CHAPTER - III

HAMILTON'S PRINCIPLE AND HAMILTON'S FORMULATION

Unit 1: Hamilton's Principles:

- ***Introduction :***

In the Chapter II we have used the techniques of variational principles of Calculus of Variation to find the stationary path between two points. Hamilton's principle is one of the variational principles in mechanics. All the laws of mechanics can be derived by using the Hamilton's principle. Hence it is one of the most fundamental and important principles of mechanics and mathematical physics.

In this unit we define Hamilