frac{\partial V}{\partial r_i}.$$

Consequently, the generalized forces are given by $Q_j = -\dfrac{\partial V}{\partial q_j}$.

2. If the forces are non-conservative, the scalar potential U may be function of position, velocity and time. i.e., $U = U\left(q_j, \dot{q}_j, t\right)$. This is called velocity dependent potential or generalized potential. Such a potential exits in the case of a motion of a particle of charge q moving in an electromagnetic field. We will see latter in example (8) that how the generalized potential can be determined in the case of a particle moving in an electromagnetic field. In this case generalized forces are given by

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right).$$

3. If however, the system is acted upon by conservative forces F_i and non-conservative forces $F_i^{(d)}$, in this case generalized forces Q_j are given by

$$Q_j = \sum \left(F_i + F_i^{(d)}\right)\frac{\partial r_i}{\partial q_j} \quad \Rightarrow \quad Q_j = -\frac{\partial V}{\partial q_j} + Q_j^{(d)} \ ,$$

where

$$Q_j^{(d)} = \sum_i F_i^{(d)} \frac{\partial r_i}{\partial q_j}$$

are non-conservative forces which are not derivable from the potential V. Such a situation often arises when frictional forces or dissipative forces are present in the system.

It is found by experiment that in general the dissipative or frictional forces are proportional to the velocity of the particle.

$$\Rightarrow \quad F_i^{(d)} = -\lambda_i \dot{r}_i \,,$$

where λ_i are constants. In such cases the generalized forces are obtained as

$$Q_j^{(d)} = \sum_i F_i^{(d)} \frac{\partial r_i}{\partial q_j},$$

$$= -\sum_i \lambda_i \dot{r}_i \frac{\partial r_i}{\partial q_j}$$

However, from transformation equation we obtain

$$\frac{\partial r_i}{\partial q_j} = \frac{\partial \dot{r}_i}{\partial \dot{q}_j}.$$

Thus we write

$$Q_j^{(d)} = \sum \frac{\partial}{\partial \dot{q}_j}\left(-\frac{1}{2}\lambda_i \dot{r}_i^2\right),$$

$$= -\frac{\partial R}{\partial \dot{q}_j},$$

where $$R = \frac{1}{2}\sum \lambda_i \dot{r}_i^2$$

is called the Rayleigh's dissipative function.

Unit 2: **Lagrange's Equations of motion**:

Newtonian approach for the description of particle involves vector quantities. We now introduce another formulation called the Lagrangian formulation for the description of mechanics of a particle or a system of particles in terms of generalized coordinates, generalized velocities with time t as a parameter. This formulation involves scalar quantities such as kinetic energy and the potential energy and hence proves to be easier then the Newtonian approach, because to deal with scalars is easy than to deal with vectors.

- ***Lagrange's Equations of motion from D'Alembert's Principle :***

Theorem 3 : Obtain Lagrange's equations of motion from D'Alembert's principle.

Proof : Consider a system of n particles of masses m_i and position vectors r_i. We know the position vectors r_i are expressed as the functions of n generalized co-ordinates $q_1, q_2, q_3, ..., q_n$ and time t as

$$r_i = r_i\left(q_1, q_2, q_3, ...q_n, t\right), \qquad ...(1)$$

If F_i are the forces acting on the system, then by D'Alembert's principle we have

$$\sum_i (F_i - \dot{p}_i)\delta r_i = 0, \qquad ...(2)$$

where, $\dot{p}_i = m_i \ddot{r}_i$ is the linear momentum of the i^{th} particle of the system . From the transformation equations we obtain the expression for the virtual displacement

$$\delta r_i = \sum \frac{\partial r_i}{\partial q_j}\delta q_j,$$

where the term δt is absent because the virtual displacement is assumed to take place only in the co-ordinates and at the particular instant. Hence equation (2) becomes

$$\sum_i \sum_j F_i \frac{\partial r_i}{\partial q_j} \delta q_j = \sum_i \sum_j m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} \delta q_j .$$

$$\sum_j \left(\sum_i F_i \frac{\partial r_i}{\partial q_j} \right) \delta q_j = \sum_{i,j} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} \delta q_j ,$$

or
$$\sum_j Q_j \delta q_j = \sum_{i,j} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} \delta q_j , \qquad \ldots (3)$$

where

$$Q_, = \sum F_i \frac{\partial r_i}{\partial q_j} . \qquad \ldots (4)$$

are called the components of generalized forces.

Consider

$$\frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_j} \right) = \ddot{r}_i \frac{\partial r_i}{\partial q_j} + \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j} \right).$$

Substituting this in equation (3) we get

$$\sum_j Q_j \delta q_j = \sum_{i,j} m_i \left[\frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_j} \right) - \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j} \right) \right] \delta q_j . \qquad \ldots (5)$$

Now from equation (1) we have

$$\dot{r}_i = \sum_k \frac{\partial r_i}{\partial q_k} \dot{q}_k + \frac{\partial r_i}{\partial t} . \qquad \ldots (6)$$

Differentiating this with respect to $\dot{q}_j$ we get

$$\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j} . \qquad \ldots (7)$$

Further, differentiating equation (6) w. r. t. q_j, we get

$$\frac{\partial \dot{r}_i}{\partial q_j} = \sum_k \frac{\partial^2 r_i}{\partial q_k \partial q_j} \dot{q}_k + \frac{\partial^2 r_i}{\partial t \partial q_j} . \qquad \text{. . . (8)}$$

Also we have

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \sum_k \frac{\partial^2 r_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 r_i}{\partial q_j \partial t} . \qquad \text{. . . (9)}$$

We notice from equations (8) and (9) that

$$\frac{\partial \dot{r}_i}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)$$

In general we have

$$\frac{\partial}{\partial q_j}\left(\frac{d}{dt}\right) = \frac{d}{dt}\left(\frac{\partial}{\partial q_j}\right). \qquad \text{. . . (10)}$$

On using equation (10) in equation (5) we get

$$\sum_j Q_j \delta q_j = \sum_{i,j} \left[\frac{d}{dt}\left(m_i v_i \frac{\partial v_i}{\partial \dot{q}_j} \right) - m_i v_i \frac{\partial v_i}{\partial q_j} \right] \delta q_j .$$

We write this as

$$\sum_j Q_j \delta q_j = \sum_j \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right)\right) - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right] \delta q_j ,$$

or $$\sum_j Q_j \delta q_j = \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j ,$$

where $$T = \frac{1}{2} \sum_i m_i v_i^2$$

is the total kinetic energy of the system of particles.

$$\Rightarrow \qquad \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0 . \qquad \text{. . . (11)}$$

If the constraints on the motion of particles in the system are holonomic then δq_j are independent. In this case we infer from equation (11) that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}-Q_j=0,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=Q_j, \quad j=1,2,3,....,n. \qquad \text{. . . (12)}$$

These are called the Lagrange's equations of motion. We see that, to derive the Lagrange's equations of motion the knowledge of forces acting on the system of particles will not be necessary.

Note : If the constraints are non-holonomic then the generalized co-ordinates are not all independent of each other. Hence we can't conclude equation (12) from equation (11).

Note: In deriving Lagrange's equations of motion the requirement of holonomic constraints does not appear until the last step.

Case (1) : Conservative system :

If the system is conservative so that particles move under the influence of a potential which is dependent on co-ordinates only, then the forces are derived from the potential V given by

$$F_i=-\nabla_i V=\frac{\partial V}{\partial r_i}.$$

In this case the components of generalized forces becomes

$$Q_j=\sum_i \frac{\partial V}{\partial r_i}\frac{\partial r_i}{\partial q_j}=\frac{\partial V}{\partial q_j}, \text{ and } \quad V\neq V\left(\dot{q}_j\right).$$

Hence equation (12) becomes

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right)-\frac{\partial (T-V)}{\partial q_j}=0.$$

Define a new function $\quad L=T-V$,

where L which is a function of $q_1, q_2, q_3, ..., q_n, \dot{q}_1, \dot{q}_2, \dot{q}_3, ..., \dot{q}_n$ and time t is called a Lagrangian function of the system of particles. Then the equations of motion become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \text{. j = 1, 2, ..., n} \qquad ...(13)$$

These are called the Lagrange's equations for motion for conservative holonomic system.

Note : The Lagrangian L satisfying equation (13) is not unique. Refer Example (13) bellow.

Case (2) : Non-conservative system :

In the case of non-conservative system the scalar potential U may be function of both position and velocity. i.e., $U = U\left(q_j, \dot{q}_j, t\right)$. Such a potential is called as velocity dependent potential. In this case the associated generalized forces are given by

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right).$$

Substituting this in the equation (12) we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, 2, ..., n$$

which are the Lagrange's equations of motion for non-conservative forces.

Case (3) : Partially conservative and partially non-conservative system :

Consider the system is acted upon by conservative forces F_i and non-conservative forces $F_i^{(d)}$. Such a situation often occurs when frictional forces or dissipative forces are present in the system. In this case the components of generalized force are given by

$$Q_j = \sum \left(F_i + F_i^{(d)} \right) \frac{\partial r_i}{\partial q_j} \quad \Rightarrow \quad Q_j = -\frac{\partial V}{\partial q_j} + Q_j^{(d)},$$

where the non-conservative forces which are not derivable from potential function V are represented in $Q_j^{(d)}$. Substituting this in equation (12) we readily obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j^{(d)}, \; j = 1, 2, ..., n \qquad ... (14)$$

where the Lagrangian L contains the potential of the conservative forces, and $Q_j^{(d)}$ represents the forces not arising from the potential V. However, it is found by experiment that, in general the dissipative or frictional forces are proportional to the velocity of the particles.

$$F_i^{(d)} = -\lambda_i \dot{r}_i, \; \lambda_i \text{ are constants.}$$

Hence we have

$$Q_j^{(d)} = \sum_i F_i^{(d)} \frac{\partial r_i}{\partial q_j},$$

$$= -\sum_i \lambda_i \dot{r}_i \frac{\partial r_i}{\partial q_j}.$$

But we know that

$$\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}.$$

Hence

$$Q_j^{(d)} = \sum \frac{\partial}{\partial \dot{q}_j} \left(-\frac{1}{2} \lambda_i \dot{r}_i^2 \right) = -\frac{\partial R}{\partial \dot{q}_j} \; ,$$

where $R = \frac{1}{2} \sum_i \lambda_i \dot{r}_i^2$

is called Rayleigh's dissipation function. Hence the Lagrange's equations of motion become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}+\frac{\partial R}{\partial \dot{q}_j}=0. \qquad \text{. . . (15)}$$

Worked Examples

- ***Conservation of Energy:***

Example 7: Show that the total energy of a particle moving in a conservative force field remains constant, if the potential energy is not an explicit function of time.

Solution : Let a particle of mass m be moving in the conservative field of force F. Let r be the position vector of the particle at any instant. The total energy of the particle is

$$E=T+V, \qquad \text{. . . (1)}$$

where T = kinetic energy,

V = potential energy.

Differentiating (1) with respect to t we get

$$\frac{dE}{dt}=\frac{dT}{dt}+\frac{dV}{dt}, \qquad \text{. . . (2)}$$

where the force

$$F=m\frac{dv}{dt}$$

Therefore

$$Fdr=m\frac{dv}{dt}\frac{dr}{dt}dt$$

$$Fdr=mvdv,$$

$$Fdr=d\left(\frac{1}{2}mv^2\right)$$

$$\Rightarrow \quad Fdr=dT,$$

$$\Rightarrow \quad F\frac{dr}{dt}=\frac{dT}{dt}. \qquad \text{. . . (3)}$$

Similarly, we have the potential energy V =V(r, t), therefore,

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz + \frac{\partial V}{\partial t}dt \,,$$

$$\Rightarrow \quad dV = \nabla V.dr + \frac{\partial V}{\partial t}dt \,,$$

$$\Rightarrow \quad \frac{dV}{dt} = \nabla V.\frac{dr}{dt} + \frac{\partial V}{\partial t}. \qquad \text{. . . (4)}$$

Substituting this in equation (2) we get

$$\frac{dE}{dt} = F\frac{dr}{dt} + \nabla V\frac{dr}{dt} + \frac{\partial V}{\partial t} \,,$$

$$\frac{dE}{dt} = \left(F + \nabla V\right)\frac{dr}{dt} + \frac{\partial V}{\partial t}.$$

Since F is conservative

$$\Rightarrow \quad F = -\nabla V,$$

$$\Rightarrow \quad \frac{dE}{dt} = \frac{\partial V}{\partial t}.$$

Now if the potential energy V is independent of time t then

$$\frac{dE}{dt} = 0.$$

This implies that E is conserved.

Theorem 4 : If the force acting on a particle is conservative then the total energy is conserved.

Proof : If the particle is acted upon by the force F, then if it moves from position P_1 to P_2 . Hence the work done by the force is given by

$$W = \int_{P_1}^{P_2} F.dr \qquad \text{. . . (1)}$$

where $\qquad F = \dot{p} = m\frac{dv}{dt},$

Therefore,

$$W = \int_{P_1}^{P_2} m \frac{dv}{dt} . \frac{dr}{dt} dt = \int_{P_1}^{P_2} m \frac{dv}{dt} . v \, dt$$

$$= \int_{P_1}^{P_2} \frac{d}{dt}\left(\frac{1}{2} m v^2\right) dt,$$

$$= \left(\frac{1}{2} m v^2\right)_{P_1}^{P_2},$$

$$W = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 .$$

Thus $$W = T_2 - T_1 . \qquad \ldots (2)$$

Now, if the force F is conservative then it is derivable from a scalar potential function V, which is a function of position only. Therefore, we have $F = -\nabla V = -\frac{\partial V}{\partial r}$, where V is the potential energy. Substituting this value in equation (1) we get

$$W = -\int_{P_1}^{P_2} \frac{\partial V}{\partial r} dr ,$$

$$= -\int_{P_1}^{P_2} dV ,$$

$$= -(V)_{P_1}^{P_2} ,$$

$$W = V_1 - V_2 . \qquad \ldots (3)$$

From equations (2) and (3) we have

$$T_2 - T_1 = V_1 - V_2$$

$$T_1 + V_1 = T_2 + V_2 = \text{ constant}$$

$$\Rightarrow \quad T + V = \text{ constant.}$$

This shows that the total energy of the particle is conserved.

Aliter : The force field is conservative. This implies that

$$\bar{F} = -\nabla V \,, \qquad \ldots (1)$$

where V is the potential energy. Newton's second law of motion defines the force by

$$\bar{F} = m\ddot{r} \,. \qquad \ldots (2)$$

Thus we have

$$m\ddot{r} = -\frac{\partial V}{\partial r} \,.$$

Multiply this equation by $\dot{r}$, we get

$$m\ddot{r}\,\dot{r} = -\frac{\partial V}{\partial r}\dot{r} \,.$$

This we write as

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + V\right) = 0 \,.$$

Integrating we get

$$\frac{1}{2}m\dot{r}^2 + V = const.$$

This shows that the total energy of the particle moving in the conservative field of force is constant.

Theorem 5 : If the external and internal forces are both conservative, then show that the total potential energy V of the system is given by

$$V = \sum_i V_i^{(e)} + \frac{1}{2}\sum_{i,j} V_{ij}^{(\text{int})} \,,$$

where $V_i^{(e)}$ is the potential energy arises due to the external forces $\bar{F}_i^{(e)}$ and $V_{ij}^{(\text{int})}$ is the internal energy arises due to internal forces $\bar{F}_{ji}^{(\text{int})}$. Further show that the total energy of the system is conserved.

Proof : Two types of forces viz., external and internal forces are acting on the system of particles. To find the total energy of the system, we find the work done by all the forces external as well as internal in moving the system from initial configuration 1 to the final configuration 2. It is given by

$$W = \sum_i \int_1^2 \overline{F}_i d\overline{r}_i ,$$

where

$$\overline{F}_i = \overline{F}_i^{(e)} + \sum_j \overline{F}_{ji}^{(\text{int})}$$

$$\Rightarrow \quad W = \sum_i \int_1^2 \overline{F}_i^{(e)} d\overline{r}_i + \sum_{i,j,i\neq j} \int_1^2 \overline{F}_{ji}^{(\text{int})} d\overline{r}_i \qquad \text{. . . (1)}$$

Let $\overline{F}_i^{(e)}$ be conservative, then there exists a potential $V_i^{(e)}$ such that

$$\overline{F}_i^{(e)} = -\nabla_i V_i^{(e)} = -\frac{\partial V_i^{(e)}}{\partial r_i}$$

$$\sum_i \int_1^2 \overline{F}_i^{(e)} d\overline{r}_i = -\sum_i \int_1^2 \frac{\partial V_i^{(e)}}{\partial r_i} d\overline{r}_i = -\sum_i \int_1^2 dV_i^{(e)},$$

$$\sum_i \int_1^2 \overline{F}_i^{(e)} d\overline{r}_i = -\left[\sum_i V_i^{(e)}\right]_1^2 \qquad \text{. . . (2)}$$

Now consider the second term on the r. h. s. of equation (1)

$$\sum_{i,j,i\neq j} \int_1^2 \overline{F}_{ji}^{(\text{int})} d\overline{r}_i = \sum_{i,j,i\neq j} \int_1^2 \overline{F}_{ji}^{(\text{int})} d\overline{r}_i \qquad \text{. . . (3)}$$

Interchanging i *and* j on the r. h. s. of equation (3) we get

$$\sum_{i,j,i\neq j} \int_1^2 \overline{F}_{ji}^{(\text{int})} d\overline{r}_i = \sum_{i,j,i\neq j} \int_1^2 \overline{F}_{ij}^{(\text{int})} d\overline{r}_j$$

$$\sum_{i,j,i\neq j} \int_1^2 \overline{F}_{ji}^{(\text{int})} d\overline{r}_i = -\sum_{i,j,i\neq j} \int_1^2 \overline{F}_{ji}^{(\text{int})} d\overline{r}_j \qquad (\overline{F}_{ji}^{(\text{int})} = -\overline{F}_{ij}^{(\text{int})}) \qquad \text{. . . (4)}$$

Adding equations (3) and (4) we get

$$\sum_{i,j,i\neq j}\int_1^2 \overline{F}_{ji}^{(\text{int})} d\overline{r}_i = \frac{1}{2}\sum_{i,j,i\neq j}\int_1^2 \overline{F}_{ji}^{(\text{int})}\left(d\overline{r}_i - d\overline{r}_j\right)$$

$$\sum_{i,j,i\neq j}\int_1^2 \overline{F}_{ji}^{(\text{int})} d\overline{r}_i = \frac{1}{2}\sum_{i,j,i\neq j}\int_1^2 \overline{F}_{ji}^{(\text{int})} d\overline{r}_{ij}, \quad for \quad d\overline{r}_{ij} = d\overline{r}_i - d\overline{r}_j$$

Now if the internal forces $\overline{F}_{ji}^{(\text{int})}$ are conservative, there exists a potential $V_{ij}^{(\text{int})}$ such that

$$\overline{F}_{ji}^{(\text{int})} = -\nabla_{ji} V_{ji}^{(\text{int})} = -\frac{\partial V_{ij}^{(\text{int})}}{\partial r_{ij}},$$

where ∇_{ji} is the gradient with respect to r_{ji}. Thus the above equation becomes

$$\sum_{i,j,i\neq j}\int_1^2 \overline{F}_{ji}^{(\text{int})} d\overline{r}_i = -\frac{1}{2}\sum_{i,j,i\neq j}\int_1^2 \frac{\partial V_{ij}^{(\text{int})}}{\partial r_{ij}} d\overline{r}_{ij},$$

$$\sum_{i,j,i\neq j}\int_1^2 \overline{F}_{ji}^{(\text{int})} d\overline{r}_i = -\frac{1}{2}\sum_{i,j,i\neq j}\int_1^2 dV_{ij}^{(\text{int})},$$

$$\sum_{i,j,i\neq j}\int_1^2 \overline{F}_{ji}^{(\text{int})} d\overline{r}_i = -\left[\frac{1}{2}\sum_{i,j} V_{ij}^{(\text{int})}\right]_1^2. \qquad \ldots (5)$$

Substituting the values from equations (2) and (5) we get

$$W = -\left[V\right]_1^2 = V_1 - V_2, \qquad \ldots (6)$$

where

$$V = \sum_i V_i^{(e)} + \frac{1}{2}\sum_{i,j} V_{ij}^{(\text{int})} \qquad \ldots (7)$$

represents the total potential energy of the system of particles. Similarly the total work done by the force on the system in terms of kinetic energy is given by

$$W = \sum_i \int_1^2 \overline{F}_i dr_i$$

$$W = \sum_i \int_1^2 \frac{d}{dt}(m_i v_i) \frac{d\overline{r}_i}{dt} dt$$

$$= \sum_i \int_1^2 m_i \frac{dv_i}{dt} v_i dt,$$

$$W = \sum_i \int_1^2 m_i \frac{d}{dt}\left(\frac{1}{2} v_i^2\right) dt$$

$$W = \sum_i \int_1^2 d\left(\frac{1}{2} m_i v_i^2\right)$$

$$W = \sum_i \left(\frac{1}{2} m_i v_i^2\right)_1^2 = T_2 - T_1 \qquad \ldots (8)$$

From equations (6) and (8) we have

$$T_1 + V_1 = T_2 + V_2 .$$

This shows that the total energy of the system is conserved.

Example 8 : Find the velocity dependent potential and hence the Lagrangian for a particle of charge q moving in an electromagnetic field.

Solution : Consider a charge particle of charge q moving with velocity v in an electric field $\overline{E}$ and magnetic field $\overline{B}$. The force acting on the particle is called Lorenz force and is given by

$$\overline{F} = q\left(\overline{E} + \overline{v} \times \overline{B}\right), \qquad \ldots (1)$$

where $\overline{E}$ and $\overline{B}$ satisfy the Maxwell's field equations

$$\nabla \cdot \overline{B} = 0,$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}. \qquad \ldots (2)$$

We know the vector identity $\nabla \cdot \nabla \times \bar{A} = 0$. Thus the Maxwell equation implies that there exists the magnetic vector potential $\bar{A}$ which is a function of co-ordinates and velocities such that

$$\bar{B} = \nabla \times \bar{A}. \qquad \ldots (3)$$

Substituting this in the second Maxwell equation we get

$$\nabla \times \bar{E} + \frac{\partial}{\partial t}\left(\nabla \times \bar{A}\right) = 0,$$

$$\Rightarrow \quad \nabla \times \bar{E} + \left(\nabla \times \frac{\partial \bar{A}}{\partial t}\right) = 0,$$

$$\Rightarrow \quad \nabla \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t}\right) = 0. \qquad \ldots (4)$$

We also know the vector identity

$$\nabla \times \nabla \phi = 0 \qquad \ldots (5)$$

Comparing equations (4) and (5) we see that, there exists a scalar potential ϕ which is function of co-ordinates and not involving velocities such that

$$\bar{E} + \frac{\partial \bar{A}}{\partial t} = -\nabla \phi$$

$$\Rightarrow \quad \bar{E} = -\nabla \phi - \frac{\partial \bar{A}}{\partial t} \qquad \ldots (6)$$

Using equations (3) and (6) in equation (1) we get

$$\bar{F} = q\left[-\nabla \phi - \frac{\partial \bar{A}}{\partial t} + \bar{v} \times \nabla \times \bar{A}\right] \qquad \ldots (7)$$

where we have

$$\nabla \phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z},$$

$$\frac{\partial \bar{A}}{\partial t} = i\frac{\partial A_x}{\partial t} + j\frac{\partial A_y}{\partial t} + k\frac{\partial A_z}{\partial t},$$

$$\nabla \times \bar{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = i\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + j\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + k\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right),$$

$$\bar{v} \times \nabla \times \bar{A} = i\left[v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right] + j\left[v_x\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right] +$$
$$+ k\left[v_x\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - v_y\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right]$$

Therefore the x-component of the Lorentz force (7) becomes

$$F_x = q\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right] \qquad \text{. . . (8)}$$

Now consider

$$v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} - \left(v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z}\right) \qquad \text{. . . (9)}$$

Also we have

$$\frac{dA_x}{dt} = v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial t},$$

$$v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z} = \frac{dA_x}{dt} - \frac{\partial A_x}{\partial t} \qquad \text{. . . (10)}$$

Also

$$\frac{\partial}{\partial x}\left(\bar{v} \cdot \bar{A}\right) = \frac{\partial}{\partial x}\left(v_x A_x + v_y A_y + v_z A_z\right)$$

$$\frac{\partial}{\partial x}\left(\bar{v}\cdot\bar{A}\right)=v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_y}{\partial x}+v_z\frac{\partial A_z}{\partial x} \quad \ldots (11)$$

Substituting from equations (10) and (11) in equation (9) we get

$$v_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-v_z\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\bar{v}\cdot\bar{A}\right)-\frac{dA_x}{dt}+\frac{\partial A_x}{\partial t}. \quad \ldots (12)$$

Hence equation (8) becomes

$$F_x=q\left[-\frac{\partial}{\partial x}\left(\phi-\bar{v}\cdot\bar{A}\right)-\frac{dA_x}{dt}\right] \quad \ldots (13)$$

Also

$$\frac{\partial}{\partial v_x}\left(\bar{v}\cdot\bar{A}\right)=\frac{\partial}{\partial v_x}\left(v_xA_x+v_yA_y+v_zA_z\right)=A_x$$

As ϕ is independent of v_x, therefore we write

$$\frac{\partial}{\partial v_x}\left(\phi-\bar{v}\cdot\bar{A}\right)=-A_x$$

$$\Rightarrow \quad \frac{d}{dt}\cdot\frac{\partial}{\partial v_x}\left(\phi-\bar{v}\cdot\bar{A}\right)=-\frac{dA_x}{dt} \quad \ldots (14)$$

Substituting this in equation (13) we get

$$F_x=q\left[-\frac{\partial}{\partial x}\left(\phi-\bar{v}\cdot\bar{A}\right)-\frac{d}{dt}\left\{\frac{\partial}{\partial v_x}\left(\phi-\bar{v}\cdot\bar{A}\right)\right\}\right] \quad \ldots (15)$$

Define the generalized potential

$$U=q\left(\phi-\bar{v}\cdot\bar{A}\right) \quad \ldots (16)$$

Hence we write equation (15) as

$$F_x=\left[-\frac{\partial U}{\partial x}-\frac{d}{dt}\left\{\frac{\partial U}{\partial \dot{x}}\right\}\right] \qquad \text{for } \dot{x}=v_x \quad \ldots (17)$$

Hence the Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F_x$$

becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = -\frac{\partial U}{\partial x} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right)$$

$$\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{x}}\right) - \frac{\partial (T-U)}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

where the Lagrangian of the particle $L = T - U$ becomes

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\phi + q\,\bar{v}\cdot\bar{A}. \qquad \text{. . . (18)}$$

Example 9: Show that the Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

can also be written in the form

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j.$$

Solution: The kinetic energy T is in general a function of generalized co-ordinates, generalized velocities and time. Thus we have

$$T = T\left(q_j, \dot{q}_j, t\right). \qquad \text{. . . (1)}$$

Differentiating this w. r. t. t we get

$$\frac{dT}{dt} = \dot{T} = \sum_k \frac{\partial T}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial T}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial T}{\partial t} \ . \qquad \text{. . . (2)}$$

Differentiating equation (2) partially w. r. t. $\dot{q}_j$ we get

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} = \sum_k \left(\frac{\partial^2 T}{\partial \dot{q}_j \partial q_k}\dot{q}_k + \frac{\partial T}{\partial q_k}\delta_k^j\right) + \sum_k \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k}\ddot{q}_k + \frac{\partial^2 T}{\partial \dot{q}_j \partial t}$$

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} = \sum_k \frac{\partial^2 T}{\partial \dot{q}_j \partial q_k} \dot{q}_k + \frac{\partial T}{\partial q_j} + \sum_k \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_k} \ddot{q}_k + \frac{\partial^2 T}{\partial \dot{q}_j \partial t}. \qquad \ldots (3)$$

Also we find the expression

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \sum_k \frac{\partial^2 T}{\partial q_k \partial \dot{q}_j} \dot{q}_k + \sum_k \frac{\partial^2 T}{\partial \dot{q}_k \partial \dot{q}_j} \ddot{q}_k + \frac{\partial^2 T}{\partial t \partial \dot{q}_j}. \qquad \ldots (4)$$

From equations (3) and (4) we have

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \frac{\partial T}{\partial q_j}. \qquad \ldots (5)$$

But it is given that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \frac{\partial T}{\partial q_j} + Q_j.$$

Consequently equation (5) becomes

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - \left(\frac{\partial T}{\partial q_j} + Q_j\right) = \frac{\partial T}{\partial q_j}.$$

$$\Rightarrow \qquad \frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j.$$

Example10: A particle of mass M moves on a plane in the field of force given by $F = -\hat{i}_r kr\cos\theta$, where k is constant and $\hat{i}_r$ is the radial unit vector. Show that angular momentum of the particle about the origin is conserved and obtain the differential equation of the orbit of the particle.

Solution: Let (x, y) and (r, θ) be the Cartesian and polar co-ordinates of a particle of mass M moving on a plane under the action of the given field of force

$$F = -\hat{i}_r kr\cos\theta, \qquad \ldots (1)$$

Since the force is explicitly given, hence the Lagrange's equation motion corresponding to the generalized coordinates r and θ are given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)-\frac{\partial T}{\partial \theta}=Q_\theta, \qquad \text{. . . (2)}$$

and $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right)-\frac{\partial T}{\partial r}=Q_r, \qquad \text{. . . (3)}$$

where T is the kinetic energy of the particle and is given by

$$T=\frac{1}{2}M\left(\dot{x}^2+\dot{y}^2\right)=\frac{1}{2}M\left(\dot{r}^2+r^2\dot{\theta}^2\right), \qquad \text{. . . (4)}$$

The components of generalized force along the radial direction and in the direction of θ are given by

$$Q_r=-\hat{i}_r kr\cos\theta,$$
$$Q_\theta=0.$$

Hence equations (1) and (2) become

$$\frac{d}{dt}\left(Mr^2\dot{\theta}\right)=0,$$
$$\Rightarrow \quad Mr^2\dot{\theta}=const.$$

and $$M\ddot{r}-Mr\dot{\theta}^2+kr\cos\theta=0.$$

This is the equation of motion of the orbit of the particle.

Example 11: Show that the Lagrange's equation of motion can also be written as

$$\frac{\partial L}{\partial t}-\frac{d}{dt}\left(L-\sum \dot{q}_j\frac{\partial L}{\partial \dot{q}_j}\right)=0,$$

Solution: A Lagrangian of a particle is

$$L=L\left(q_{j,}\,\dot{q}_{j,}\,t\right),$$

Differentiating this w. r. t. we obtain

$$\frac{dL}{dt}=\sum_j\frac{\partial L}{\partial q_j}\dot{q}_j+\sum_j\frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j+\frac{\partial L}{\partial t} \qquad \text{. . . (1)}$$

Consider the expression

$$\frac{d}{dt}\left(\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \sum_j \dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \qquad \ldots (2)$$

Subtracting equation (2) from (1) we get

$$\frac{d}{dt}\left(L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial t} + \sum_j \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right)\dot{q}_j . \qquad \ldots (3)$$

But from Lagrange's equation we have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 . \qquad \ldots (4)$$

Consequently, equation (3) becomes

$$\frac{d}{dt}\left(L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial t} = 0 .$$

This is the required form.

Example 12: A particle of mass m moves in a plane under the action of a conservative force F with components $F_x = -k^2(2x + y)$, $F_y = -k^2(x + 2y)$, k is a constant. Find the total energy of the motion, the Lagrangian, and the equations of motion of the particle.

Solution: A particle is moving in a plane. Let (x, y) be the co-ordinates of the particle at any instant t. If T and V are the kinetic and potential energies of the particle then we have

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right), \qquad \ldots (1)$$

and

$$V = V(x, y). \qquad \ldots (2)$$

Since the force is given by

$$F = -\nabla V ,$$

$$\Rightarrow \quad iF_x + jF_y = -\left(i\frac{\partial V}{\partial x} + j\frac{\partial V}{\partial y} + k\frac{\partial V}{\partial z}\right)$$

$$-k^2(2x+y)i-k^2(x+2y)j=-\left(i\frac{\partial V}{\partial x}+j\frac{\partial V}{\partial y}\right),$$

$$\Rightarrow \quad \frac{\partial V}{\partial x}=k^2(2x+y),$$

$$\Rightarrow \quad \frac{\partial V}{\partial y}=k^2(x+2y).$$

We write

$$dV=\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy,$$

$$dV=k^2(2x+y)dx+k^2(x+2y)dy,$$

$$dV=k^2(2xdx+d(xy)+2ydy).$$

On integrating we get,

$$V=k^2(x^2+xy+y^2) \qquad \ldots (3)$$

The total energy of motion of the particle is therefore given by

$$E=T+V$$

While the Lagrangian of the motion is given by

$$L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)-k^2(x^2+xy+y^2) \qquad \ldots (4)$$

The Lagrange's equations of motion corresponding to the generalized co-ordinates x and y are respectively given by

$$m\ddot{x}+k^2(2x+y)=0,$$

$$m\ddot{y}+k^2(x+2y)=0.$$

- ***Kinetic Energy as a Homogeneous Quadratic Function of Generalized Velocities :***

Theorem 6: Find the expression for the kinetic energy as the quadratic function of generalized velocities. Further show that

i) when the constraints are scleronomic, the kinetic energy is a homogeneous function of generalized velocities and $\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$,

ii) when the constraints are rheonomic then $\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T_2 + T_1$,

where T_1, T_2 have usual meaning.

Proof: Consider a system of particles of masses m_i and position vectors r_i . The kinetic energy of the system is given by

$$T = \frac{1}{2}\sum_i m_i \dot{r}_i^2 \quad , \qquad \text{. . . (1)}$$

where

$$r_i = r_i\left(q_1, q_2, q_3, ...q_n, t\right),$$

$$\dot{r}_i = \sum_k \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t}.$$

Substituting this value in equation (1) we get

$$T = \frac{1}{2}\sum_i m_i \left[\sum_j \left(\frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}\right)\sum_k \left(\frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t}\right)\right]$$

$$T = \frac{1}{2}\sum_i m_i \left[\sum_{j,k} \frac{\partial r_i}{\partial q_j}.\frac{\partial r_i}{\partial q_k}\dot{q}_j\dot{q}_k + 2\sum_j \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial t}\dot{q}_j + \left(\frac{\partial r_i}{\partial t}\right)^2\right],$$

$$T = \sum_{j,k}\left[\frac{1}{2}\sum_i m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial q_k}\right]\dot{q}_j\dot{q}_k + \sum_j\left[\sum_i m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial t}\right]\dot{q}_j + \sum_i \frac{1}{2}m_i\left(\frac{\partial r_i}{\partial t}\right)^2$$

or

$$T = \sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k + \sum_j a_j\dot{q}_j + a \qquad \text{. . . (2)}$$

where

$$a_{jk} = \sum_i \frac{1}{2}m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial q_k},$$

$$a_j = \sum_i m_i \frac{\partial r_i}{\partial q_j}\frac{\partial r_i}{\partial t}, \qquad \text{... (3)}$$

$$a = \sum_i \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial t} \right)^2$$

are definite functions of r and t and hence functions of $q's$ and t. From equation (2) we observe that the kinetic energy is a quadratic function of the generalized velocities.

<u>Case 1 :</u> If the constraints are scleronomic. This implies equivalently that the transformation equations do not contain time t explicitly, and then we have

$$\frac{\partial r_i}{\partial t} = 0,$$

and consequently $a \quad and \quad a_j$ vanish. Therefore equation (2) reduces to

$$T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k . \qquad \text{. . . (4)}$$

This shows that the kinetic energy is a homogeneous quadratic function of generalized velocities. Now applying Euler's theorem for the homogeneous quadratic function of generalized velocities we have

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T \qquad \text{. . . (5)}$$

<u>Case 2 :</u> If the constraints are rheonomic then we write equation (2) in the form

$$T = T_2 + T_1 + T_0 , \qquad \text{. . . (6)}$$

where

$$\begin{aligned} T_2 &= \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k , \\ T_1 &= \sum_j a_j \dot{q}_j , \end{aligned} \qquad \text{. . . (7)}$$

and $$T_0 = a = \sum_i \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial t} \right)^2$$

are homogeneous function of generalized velocities of degree two, one and zero respectively.

Now we consider

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = \sum_j \dot{q}_j \frac{\partial T_2}{\partial \dot{q}_j} + \sum_j \dot{q}_j \frac{\partial T_1}{\partial \dot{q}_j} + \sum_j \dot{q}_j \frac{\partial T_0}{\partial \dot{q}_j}$$

On applying Euler's theorem for the homogeneous function to each term on the right hand side we readily get

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T_2 + T_1 . \qquad \text{. . . (8)}$$

This completes the proof.

Note : However, the result (8) can also be obtained by direct differentiating equation (2) w. r. t. $\dot{q}_j$. Thus

$$\frac{\partial T}{\partial \dot{q}_j} = 2\sum_k a_{jk}\dot{q}_k + a_j .$$

Next multiplying this equation by $\dot{q}_j$ and summing over j we get

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2\sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k + \sum_j a_j\dot{q}_j$$

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T_2 + T_1 .$$

The result (5) can similarly be derived by direct differentiating equation (4).

- ***Another way of proving conservation theorem for energy :***

Theorem (7): If the Lagrangian does not contain time t explicitly, the total energy of the conservative system is conserved.

Proof : Consider a conservative system, in which the forces are derivable from a potential V which is dependent on position only. The Lagrangian of the system is defined as

$$L = T - V , \qquad \text{. . . (1)}$$

where

$$L = L\left(q_j, \dot{q}_j, t\right) \qquad \ldots (2)$$

satisfies the Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0. \qquad \ldots (3)$$

Differentiating equation (2) we obtain

$$\frac{dL}{dt} = \sum_j \left[\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right] + \frac{\partial L}{\partial t}.$$

Since L does not contain time t explicitly implies $\frac{\partial L}{\partial t} = 0$.

$$\frac{dL}{dt} = \sum_j \left[\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right].$$

On using equation (3) we write

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j.$$

$$\frac{dL}{dt} = \frac{d}{dt}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right),$$

$$\Rightarrow \quad \frac{d}{dt}\left[L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right] = 0$$

$$\Rightarrow \quad L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = const.. \qquad \ldots (4)$$

Since the potential energy V for the conservative system depends upon the position co-ordinates only and does not involve generalized velocities. Hence we have

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}.$$

The generalized momentum is defined as

$$p_j = \frac{\partial T}{\partial \dot{q}_j}.$$

Thus we have from equation (4) that

$$\sum_j p_j \dot{q}_j - L = const\,(H). \qquad \ldots (5)$$

L does not contain time t means neither the kinetic energy nor the potential energy of the particle involves time t. In this case the transformation equations do not contain time t. consequently the constraints are scleronomic. Hence the kinetic energy T is a homogeneous quadratic function of generalized velocities.

$$T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k \qquad \ldots (6)$$

where

$$a_{jk} = \sum_i \frac{1}{2} m_i \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial q_k},$$

Hence by Euler's formula we have

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T \qquad \ldots (7)$$

Hence from equation (4) we have

$$\begin{aligned} & 2T - L = H \\ \Rightarrow \quad & 2T - T + V = H \\ \Rightarrow \quad & T + V = H, \\ \Rightarrow \quad & E = H\ (Const) \end{aligned}$$

This proves the total energy E is conserved for conservative system.

Theorem (8): Show that non-conservation of total energy is directly associated with the existence of non-conservative forces even if the transformation equation does not contain time t.

Proof: We know the Lagrange's equations of motion for a system in which conservative forces F_i and non-conservative forces $F_i^{(d)}$ are present are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=Q_j^{(d)}, \qquad \text{j=1, 2, 3, …, n.} \qquad \text{. . . (1)}$$

where the Lagrangian L contains the potential of the conservative forces and the forces which are not arising from potential V are represented by $Q_j^{(d)}$.

Since $\qquad L=L\left(q_j,\dot{q}_j,t\right)$

$$\frac{dL}{dt}=\sum_j\left[\frac{\partial L}{\partial q_j}\dot{q}_j+\frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right]+\frac{\partial L}{\partial t}. \qquad \text{. . . (2)}$$

From equation (1) we have $\qquad \frac{\partial L}{\partial q_j}=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-Q_j^{(d)}$,

Therefore

$$\frac{dL}{dt}=\sum_j\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j+\sum_j\frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j-\sum_j Q_j^{(d)}\dot{q}_j+\frac{\partial L}{\partial t},$$

$$=\sum_j\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right)-\sum_j Q_j^{(d)}\dot{q}_j+\frac{\partial L}{\partial t},$$

$$\frac{dL}{dt}\quad=\frac{d}{dt}\left(\sum_j\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right)-\sum_j Q_j^{(d)}\dot{q}_j+\frac{\partial L}{\partial t} \qquad \text{. . . (3)}$$

Since L contains the potential of the conservative forces

This implies that $\frac{\partial L}{\partial \dot{q}_j}=\frac{\partial T}{\partial \dot{q}_j}$

$$\frac{dL}{dt}=\frac{d}{dt}\left(\sum_j\frac{\partial T}{\partial \dot{q}_j}\dot{q}_j\right)-\sum_j Q_j^{(d)}\dot{q}_j+\frac{\partial L}{\partial t}. \qquad \text{. . . (4)}$$

where T here is a quadratic function of generalized velocities and hence in this case we have

$$\sum_j \dot{q}_j\frac{\partial T}{\partial \dot{q}_j}=2T. \qquad \text{. . . (5)}$$

Substituting this in equation (4) we get

$$\frac{dL}{dt} = 2\frac{dT}{dt} - \sum_j Q_j^{(d)} \dot{q}_j + \frac{\partial L}{\partial t}.$$

Hence

$$\frac{dE}{dt} = \sum_j Q_j^{(d)} \dot{q}_j - \frac{\partial L}{\partial t}. \qquad \text{. . . (6)}$$

If the transformation equations do not contain time t explicitly, then the kinetic energy does not contain time t. This implies that $\frac{\partial T}{\partial t} = 0$. Also Lagrangian contains the potential of conservative forces, we have therefore V=V (q_j) and hence

$$\frac{\partial V}{\partial t} = 0.$$

Consequently, we have $\frac{\partial L}{\partial t} = 0$. Hence equation (6) becomes

$$\frac{dE}{dt} = \sum_j Q_j^{(d)} \dot{q}_j. \qquad \text{...(7)}$$

This shows that the non-conservation of total energy is directly associated with the existence of non-conservative forces $Q_j^{(d)}$. However, if the system is conservative and the transformation equations do not contain time t then the total energy is conserved.

Example 13: Show that the new Lagrangian L' defined by

$$L' = L + \frac{df(q_j, t)}{dt}, \quad j = 1, 2, ..., n$$

satisfies Lagrange's equation of motion, where f is an arbitrary differentiable functions of q_j and t , and L is a Lagrangian for a system of n degrees of freedom.

Solution : Given that

$$L' = L + \frac{df(q_j, t)}{dt}, \qquad \text{. . . (1)}$$

where L satisfies

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0, \qquad \ldots (2)$$

We prove that $\quad \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right)-\frac{\partial L'}{\partial q_j}=0.$

Since $\quad f = f\left(q_j, t\right),$

Therefore we have

$$\frac{df}{dt}=\sum_k \frac{\partial f}{\partial q_k}\dot{q}_k+\frac{\partial f}{\partial t}. \qquad \ldots (3)$$

Differentiating this partially w. r. t. q_j we get

$$\frac{\partial}{\partial q_j}\left(\frac{df}{dt}\right)=\sum_k \frac{\partial^2 f}{\partial q_k \partial q_j}\dot{q}_k+\frac{\partial^2 f}{\partial t \partial q_j}. \qquad \ldots (4)$$

Also from equation (3) we have

$$\frac{\partial}{\partial \dot{q}_j}\left(\frac{df}{dt}\right)=\frac{\partial f}{\partial q_j}.$$

Differentiating this w. r. t. t we get

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{df}{dt}\right)\right]=\sum_k \frac{\partial^2 f}{\partial q_k \partial q_j}\dot{q}_k+\frac{\partial^2 f}{\partial t \partial q_j}. \qquad \ldots (5)$$

Subtracting equation (4) from (5) we get

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{df}{dt}\right)\right]-\frac{\partial}{\partial q_j}\left(\frac{df}{dt}\right)=0,$$

i.e., $\quad \frac{d}{dt}\left[\frac{\partial L'}{\partial \dot{q}_j}\right]-\frac{\partial L'}{\partial q_j}=0$

This proves that the Lagrangian of the system is not unique.

Example 14 : Deduce the principle of energy from the Lagrange's equation of motion.

Solution : We know the Lagrange's equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0, \qquad \ldots (1)$$

where $L=T-V$ is a Lagrangian and $V=V\left(q_j\right), \quad T=T\left(q_j,\dot{q}_j\right)$

Hence equation (1) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=-\frac{\partial V}{\partial q_j}, \qquad \ldots (2)$$

We also know

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j}=2T\,. \qquad \ldots (3)$$

Also we obtain

$$\frac{dT}{dt}= \quad \sum_j \frac{\partial T}{\partial q_j}\dot{q}_j+\sum_j \frac{\partial T}{\partial \dot{q}_j}\ddot{q}_j\,. \qquad \ldots (4)$$

Multiply equation (2) by $\dot{q}_j$ and summing over j we obtain

$$\frac{d}{dt}\left[\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j}\right]-\sum_j \ddot{q}_j \frac{\partial T}{\partial \dot{q}_j}-\sum_j \dot{q}_j \frac{\partial T}{\partial q_j}=-\sum_j \dot{q}_j \frac{\partial V}{\partial q_j}. \quad \ldots (5)$$

On using (3) and (4) we write equation (5) as

$$\frac{d(2T)}{dt}-\frac{dT}{dt}=-\frac{dV}{dt},$$

$$\Rightarrow \quad \frac{d}{dt}\left(T+V\right)=0,$$

$$\Rightarrow \quad T+V=const.$$

This implies that total energy is conserved.

Unit 3: ***Lagrange's Equations for Non-holonomic Constraints:***

Introduction:

We have seen that the constraints, which are not expressible in the form of equations are called non-holonomic constraints. We have also seen that this is not the only way to describe the non-holonomic system. A system is also said to be non-holonomic, if it corresponds to non-integrable differential equations of constraints. Such constraints can not be expressed in the form of equation of the type

$$f_l\left(q_j, t\right) = 0, l = 1, 2, 3, ..., m. \qquad ...(1)$$

Hence such constraints are called non-holonomic constraints. Obviously, holonomic system has integrable differential equations of constraints expressible in the form of equation.

Consider non-integrable differential constraints of the type

$$\sum_{k=1}^{n} a_{lk} dq_k + a_{lt} dt = 0, \qquad ...(2)$$

where a_{lk} and a_{lt} are functions of q_j and t . Constraints of this type will be holonomic only if, an integrating factor can be found that turns it in to an exact differential, and hence the constraints can be reduced to the form of equations.

However, neither equations (2) can be integrated nor one can find an integrating factor that will turn either of the equations in to perfect differentials. Hence the constraints cannot be reduced to the form (1). Hence the constraints of the type (2) are therefore non-holonomic.

Note also that non-integrable differential constraints of the type (2) are not the only type of non-holonomic constraints. The non-holonomic constraint conditions may involve higher order derivatives or may appear in the form of inequalities.

There is no general way of attacking non-holonomic problems. However, the constraints are not integrable, the differential equations of the constraint can be introduced in to the problem along with the differential equations of motion and the dependent equations are eliminated by the method of Lagrange's multipliers. The method is illustrated in the following theorem.

Theorem 9: Explain the method of Lagrange's undetermined multipliers to construct equations of motion of the system with non-holonomic constraints.

Proof: Consider a conservative non-holonomic system, where the equations of the non-holonomic constraints are given by

$$\sum_{k=1}^{n} a_{lk} dq_k + a_{lt} dt = 0, \qquad \text{. . . (1)}$$

where l = 1, 2, 3, …., m represents the number of constraints, and a_{lk}, a_{lt} are functions of q_j and t .

Since the constraints are non-holonomic, hence the equations expressing the constraints (1) cannot be used to eliminate the dependent co-ordinates and hence all the generalized co-ordinates are not independent, but are related by constraint relations.

In the variational (Hamilton's) principle, the time for each path is held fixed ($\delta t = 0$). Hence the virtual displacement δq_k must satisfy the following equations of constraints.

$$\sum_{k=1}^{n} a_{lk} \delta q_k = 0, \quad l = 1, 2, 3, ..., m \qquad \text{. . . (2)}$$

We can use these m-equations (2) to eliminate the dependent virtual displacement and reduce the number of virtual displacement to n-m independent one by the method of Lagrange's multipliers. Hence we multiply equations (2) by $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_m$ respectively and summing over l and integrating it between the limits t_0 to t_1 we get

$$\int_{t_0}^{t_1} \sum_{l=1}^{m} \sum_{k=1}^{n} \lambda_l a_{lk} \delta q_k dt = 0 \qquad \text{. . . (3)}$$

Hamilton's principle is assumed to hold for non-holonomic system, (see chapter 3) we therefore have

$$\int_{t_0}^{t_1} \delta L dt = 0.$$

$$\Rightarrow \int_{t_0}^{t_1} \sum_{k=1}^{n} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \right] \delta q_k dt = 0. \qquad \ldots (4)$$

Adding equations (3) and (4) we get

$$\int_{t_0}^{t_1} \left\{ \sum_{k=1}^{n} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_{l=1}^{m} \lambda_l a_{lk} \right] \delta q_k \right\} dt = 0 \qquad \ldots (5)$$

Note all the virtual displacement δq_k , $k = 1, 2, ..., n$ are not independent but connected by m equations (2). Now to eliminate the extra dependent virtual displacements we choose the multipliers $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_m$ such that the coefficients of m-dependent virtual displacements, in equation (5) are zero. i.e.,

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_{l=1}^{m} \lambda_l a_{lk} = 0, \text{ for } k = n-(m-1), ..., (n-1), n. \qquad \ldots (6)$$

Hence from equation (6) we have

$$\int_{t_0}^{t_1} \left\{ \sum_{k=1}^{n-m} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_{l=1}^{m} \lambda_l a_{lk} \right] \delta q_k \right\} dt = 0, \qquad \ldots (7)$$

where $\delta q_1, \delta q_2, \delta q_3, ..., \delta q_{n-m}$ are all independent. Hence it follows that

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_{l=1}^{m} \lambda_l a_{lk} = 0, \qquad \text{for } k = 1, 2, ..., n-m. \qquad \ldots (8)$$

Combining equations (6) and (8) we have finally the complete set of Lagrange's equations of motion for non-holonomic system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = \sum_{l=1}^{m} \lambda_l a_{lk}, \qquad k = 1, 2, ..., n-m, ..., n. \qquad \ldots (9)$$

Remarks:

1. The n-equations in (9) together with m-equations of constraints (1) are sufficient to determine (n + m) unknowns viz., the n-generalized co-ordinates q_j and m Lagrange's multipliers λ_l.

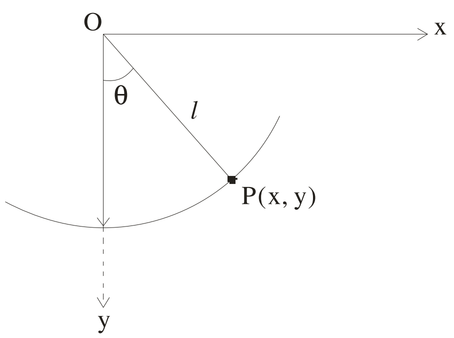
2. Lagrange's multiplier method can also be used for holonomic constraints, when it is inconvenient to reduce all the $q's$ to independent co-ordinates, and then obtain the forces of constraints.

Worked Examples

Example 15: Use Lagrange's undetermined multipliers to construct the equation of motion of simple pendulum and obtain the force of constraint.

Solution : Consider a simple pendulum of mass m and of constant length l. Let P (x, y) be the position co-ordinates of the pendulum. Then the equation of the constraint is

$$x^2 + y^2 = l^2. \qquad \ldots (1)$$



This shows that x, y are not the generalized co-ordinates. If (r, θ) are the polar co-ordinates of the pendulum, then the equation of constraint is (say)

$$f_1 \equiv r - l = 0. \qquad \ldots (2)$$

If this constraint is not used to eliminate the dependent variable r, then r, θ are the generalized co-ordinates. Hence the kinetic energy and potential energy of the pendulum are respectively given by

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2 \theta^2\right),$$

$$V = -mgr\cos\theta.$$

The Lagrangian of the pendulum L = T – V becomes

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta . \qquad \text{. . . (3)}$$

Differentiating the equation of the constraint, we get

$$dr = 0 .$$

Comparing this with the standard equation

$$a_{1r}dr + a_{1\theta}d\theta = 0, \qquad \left(\text{viz., } \sum_k a_{lk}dq_k = 0 \quad l = 1, k = 1, 2\right)$$

we get

$$a_{1r} = 1, \quad a_{1\theta} = 0 . \qquad \text{. . . (4)}$$

The Lagrange's equations of motion viz.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{l=1}^{m} \lambda_l a_{lk} , \qquad k = 1, 2, ..., n$$

become $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda_1 a_{1\theta} \quad ,$$

and $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda_1 a_{1r} .$$

These equations after solving become

$$\ddot{\theta} + \frac{g}{r}\sin\theta = 0 , \qquad \text{. . . (5)}$$

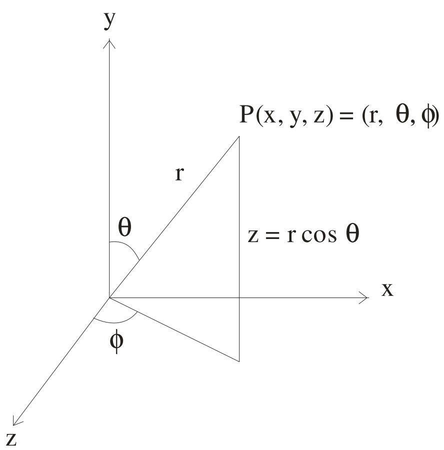
$$ml\dot{\theta} + mg\cos\theta = -\lambda_1 , \qquad \text{. . . (6)}$$

where λ_1 is the force of constraint, in this case it is the tension in the string. Equation (5) determines the motion of the pendulum under the constraint force given in (6).

Example 16 : Use Lagrange's undetermined multipliers to construct the equations of motion of spherical pendulum.

Solution : Let a particle of mass m move on a frictionless surface of radius r under the action of gravity. Let P (x, y, z) be the position co-ordinates of the pendulum. If

(r,θ,ϕ) are the spherical polar co-ordinates of the pendulum, then we have the relations



$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi, \qquad \ldots (1)$$
$$z = r\cos\theta,$$

where $x^2 + y^2 + z^2 = r^2$.

This shows that x, y, z are not the generalized co-ordinates. The kinetic and potential energies of the spherical pendulum are given by respectively

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right), \qquad \ldots (2)$$

$$V = mgr\cos\theta. \qquad \ldots (3)$$

Hence the Lagrangian of the system becomes

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - mgr\cos\theta. \qquad \ldots (4)$$

The equation of the constraint on the motion of the particle moving on the sphere is

$$f_1 \equiv r - l = 0. \qquad \ldots (5)$$

If this constraint is not used to eliminate the dependent variable r, then the generalized co-ordinates are (r,θ,ϕ). Differentiating equation (5) we get

$$dr = 0$$

Comparing this with the standard equation

$$a_{1r}dr + a_{1\theta}d\theta = 0, \qquad \left(\text{viz., } \sum_k a_{lk}dq_k = 0 \quad l = 1,\ k = 1,2,3\right).$$

we get

$$a_{1r} = 1, \quad a_{1\theta} = 0, \quad a_{1\phi} = 0. \qquad \ldots (6)$$

In this case the Lagrange's equations of motion viz.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=\sum_{l=1}^{m}\lambda_l a_{lk}, \qquad k=1,2,...,n$$

become
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=\lambda_1 a_{1r}, \qquad ...(7)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=\lambda_1 a_{1\theta}, \qquad ...(8)$$

and
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi}=\lambda_1 a_{1\phi}. \qquad ...(9)$$

Consequently, these equations reduce to

$$ml\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2-\frac{g}{l}\cos\theta\right)=-\lambda_1. \qquad ...(10)$$

This equation determines the constraint force. Similarly, from equations (8) and (9) we obtain

$$\ddot{\theta}-\sin\theta\cos\theta\dot{\phi}^2-\frac{g}{l}\sin\theta=0, \qquad ...(11)$$

and
$$\sin^2\theta\dot{\phi}^2=p_\phi\ (const.) \qquad ...(12)$$

Eliminating $\dot{\phi}$ between (11) and (12) we get

$$\ddot{\theta}-\frac{p_\phi^{\,2}}{\sin^3\theta}\cos\theta-\frac{g}{l}\sin\theta=0. \qquad ...(13)$$

Equations (10) and (13) determine the motion of the spherical pendulum.

Example 17 : A particle is constrained to move on the plane curve $xy=c$, where c is a constant, under gravity. Obtain the Lagrangian and hence the equation of motion.

Solution : Given that a particle is constrained to move on the plane curve

$$xy=c, \qquad ...(1)$$

The kinetic energy of the particle is given by

$$T=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) \qquad ...(2)$$

The potential energy is given by

$$V = mgy, \quad y \quad is \quad vertical \qquad \ldots (3)$$

We see that x and y are not linearly independent as they are related by the equation of constraint (1) and hence they are not the generalized co-ordinates. However, we eliminate the variable y by putting $y = \frac{c}{x}$ and hence $\dot{y} = -\frac{c}{x^2}\dot{x}$ in equations (2) and (3), we get

$$T = \frac{1}{2} m\, \dot{x}^2 \left(1 + \frac{c^2}{x^4}\right),$$

$$V = mg\,\frac{c}{x}$$

Here x is the generalized co-ordinate. Hence the Lagrangian of the particle becomes

$$L = \frac{1}{2} m\, \dot{x}^2 \left(1 + \frac{c^2}{x^4}\right) - \frac{mgc}{x}. \qquad \ldots (4)$$

The Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad ,$$

becomes

$$m\ddot{x}\left(1 + \frac{c^2}{x^4}\right) - 2\frac{c^2 m}{x^5}\dot{x}^2 - \frac{mgc}{x^2} = 0. \qquad \ldots (5)$$

Example 18 : A particle is constrained to move on the surface of a cylinder of fixed radius. Obtain the Lagrange's equation of motion.

Solution : The surface of the cylinder is characterized by the parametric equations given by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \qquad \ldots (1)$$

However, x, y, z are not the generalized co-ordinates as x and y are related by the equation of constraint $x^2 + y^2 = r^2$, r is a constant radius of the circle. Hence the generalized co-ordinates are θ and z. In terms of these generalized co-ordinates the kinetic and potential energies become

$$T = \frac{1}{2}m\left(r^2\dot{\theta}^2 + \dot{z}^2\right),$$
$$V = mgz.$$

Hence the Lagrangian is given by

$$L = \frac{1}{2}m\left(r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz. \qquad \ldots (2)$$

Therefore the $\theta-$Lagrange's equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

yields $\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \Rightarrow \quad mr^2\dot{\theta} = const(l)$

Integrating we get

$$\theta = \frac{l}{mr^2}t + \theta_0, \qquad \ldots (3)$$

where θ_0 is a constant of integration. Similarly, z- Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0,$$

gives $z = ut - \frac{1}{2}gt^2, \qquad \ldots (4)$

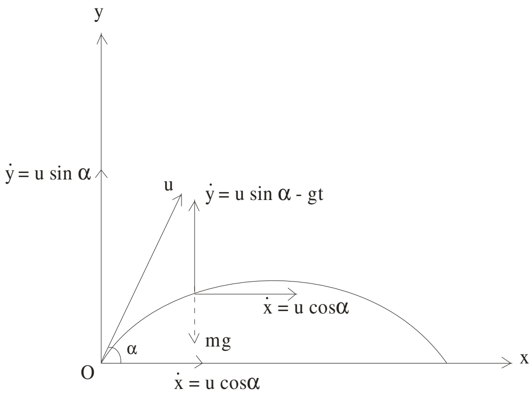
where $\dot{z} = u \quad at \quad t = 0$.

Example 19 : A particle of mass m is projected with initial velocity u at an angle α with the horizontal. Use Lagrange's equation to describe the motion of the projectile.

Solution : Let a particle of mass m be projected from O with an initial velocity u unit making an angle α with the horizontal line referred as x-axis. Let P (x, y) be the position of the particle at any instant t. Since x and y are independent and hence the generalized co-ordinates. The kinetic of the projectile is given by

$$T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right),$$

and the potential energy is $V = mgy$



Thus the Lagrangian function of the projectile is

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - mgy \,. \qquad \text{. . . (1)}$$

The x- Lagrange's equation of motion and y-Lagrange's equation of motion respectively give

$$\ddot{x} = 0 \quad and \quad \ddot{y} + g = 0 \qquad \text{. . . (2)}$$

To find the velocity of the projectile and its path at any instant we integrate equations (2) and using boundary conditions we readily obtain

$$\dot{x} = u\cos\alpha, \quad \dot{y} = u\sin\alpha - gt \qquad \text{. . . (3)}$$

These equations determine velocity at any time t. Integrating (3) once again and using boundary conditions we get

$$x = u\cos\alpha.t \quad and \quad y = u\sin\alpha.t - \frac{1}{2}gt^2 \qquad \text{. . . (4)}$$

Eliminating t between equations (4) we get

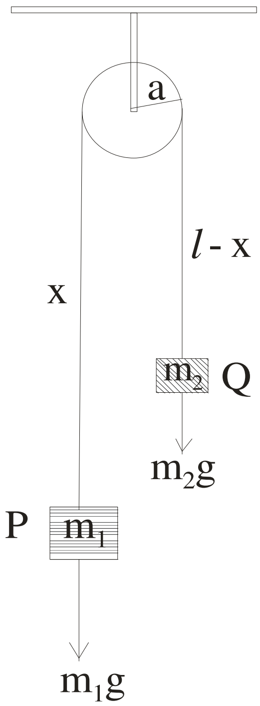
$$y = x\tan\alpha - \frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha} \qquad \text{.. . (5)}$$

This represents the path of the projectile and it is a parabola.

- ***Atwood's Machine :***

Example 20: Explain Atwood Machine and discuss it's motion.

Solution : Atwood machine consists of two masses m_1 *and* m_2 suspended over a frictionless pulley of radius 'a'. Both the ends of the string are attached the masses m_1 *and* m_2 respectively. Let the length of the string between m_1 *and* m_2 be l. Then we have from fig. that PA = x and QA = l - x. The system has only one degrees of freedom and x is the only generalized co-ordinate. Hence the kinetic energy of the system is given by



$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2. \qquad \ldots (1)$$

Considering the reference level as a horizontal plane passing through A, the potential energy of both masses is given by

$$V = -m_1 gx - m_2 g(l - x). \qquad \ldots (2)$$

Hence the Lagrangian of the system becomes

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx + m_2 gl. \qquad \ldots (3)$$

The corresponding Lagrange's equation of motion gives

$$\ddot{x} = \frac{(m_1 - m_2)}{m_1 + m_2} g \qquad \ldots (4)$$

The solution of this equation gives

$$x = \frac{1}{2}\frac{(m_1 - m_2)}{m_1 + m_2} gt^2 + x_0 t + y_0 , \qquad \ldots (5)$$

where x_0, y_0 are constants of integration.

Example 21: A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2 V(x) - V^2(x),$$

where V is some differentiable function of x. Find equation of motion for x (t).

Solution : Here the Lagrangian of the system is

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2 V(x) - V^2(x), \qquad \ldots (1)$$

We see from equation (1) that x is the only generalized co-ordinate. Therefore the corresponding Lagrange's equation of motion becomes

$$m\ddot{x} + \frac{\partial V}{\partial x} = 0. \qquad \ldots (2)$$

This equation of motion shows that the particle moves in a straight line under the action of a force $F = -\frac{\partial V}{\partial x}$.

Example 22 : Let a particle be moving in a field of force given by

$$F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2r\ddot{r}}{c^2}\right).$$

Find the Lagrangian of motion and hence the equation of motion.

Solution: One can check that $\nabla \times F \neq 0$, hence the force is non-conservative; consequently, the corresponding potential is generalized potential or velocity dependent potential. We know the component of generalized force corresponding to the generalized coordinate r is given by

$$Q_r = F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2r\ddot{r}}{c^2}\right).$$

We write this force as

$$Q_r = \frac{1}{r^2} - \frac{\dot{r}^2}{c^2 r^2} + \frac{2\ddot{r}}{c^2 r},$$

$$Q_r = \frac{1}{r^2} + \frac{\dot{r}^2}{c^2 r^2} + \frac{2\ddot{r}}{c^2 r} - \frac{2\dot{r}^2}{c^2 r^2},$$

$$Q_r = -\frac{\partial}{\partial r}\left(\frac{1}{r}+\frac{\dot{r}^2}{c^2 r}\right)+\frac{d}{dt}\frac{\partial}{\partial \dot{r}}\left(\frac{1}{r}+\frac{\dot{r}^2}{c^2 r}\right),$$

$$Q_r = -\frac{\partial U}{\partial r}+\frac{d}{dt}\frac{\partial U}{\partial \dot{r}},$$

$$U = \frac{1}{r}\left(1+\frac{\dot{r}^2}{c^2}\right). \qquad \ldots(1)$$

We notice that the potential energy U is the velocity dependent potential. The kinetic energy of the particle is given by

$$T = \frac{1}{2}m\dot{r}^2. \qquad \ldots(2)$$

Hence the Lagrangian of the particle becomes

$$L = \frac{1}{2}m\dot{r}^2 - \frac{1}{r}\left(1+\frac{\dot{r}^2}{c^2}\right). \qquad \ldots(3)$$

We see that r is the only generalized co-ordinate; hence the corresponding Lagrange's equation yields the equation of motion in the form

$$\ddot{r}\left(m-\frac{2}{rc^2}\right)+\frac{\dot{r}^2}{r^2c^2}-\frac{1}{r^2}=0. \qquad \ldots(4)$$

Example 23 : Derive the equation of motion of a particle falling vertically under the influence of gravity, when frictional forces obtainable from dissipation function $\frac{1}{2}Kv^2$ are present. Integrate the equation to obtain the velocity as a function of time.

Show also that the maximum possible velocity for fall from rest is $v = \frac{mg}{K}$.

Solution : Let a particle of mass m be falling vertically under the influence of gravity. Let z be the height of the particle at any instant t. Therefore the only generalized co-ordinate is z. Thus the Kinetic energy and the potential energy of the particle are given by

$$T = \frac{1}{2} m\dot{z}^2 \qquad \ldots (1)$$

and $$V = -mgz \qquad \ldots (2)$$

Hence the Lagrangian function becomes

$$L = \frac{1}{2} m\dot{z}^2 + mgz\,. \qquad \ldots (3)$$

We know the Lagrange's equation of motion for a system containing the frictional forces obtainable from a dissipation function $R = \frac{1}{2} Kv^2$ is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} + \frac{\partial R}{\partial \dot{z}} = 0\,. \qquad \ldots (4)$$

Solving this equation we get

$$m\ddot{z} + K\dot{z} - mg = 0\,. \qquad \ldots (5)$$

On integrating equation (5) we get

$$m\dot{z} + Kz - mgt + c_1 = 0\,, \qquad \ldots (6)$$

where c_1 is a constant of integration. Using initial conditions viz., when

$$t = 0, \quad \dot{z} = v = 0, \quad z = 0 \Rightarrow \quad c_1 = 0\,.$$

We have therefore

$$\dot{z} + \left(\frac{K}{m}\right) z = gt\,. \qquad \ldots (7)$$

This is a linear differential equation of first order whose solution is given by

$$z = \left(\frac{mg}{K}\right) t - \left(\frac{m}{K}\right)^2 g + c_2 e^{-\frac{Kt}{m}}\,.$$

As $$t = 0 \Rightarrow \quad z = 0 \Rightarrow \quad c_2 = \left(\frac{m}{K}\right)^2 g\,,$$

Hence

$$z = \left(\frac{mg}{K}\right) t - \left(\frac{m}{K}\right)^2 g + \left(\frac{m}{K}\right)^2 g\, e^{-\frac{Kt}{m}}\,. \qquad \ldots (8)$$

Differentiating equation (8) we obtain

$$\dot{z}=\left(\frac{mg}{K}\right)-\left(\frac{m}{K}\right)g\,e^{-\frac{Kt}{m}}. \qquad \ldots (9)$$

This shows that the velocity $\dot{z}$ is the function of time only. For maximum velocity we have

$$\frac{d\dot{z}}{dt}=0.$$

Hence the maximum velocity is obtained from (5) by putting $\ddot{z}=0$ and is given by

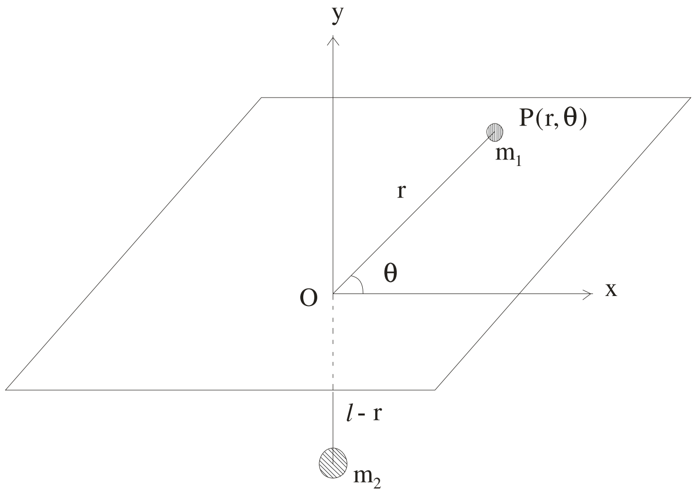
$$\dot{z}=\frac{mg}{K}.$$

Example 24: Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized co-ordinates for the system? Write down the Lagrangian for the system. Reduce the problem to a single second order differential equation and obtain a first integral of the equation.

Solution: Let the two mass points m_1 and m_2 be connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. We assume that m_2 moves only in a vertical line. The system is shown in the fig.

Let l be the length of a string. Consider OX as an initial line. Let (r,θ) be the position of the particle of mass m_1.

$$\Rightarrow \quad Om_2 = l - r.$$



Thus the system is specified by two generalized co-ordinates r and θ. The kinetic energy of the system is the sum of the kinetic energies of the two masses and is given by

$$T = \frac{1}{2} m_1 \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{1}{2} m_2 \dot{r}^2 \ . \qquad \text{. . . (1)}$$

Potential energy of mass m_1 is zero while that of mass m_2 is $-m_2 g \left(l - r \right)$. Hence the Lagrangian of the system becomes

$$L = \frac{1}{2} m_1 \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{1}{2} m_2 \dot{r}^2 + m_2 g \left(l - r \right). \qquad \text{. . . (2)}$$

The Lagrange's equations corresponding to the generalized co-ordinates r and θ respectively reduce to

$$\left(m_1 + m_2 \right) \ddot{r} - m_1 r \dot{\theta}^2 + m_2 g = 0 \qquad \text{. . . (3)}$$

and $$m_1 r^2 \dot{\theta} = const. \quad h \left(say \right) \qquad \text{. . . (4)}$$

These are the required equations of motion. Now eliminating $\dot{\theta}$ between (3) and (4) we obtain

$$\left(m_1 + m_2 \right) \ddot{r} - \frac{h^2}{m_1 r^3} + m_2 g = 0 . \qquad \text{. . . (5)}$$

This is the required single second order differential equation of motion. Now to find the first integral of (5), multiply equation (5) by $2\dot{r}$ and integrating it w. r. t. time t, we get

$$(m_1+m_2)\int 2\dot{r}\ddot{r}dt - \frac{h^2}{m_1}\int\left(\frac{\dot{r}}{r^3}\right)dt + 2m_2 g\int \dot{r}dt = const.$$

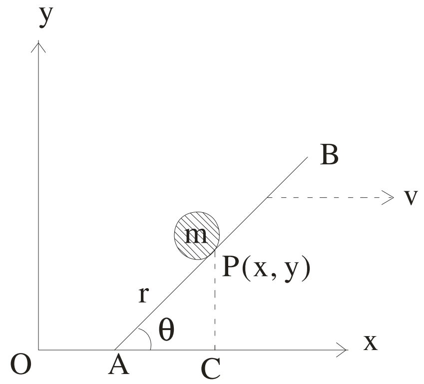
$$(m_1+m_2)\int d\left(\dot{r}^2\right) - \frac{2h^2}{m_1}\int d\left(-\frac{1}{2r^2}\right) + 2m_2 g\int dr = const.$$

$$(m_1+m_2)\dot{r}^2 + \frac{h^2}{m_1 r^2} + 2m_2 gr = const. \qquad \ldots (6)$$

This is required first integral of motion which represents total energy of the particle.

Example 25 : A body of mass m is thrown up an inclined plane which is moving horizontally with a constant velocity v. Use Lagrangian equation to find the locus of the position of the body at any time t after the motion sets in.

Solution: Let AB be an inclined plane moving horizontally with constant velocity v.



Therefore at some instant t the distance moved by the plane AB is given by

$$OA = v\cdot t \qquad \ldots (1)$$

Let at t = 0 a body of mass m be thrown up an inclined plane AB. Let P be the position of the particle at that instant t, where AP = r. If (x, y) are the co-ordinates of the particle at P, then we have

$$x = OA + AP\cos\theta,$$

$$\Rightarrow \quad x = vt + r\cos\theta \qquad \ldots (2)$$

and $y = r\sin\theta$, (note θ is a fixed angle) . . . (3)

The kinetic energy of the particle is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right).$$

We notice that x and y are related by equations (2) and (3) and hence will not be the generalized co-ordinates. The only generalized co-ordinate is r. Hence using equations (2) and (3) we write the expression for the kinetic energy in terms of generalized co-ordinate r as

$$T = \frac{1}{2}m\left(v^2 + \dot{r}^2 + 2\dot{r}v\cos\theta\right). \quad ...(4)$$

The potential energy of the particle is given by

$$V = mgr\sin\theta. \quad ...(5)$$

Hence the Lagrangian of the system is

$$L = \frac{1}{2}m\left(v^2 + \dot{r}^2 + 2\dot{r}v\cos\theta\right) - mgr\sin\theta. \quad ...(6)$$

Hence the corresponding r- Lagrange's equation reduces to the form

$$\ddot{r} = -g\sin\theta. \quad ...(7)$$

Integrating we get

$$\dot{r} = -g\sin\theta t + c_1.$$

At $t = 0$ let $\dot{r} = u$ be the initial velocity of the particle with which it is projected. This gives $c_1 = u$, hence

$$\dot{r} = u - g\sin\theta t. \quad ...(8)$$

Integrating once again we get

$$r = u.t - \frac{1}{2}gt^2\sin\theta + c_2.$$

At $t = 0,\ r = 0 \Rightarrow\ c_2 = 0.$

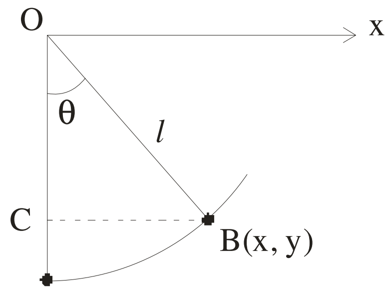
$$\Rightarrow\ r = ut - \frac{1}{2}gt^2\sin\theta.$$

Hence the locus of the position of the particle is given by

$$r^2 = \left(ut - \frac{1}{2}gt^2 \sin\theta\right)^2 = \left(x - vt\right)^2 + y^2. \qquad \text{. . . (9)}$$

Example 26 : Set up the Lagrangian and the Lagrange's equation of motion for simple pendulum.

Solution : Consider a simple pendulum of point mass m attached to one end of an inextensible light string of length l and other end is fixed at point O. The system is shown in fig. If $B(x, y)$ are the position co-ordinates of the pendulum at any instant t, then the equation of the constraint is given by



$$x^2 + y^2 = l^2, \qquad \text{. . . (1)}$$

where $x = l\sin\theta,\quad y = l\cos\theta,\quad \theta$ is the angle made by the pendulum with the vertical. This shows that x and y are not the generalized co-ordinates. We see that the angle θ determines the position of pendulum at any given time; hence it is a generalized co-ordinate. Hence the kinetic and potential energies of the pendulum become

$$T = \frac{1}{2}m\left(l\dot{\theta}\right)^2, \quad V = mgl\left(1 - \cos\theta\right). \qquad \text{. . . (2)}$$

Hence the Lagrangian of the motion becomes

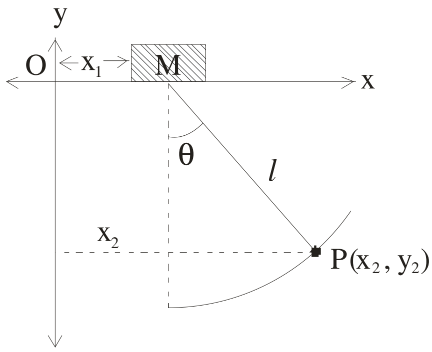
$$L = \frac{1}{2}m\left(l\dot{\theta}\right)^2 - mgl\left(1 - \cos\theta\right). \qquad \text{. . . (3)}$$

The θ-Lagrange's equation of motion gives

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0. \qquad \text{. . . (4)}$$

This is the second order differential equation that determines the motion of the simple pendulum.

Example 27 : A pendulum of mass is attached to a block of mass M. The block slides on a horizontal frictional less surface. Find the Lagrangian and equation of motion of the pendulum. For small amplitude oscillations derive an expression for periodic time.



Solution : Let a pendulum of point mass m be attached to one end of the light and inextensible string of length l and other end is attached to a block of mass M. The system is shown in fig. Let at any instant t the position co-ordinates of the block of mass M and the pendulum of mass m be $(x_1, 0)$ and (x_2, y_2) respectively,

where $$x_2 = x_1 + l\sin\theta,$$

$$y_2 = l\cos\theta.$$

We see that the position co-ordinates of the pendulum are related by the constraint equation; hence these are not the generalized co-ordinates. The generalized co-ordinates in this case are x_1 and θ. The kinetic energy of the system is the sum of the kinetic energy of the pendulum and the kinetic energy of the block. It is given by

$$T = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\left(\dot{x}_1^{\,2} + l^2\dot{\theta}^2 + 2l\dot{x}_1\dot{\theta}\cos\theta\right).$$

The potential energy of the pendulum is given by

$$V = -mgl\cos\theta.$$

Hence the Lagrangian of the system becomes

$$L = \frac{1}{2}(M+m)\dot{x}_1^2 + \frac{1}{2}m\left(l^2\dot{\theta}^2 + 2l\dot{x}_1\dot{\theta}\cos\theta\right) + mgl\cos\theta. \qquad \ldots (1)$$

Since θ and x_1 are the generalized co-ordinates, hence the corresponding Lagrange's equations of motion viz.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0 \text{ and } \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}=0,$$

respectively reduces to

$$ml^2\ddot{\theta}+ml\cos\theta\ddot{x}_1+mgl\sin\theta=0, \qquad ...(2)$$

$$(M+m)\ddot{x}_1+ml\ddot{\theta}=0. \qquad ...(3)$$

However, if θ is small then we have $\sin\theta=\theta$ and $\cos\theta=1$. Consequently equation (2) becomes

$$\ddot{\theta}+\frac{\ddot{x}_1}{l}+\frac{g}{l}\theta=0 \qquad ...(4)$$

Eliminating $\ddot{x}_1$ between equations (3) and (4) we get

$$\ddot{\theta}=\frac{(M+m)g}{Ml}\theta. \qquad ...(5)$$

This is the required equation of simple harmonic motion. The periodic time T is given by

$$T=\frac{2\pi}{\sqrt{accel^n.per\ unit\ displacement}}$$

$$\Rightarrow \quad T=2\pi\sqrt{\frac{Ml}{(M+m)g}} .$$

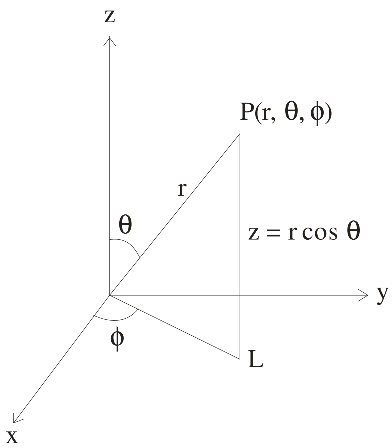
- ***Spherical Pendulum:*** A point mass constrained to move on the surface of a sphere is called spherical pendulum.

Example 28 : In a spherical pendulum a particle of mass m moves on the surface of a sphere of radius r in a gravitational field. Show that the equation of motion of the particle may be written as

$$\ddot{\theta}-\frac{p_\phi^2\cos\theta}{m^2r^4\sin^3\theta}-\frac{g}{r}\sin\theta=0,$$

where p_ϕ is the constant of angular momentum.

Solution : Let P (x, y, z) be the position co-ordinates of the particle moving on the surface of a sphere of radius r. If (r, θ, ϕ) are its spherical co-ordinates, then we have



$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi, \qquad \ldots (1)$$
$$z = r\cos\theta$$

It clearly shows that x, y, z are not the generalized co-ordinates, as they are related by the constraint equations (1). The generalized co-ordinates are (θ, ϕ). Hence the kinetic and potential energies of particle are respectively given by

$$T = \frac{1}{2}mr^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right),$$
$$V = mgr\cos\theta$$

Hence the Lagrangian function becomes

$$L = \frac{1}{2}mr^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mgr\cos\theta. \qquad \ldots (2)$$

The two Lagrange's equations of motion corresponding to the generalized co-ordinates θ and ϕ reduce to

$$mr^2\ddot{\theta} - mr^2\sin\theta\cos\theta\dot{\phi}^2 - mgr\sin\theta = 0 \qquad \ldots (3)$$

and $$mr^2\sin^2\theta\dot{\phi} = const. = p_\phi. \qquad \ldots (4)$$

Eliminating $\dot{\phi}$ between equations (3) and (4) we get

$$\ddot{\theta} - \frac{p_\phi^2\cos\theta}{m^2r^4\sin^3\theta} - \frac{g}{r}\sin\theta = 0, \qquad \ldots (5)$$

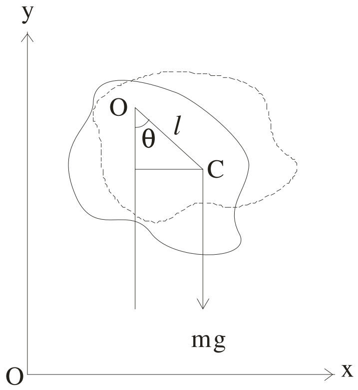
where p_ϕ is a constant of angular momentum.

- ***Compound Pendulum :***

A rigid body capable of oscillating in a vertical plane about a fixed horizontal axis under the action of gravity is called a compound pendulum.

Example 29 : Set up the Lagrangian and the Lagrange's equation of motion for the compound pendulum.

Solution : Let O be a fixed point of a rigid body through which axis of rotation passes. Let C be the center of mass, and $OC = l$ .Let m be the mass of the pendulum and I the moment of inertia about the axis of rotation. If θ is the angle of deflection of the body then the rotational kinetic energy of the pendulum is given by



$$T = \frac{1}{2} I \dot{\theta}^2 \qquad \text{. . . (1)}$$

The potential energy relative to the horizontal plane through O is

$$V = -mgl \cos\theta \qquad \text{. . . (2)}$$

Hence the Lagrangian of compound pendulum becomes

$$L = \frac{1}{2} I \dot{\theta}^2 + mgl \cos\theta \qquad \text{. . . (3)}$$

Thus the Lagrange's equation of motion corresponding to the generalized co-ordinate θ becomes

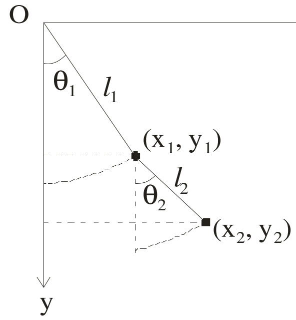
$$\ddot{\theta} + \frac{mgl}{I} \sin\theta = 0 . \qquad \text{. . . (4)}$$

The periodic time of oscillation is given by

$$T = 2\pi \sqrt{\frac{I}{mgl}} . \qquad \text{. . . (5)}$$

Example 30 : Obtain the Lagrangian and equations of motion for a double pendulum vibrating in a vertical plane.

Solution : A double pendulum moving in a plane consists of two particles of masses m_1 and m_2 connected by an inextensible string. The system is suspended by another inextensible and weightless string fastened to one of the masses as shown in the fig. Let θ_1 and θ_2 be the deflections of the pendulum from vertical. These are the generalized co-ordinates of the system. Let l_1 and l_2 be the lengths of the strings and $(x_1, y_1), (x_2, y_2)$ be the rectangular position co-ordinates of the masses m_1 and m_2 respectively at any instant t. From the fig. we have



$$\begin{aligned} x_1 &= l_1 \sin\theta_1, \quad y_1 = l_1 \cos\theta_1; \\ x_2 &= l_1 \sin\theta_1 + l_2 \sin\theta_2, \\ y_2 &= l_1 \cos\theta_1 + l_2 \cos\theta_2 \end{aligned} \qquad . . (1)$$

The total kinetic energy of the system is given by

$$T = \frac{1}{2} m_1 \left(\dot{x}_1^{\,2} + \dot{y}_1^2 \right) + \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2 \right). \qquad . . . (2)$$

Using equation (1) we obtain

$$T = \frac{1}{2} m_1 l_1^{\,2} \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^{\,2} \dot{\theta}_1^2 + l_2^{\,2} \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]. \qquad . . . (3)$$

Taking the reference level as a horizontal plane through the point of suspension O, the total potential energy of the system is given by

$$V = -m_1 g l_1 \cos\theta_1 - m_2 g \left(l_1 \cos\theta_1 + l_2 \cos\theta_2 \right). \qquad . . . (4)$$

Hence the Lagrangian of the system becomes

$$\begin{aligned} L = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 + \\ + m_1 g l_1 \cos\theta_1 + m_2 g \left(l_1 \cos\theta_1 + l_2 \cos\theta_2 \right). \qquad . . . (5) \end{aligned}$$

Solving the Lagrange's equations of motion corresponding to the generalized co-ordinates θ_1 and θ_2 we obtain

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 + m_2 l_1 l_2 \sin(\theta_1 - \theta_2) \dot{\theta}_2^2 + (m_1 + m_2) g l_1 \sin\theta_1 = 0, \quad . . (6)$$

and

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 - m_2 l_1 l_2 \sin(\theta_1 - \theta_2) \dot{\theta}_1^2 + m_2 g l_2 \sin\theta_2 = 0. \qquad . . . (7)$$

Equations (6) and (7) describe the motion of the double pendulum.

Note : If in particular, two masses are equal, the lengths of the pendula are also equal and $\theta_1 - \theta_2$ is very small, then for small angle we have $\sin\theta = \theta, \quad \cos\theta = 1$ and hence neglecting the terms involving $\dot{\theta}^2$ we get from equations (6) and (7) that

$$\begin{aligned} 2l\ddot{\theta}_1 + l\ddot{\theta}_2 + 2g\theta_1 &= 0, \\ l\ddot{\theta}_2 + l\ddot{\theta}_1 + g\theta_2 &= 0. \end{aligned} \qquad \text{. . . (8)}$$

Example 31: A particle is moving on a cycloid $s = 4a\sin\theta$ under the action of gravity. Obtain the Lagrangian and Lagrange's equation of motion.

Solution : A particle is moving on a cycloid under the action of gravity whose intrinsic equation is given by

$$s = 4a\sin\theta \qquad \text{. . . (1)}$$

From equation (1) we find

$$ds = 4a\cos\theta d\theta\,,$$

$$\Rightarrow \quad ds^2 = 16a^2\cos^2\theta d\theta^2\,. \qquad \text{. . . (2)}$$

Hence the kinetic energy of the particle is given by

$$T = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 \qquad \text{. . . (3)}$$

$$\Rightarrow \quad T = 8a^2m\cos^2\theta\dot{\theta}^2.$$

To find the potential energy of the particle, let P (x, y) be the position of the particle at any instant, where the Cartesian equations of the cycloid are given by

$$\begin{aligned} x &= a(2\theta + \sin 2\theta), \\ y &= a(1 - \cos 2\theta). \end{aligned} \qquad \text{. . . (4)}$$

The potential energy of the particle is therefore

$$V = mgy\,,$$

$$V = mga(1 - \cos 2\theta)\,. \qquad \text{. . . (5)}$$

Thus the Lagrangian of motion of the particle is

$$L = 8a^2 m\cos^2\theta\dot{\theta}^2 - mga(1-\cos 2\theta) \qquad \text{. . . (6)}$$

We see that the system has one degree of freedom and θ is the only generalized co-ordinate. Hence the Lagrange's equation of motion is obtained as

$$\ddot{\theta} - \tan\theta\,\dot{\theta}^2 + \left(\frac{g}{4a}\right)\tan\theta = 0. \qquad \text{. . . (7)}$$

Example 32: Obtain the expression for kinetic energy of a particle constrained to move on a horizontal xy plane which is rotating about the vertical z-axis with angular velocity ω. Show that

$$\dot{x}\frac{\partial T}{\partial \dot{x}} + \dot{y}\frac{\partial T}{\partial \dot{y}} = 2T_2 + T_1,$$

where

$$T_2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right),$$

$$T_1 = m\omega\left(x\dot{y} - y\dot{x}\right)$$

Show also that the Lagrangian of the particle is given by

$$L = \frac{1}{2}m\left[\left(\dot{x}-\omega y\right)^2 + \left(\dot{y}+\omega x\right)^2\right] - V(x, y).$$

Solution : A particle is moving on the xy-plane and the plane itself is rotating with respect to z-axis with angular velocity ω. Let (x_1, y_1, z_1) be the co-ordinates of the particle with respect to the fixed co-ordinate system and (x, y, z) the co-ordinates of the particle with respect to rotating axes. The co-ordinates with respect to the rotating axes are taken as the generalized co-ordinates. The transformation equations for rotation are given by

$$\begin{aligned} x_1 &= x\cos\omega t - y\sin\omega t, \\ y_1 &= x\sin\omega t + y\cos\omega t, \end{aligned} \qquad \text{. . . (1)}$$

$$z_1 = z \qquad \text{. . . (2)}$$

Since z fixed is the constraint, therefore, the system has only two degrees of freedom and hence only two generalized co-ordinates and that are x and y. We note here that the transformation equations (1) are not independent of time, though the constraint equation (2) is. Thus the kinetic energy of the particle is given by

$$T=\frac{1}{2}m\left(\dot{x}_1^2+\dot{y}_1^2\right) \qquad \ldots (3)$$

Differentiating equations (1) with respect to t and putting in (3) we obtain

$$T=\frac{1}{2}m\left[\left(\dot{x}^2+\dot{y}^2\right)+\omega^2\left(x^2+y^2\right)+2\omega\left(x\dot{y}-y\dot{x}\right)\right],$$

$$T=\frac{1}{2}m\left[\left(\dot{x}-\omega y\right)^2+\left(\dot{y}+\omega x\right)^2\right]. \qquad \ldots (4)$$

We can also write this equation as

$$T=T_2+T_1+T_0,$$

where

$$T_2=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right),$$

$$T_1=m\omega\left(x\dot{y}-y\dot{x}\right),$$

$$T_0=\frac{1}{2}m\omega^2\left(x^2+y^2\right).$$

Differentiating (4) w. r. t. $\dot{x},\dot{y}$ we get

$$\frac{\partial T}{\partial \dot{x}}=m\dot{x}-\omega m y,$$

$$\frac{\partial T}{\partial \dot{y}}=m\dot{y}+\omega m x.$$

This gives on solving

$$\dot{x}\frac{\partial T}{\partial \dot{x}}+\dot{y}\frac{\partial T}{\partial \dot{y}}=2T_2+T_1,$$

Now if V is the potential energy of the particle which is function of the generalized co-ordinates x, y then the Lagrangian of the particle is given by

$$L=T-V,$$

$$L=\frac{1}{2}m\left[\left(\dot{x}-\omega y\right)^2+\left(\dot{y}+\omega x\right)^2\right]-V\left(x,y\right). \qquad \ldots (5)$$

Example 33 : The Lagrangian of a system is

$$L=\frac{m}{2}\left(a\dot{x}^2+2b\dot{x}\dot{y}+c\dot{y}^2\right)-\frac{k}{2}\left(ax^2+2bxy+cy^2\right),\quad a,b,c$$

are arbitrary constants such that $b^2-4ac\neq 0$. Write down the equation of motion. Examine the two cases $a=0,c=0$ *and* $b=0,c=-a$ and interpret physically.

Solution: Given that

$$L=\frac{m}{2}\left(a\dot{x}^2+2b\dot{x}\dot{y}+c\dot{y}^2\right)-\frac{k}{2}\left(ax^2+2bxy+cy^2\right), \qquad \ldots (1)$$

a,b,c are arbitrary constants. We notice that x and y are the generalized co-ordinates. Hence the corresponding Lagrange's equations of motion are

$$m\left(a\ddot{x}+b\ddot{y}\right)+k\left(ax+by\right)=0, \qquad \ldots (2)$$

$$m\left(b\ddot{x}+c\ddot{y}\right)+k\left(bx+cy\right)=0. \qquad \ldots (3)$$

<u>**Case (i)**</u> If $a=0,c=0$.

Equations (2) and (3) reduce to

$$\ddot{y}+\left(\frac{k}{m}\right)y=0, \qquad \ldots (4)$$

and $$\ddot{x}+\left(\frac{k}{m}\right)x=0. \qquad \ldots (5)$$

<u>**Case (ii)**</u> $b=0,c=-a$. Putting this in equations (2) and (3) we get

$$\ddot{x}+\left(\frac{k}{m}\right)x=0, \qquad \ldots (6)$$

and $$\ddot{y}+\left(\frac{k}{m}\right)y=0. \qquad \ldots (7)$$

We see from the equations (4), (5) and (6), (7) that in both the cases we get the same set of equations of motion. These are the differential equations of particle performing a linear S. H. M. The solution of these equations gives the displacement of the particle with the frequency of oscillation $\omega = \sqrt{\frac{k}{m}}$.

Unit 4: ***Generalized Momentum and Cyclic co-ordinates:***

Introduction: We have proved some conservation theorems in the Unit 1. In this unit we will prove that the conservation theorems are continued to be true for cyclic generalized coordinates.

Definitions: In Newtonian mechanics the components of momentum (linear) are defined as the derivative of kinetic energy with respect to the corresponding components of velocity. i.e., If

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

is the kinetic energy of a particle, then the components of momentum of the particle are defined as

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial T}{\partial \dot{y}} = m\dot{y}, \quad p_z = \frac{\partial T}{\partial \dot{z}} = m\dot{z}. \qquad \text{. . . (1)}$$

- ***Generalized Momentum :***

Consider a conservative system in which the forces are derivable from a potential function V which is dependent on position only. In this case we have

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}.$$

Thus the quantity

$$p_j = \frac{\partial L}{\partial \dot{q}_j}. \qquad \text{. . . (2)}$$

is called the generalized momentum associated with the generalized co-ordinates q_j.

Note 1 : The definition of generalized momentum (2) is exactly analogous to the usual definition of momentum (1).

Note 2 : The word 'generalized' momentum subsumes linear momentum and angular momentum of the particle.

e.g. To illustrate, let a particle be moving in plane polar co-ordinates (r,θ). Then we have its kinetic energy is given by

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right).$$

Hence the generalized momentum corresponding to the generalized co-ordinate r and θ are respectively given by

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r},$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr\dot{\theta}.$$

We notice that p_r and p_θ represent respectively the linear momentum and angular momentum of the particle.

- **Cyclic or ignorable Co-ordinates:**

Co-ordinates which are absent in the Lagrangian are called cyclic or ignorable co-ordinates, although the Lagrangian may contain the corresponding generalized velocity $\dot{q}_j$ of the particle.

- ***Conservation Theorem for generalized momentum :***

Theorem 10 : Show that the generalized momentum corresponding to a cyclic co-ordinate is conserved.

Proof : The Lagrange's equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad \text{. . . (1)}$$

where $L = L\left(q_j, \dot{q}_j, t\right)$ is the Lagrangian function. If the generalized co-ordinate q_j is cyclic in L, then it must be absent in the Lagrangian. Obviously we have therefore,

$$\frac{\partial L}{\partial q_j} = 0. \qquad \ldots (2)$$

Thus the Lagrange's equation of motion (1) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \qquad \ldots (3)$$

But we have $\quad p_j = \dfrac{\partial L}{\partial \dot{q}_j}.$

This implies that

$$\frac{d}{dt}\left(p_j\right) = 0,$$

$$\Rightarrow \quad p_j = const.$$

This proves that the generalized momentum corresponding to the cyclic co-ordinate is conserved.

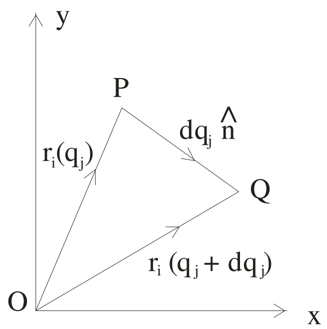
- ***Conservation Theorem for Linear momentum :***

We will show that the conservation Theorems are continued to be true for cyclic generalized co-ordinates.

Theorem 12 : If the cyclic generalized co-ordinate q_j is such that dq_j represents the translation of the system, then prove that the total linear momentum is conserved.

Proof : Consider a conservative system so that the potential energy V is a function of generalized co-ordinates only.

i.e. $V = V\left(q_j\right)$.



Hence we have

$$\frac{\partial V}{\partial \dot{q}_j} = 0. \qquad \ldots (1)$$

Let $P = \overline{r}_i\left(q_j\right)$ be the initial position of the system and let it be translated to a point $Q = \overline{r}_i\left(q_j + dq_j\right)$, so that

$$PQ = dq_j\,\hat{n}, \qquad ...(2)$$

where $\hat{n}$ is the unit vector along the direction of translation and dq_j represents the translation of the system.

We know by the first principle that

$$\frac{\partial \overline{r}_i}{\partial q_j} = \lim \frac{\overline{r}_i\left(q_j + dq_j\right) - \overline{r}_i\left(q_j\right)}{dq_j} \quad as \quad dq_j \to 0,$$

$$\frac{\partial \overline{r}_i}{\partial q_j} = \lim \frac{PQ}{dq_j} \quad as \quad dq_j \to 0,$$

$$\frac{\partial \overline{r}_i}{\partial q_j} = \lim \frac{dq_j}{dq_j}\hat{n} \quad as \quad dq_j \to 0.$$

This gives on using (1)

$$\frac{\partial \overline{r}_i}{\partial q_j} = \hat{n}. \qquad ...(3)$$

Now the generalized force is given by

$$\overline{Q}_j = \sum_i \overline{F}_i \frac{\partial \overline{r}_i}{\partial q_j}.$$

On using equation (3) we get

$$\overline{Q}_j = \sum_i \overline{F}_i \hat{n} \Rightarrow \overline{Q}_j = \overline{F}\hat{n}, \qquad ...(4)$$

where F is the total force acting on the system. Equation (4) implies that $\overline{Q}_j$ are the components of the total force in the direction of translation $\hat{n}$.

Now the generalized momentum p_j is defined by

$$p_j = \frac{\partial T}{\partial \dot{q}_j}, \qquad ...(5)$$

where T is the kinetic energy of the system and is given by

$$T = \sum_i \frac{1}{2} m_i \dot{r}_i^2 .$$

Thus we have

$$p_j = \frac{\partial}{\partial \dot{q}_j} \sum_i \frac{1}{2} m_i \dot{r}_i^2 ,$$

$$p_j = \sum_i m_i \dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_j} ,$$

$$p_j = \sum_i m_i \dot{r}_i \frac{\partial r_i}{\partial q_j} , \quad as \quad \frac{\partial r_i}{\partial q_j} = \frac{\partial \dot{r}_i}{\partial \dot{q}_j} .$$

On using equation (3) we get

$$p_j = \sum_i m_i \dot{r}_i \hat{n},$$

$$p_j = \sum_i p_i \, \hat{n},$$

$$p_j = p \, \hat{n},$$

where p is the total linear momentum of the system. This equation shows that p_j are the components of total linear momentum of the system along the displacement dq_j. Since in the translation of the system, velocity is not affected and hence the kinetic energy of the system. This means that q_j will not appear in kinetic energy expression. That is, change in the kinetic energy due to change in q_j is zero. Consequently, we have

$$\frac{\partial T}{\partial q_j} = 0 . \qquad \text{. . . (6)}$$

Thus from the Lagrange's equation of motion on using equations (1) and (6) we have,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) + \frac{\partial V}{\partial q_j} = 0$$

$$\Rightarrow \quad \dot{p}_j = -\frac{\partial V}{\partial q_j} = \bar{Q}_j . \qquad \ldots (7)$$

Now, if the co-ordinate q_j is cyclic in the Lagrangian, then

$$\frac{\partial L}{\partial q_j} = 0$$

Due to equation (6), we have

$$\frac{\partial V}{\partial q_j} = 0$$

Consequently, we have from equation (7)

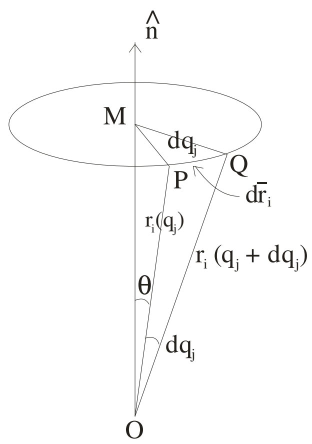
$$\dot{p}_j = 0 \quad \Rightarrow \quad p_j = const. \qquad \ldots (8)$$

This shows that corresponding to the cyclic co-ordinate q_j the total linear momentum is conserved.

Note : This can also be stated from equation (7) that if the components of total force Q_j are zero, then the total linear momentum is conserved.

Theorem 13 : If the cyclic generalized co-ordinate q_j is such that dq_j represents the rotation of the system of particles around some axis $\hat{n}$, then prove that the total angular momentum is conserved along $\hat{n}$.

Proof : Consider a conservative system so that the potential energy V is a function of generalized co-ordinates only.



$$V = V\left(q_j\right).$$

Hence we have

$$\frac{\partial V}{\partial \dot{q}_j} = 0 . \qquad \ldots (1)$$

Let the system be rotated through an angle dq_j around a unit vector $\hat{n}$. This gives the rotation of the vector

$$OP = \overline{r}_i\left(q_j\right) \text{ to } OQ = \overline{r}_i\left(q_j + dq_j\right)$$

The magnitude of change in position vector $\overline{r}_i\left(q_j\right)$ due to rotation is given by

$$\begin{aligned} \left|d\overline{r}_i\right| &= MPdq_j, \\ &= OP\sin\theta dq_j, \\ \left|d\overline{r}_i\right| &= r_i\left(q_j\right)\sin\theta dq_j \end{aligned}$$

$$\Rightarrow \quad \left|\frac{d\overline{r}_i}{dq_j}\right| = r_i\sin\theta. \qquad \text{. . . (2)}$$

Since $\overline{r}_i = \overline{r}_i\left(q_1, q_2, q_3, ..., q_n, t\right)$, we write therefore from equation (2)

$$\frac{\partial\overline{r}_i}{\partial q_j} = \hat{n}\times\overline{r}_i. \qquad \text{. . . (3)}$$

This shows that $\frac{\partial\overline{r}_i}{\partial q_j}$ is perpendicular to both $\hat{n}$ and $\overline{r}_i$.

The generalized force is given by

$$\overline{Q}_j = \sum_i \overline{F}_i \frac{\partial\overline{r}_i}{\partial q_j}.$$

On using equation (3) we get

$$\begin{aligned} \overline{Q}_j &= \sum_i \overline{F}_i\left(\hat{n}\times\overline{r}_i\right) \\ \Rightarrow \quad \overline{Q}_j &= \sum_i \hat{n}\left(\overline{r}_i\times\overline{F}_i\right) \qquad \text{. . . (4)} \\ &= \hat{n}\sum_i N_i, \end{aligned}$$

where $N_i = \overline{r}_i\times\overline{F}_i$ is the torque on the i^{th} particle. If $N = \sum_i N_i$ is the total torque acting on the system, then equation (4) shows that $\overline{Q}_j$ are the components of the total torque along the axis of rotation. Now the generalized momentum p_j is defined by

$$p_j = \frac{\partial T}{\partial \dot{q}_j}. \qquad \text{. . . (5)}$$

where T is the kinetic energy of the system and is given by

$$T = \sum_i \frac{1}{2} m_i \dot{r}_i^2 ,$$

Thus we have

$$p_j = \frac{\partial}{\partial \dot{q}_j} \sum_i \frac{1}{2} m_i \dot{r}_i^2 ,$$

$$p_j = \sum_i m_i \dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_j},$$

$$p_j = \sum_i m_i \dot{r}_i \frac{\partial r_i}{\partial q_j}, \quad as \quad \frac{\partial r_i}{\partial q_j} = \frac{\partial \dot{r}_i}{\partial \dot{q}_j}.$$

On using equation (3) we get

$$p_j = \sum_i m_i \dot{r}_i \left(\hat{n} \times \overline{r}_i \right),$$

$$p_j = \sum_i p_i \left(\hat{n} \times \overline{r}_i \right),$$

$$p_j = \sum_i \hat{n} \left(\overline{r}_i \times p_i \right),$$

$$p_j = \hat{n} \sum_i \left(\overline{r}_i \times p_i \right)$$

$$p_j = \hat{n} \sum_i L_i$$

where $L = \sum_i L_i$ is the total angular momentum of the system.

Thus we have

$$p_j = \hat{n} L . \qquad \text{. . . (6)}$$

This equation shows that p_j are the components of total angular momentum of the system along the axis of rotation. Since the rotation of the system does not change the magnitude of the velocity and hence the kinetic energy of the system. This means that T does not depend on positions q_j. That is, change in the kinetic energy due to change in q_j is zero. Consequently, we have

$$\frac{\partial T}{\partial q_j} = 0. \qquad \ldots (7)$$

Thus from the Lagrange's equation of motion on using equations (1) and (7) we have,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial V}{\partial q_j} = 0$$

$$\Rightarrow \quad \dot{p}_j = -\frac{\partial V}{\partial q_j} = \bar{Q}_j. \qquad \ldots (8)$$

Now, if the co-ordinate q_j is cyclic in the Lagrangian, then

$$\frac{\partial L}{\partial q_j} = 0.$$

Due to equation (7), this gives

$$\Rightarrow \quad \frac{\partial V}{\partial q_j} = 0.$$

Consequently, we have from equation (8)

$$\dot{p}_j = 0 \quad \Rightarrow \quad p_j = const. \qquad \ldots (9)$$

This shows that corresponding to the cyclic co-ordinate q_j the total angular momentum is conserved.

Note : From equation (8) the Theorem can also be stated as, if the applied torque is zero then the total angular momentum is conserved.

Exercise:

1. Show that

$$\text{i) } \delta L = \frac{d}{dt}\left(\sum_j p_j \delta q_j\right), \text{ ii) } 2\frac{\partial T}{\partial p_j} = \dot{q}_j, \text{ iii) } \delta T + \delta V = 0$$

for conservative scleronomic systems.

2. Derive the Newton's equation of motion from the Lagrange's equation of motion for a particle moving under the action of the force F.

Hint : The force is explicitly given, use Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j,$$

where T is the kinetic energy of the particle and is given by $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$.

3. A particle is moving on a cycloid $s = 4a\sin\left(\frac{\theta}{2}\right)$ under the action of gravity.

Obtain the Lagrangian and the equation of motion.

Ans:

$$L = 2a^2 m\cos^2\left(\frac{\theta}{2}\right)\dot{\theta}^2 - mga\left(1-\cos\theta\right),$$

$$\ddot{\theta} - \frac{1}{2}\tan\left(\frac{\theta}{2}\right)\dot{\theta}^2 + \frac{g}{2a}\tan\left(\frac{\theta}{2}\right) = 0$$

4. Show that the force F defined by

$$\bar{F} = \left(y^2z^3 - 6xz^2\right)i + 2xyz^3 j + \left(3xy^2z^2 - 6x^2z\right)k$$

is conservative and hence find the total energy of the particle.

Ans : For conservative force

$$\nabla \times \bar{F} = 0, \quad E = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + 3x^2z^2 - xy^2z^3.$$

5. Show that Newton's equation of motion is the necessary condition for the action to have the stationary value.

6. Show that the two Lagrangians $L_1 = (q+\dot{q})^2$, $L_2 = (q^2+\dot{q}^2)$ are equivalent.

Ans : Both the Lagrangians produce the same equation of motion $\ddot{q} - q = 0$.

7. For a mechanical system the generalized co-ordinates appear separately in the kinetic energy and the potential energy such that

$$T = \sum f_i(q_i)\dot{q}_i^2, \quad V = \sum V_i(q_i).$$

Show that the Lagrange's equations reduce to $2f_i\ddot{q}_i + f_i'\dot{q}_i^2 + V_i' = 0, i = 1, 2, ..., n$.

8. The length of a simple pendulum changes with time such that $l = a + bt$, where a and b are constants. Find the Lagrangian and the equation of motion.

Ans : Equation of motion $(a+bt)\ddot{\theta} + 2b\dot{\theta} + g\sin\theta = 0.$

9. Find the Lagrangian and the equation of motion of a particle of mass m moving on the surface characterized by

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = r\cot\theta.$$

Ans : $$L = \frac{1}{2}m\left(\dot{r}^2\cos ec^2\theta + r^2\dot{\phi}^2\right) - mgr\cot\theta.$$

And equation of motion is $\ddot{r} - r\sin^2\theta\dot{\phi}^2 + g\cos\theta\sin\theta = 0$.

10. Find the Lagrangian and the equation of motion of a particle moving on the surface obtained by revolving the line $x = z$ about z - axis.

Hint : Surface of revolution is a cone $x^2 + y^2 = z^2$

Ans : Lagrangian of the particle $L = \frac{1}{2}m\left(2\dot{r}^2 + r^2\dot{\phi}^2\right) - mgr$.

Equation of motion $\ddot{r} - \frac{1}{2}r\dot{\phi}^2 + \frac{1}{2}g = 0$.

11. Describe the motion of a particle of mass m moving near the surface of the earth under the earth's gravitational field by Lagrange's procedure.

Ans.: $L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz,$

and equations of motion are $\ddot{x} = 0,\ \ddot{y} = 0,\ \ddot{z} = -g$.

12. A particle of mass m can move in a frictionless thin circular wire of radius r. If the wire rotates with an angular velocity ω about a vertical diameter, deduce the differential equation of motion of the particle.

Ans : Equation of motion of the particle : $\ddot{\theta} - \omega^2 \sin\theta \cos\theta - \frac{g}{r}\sin\theta = 0.$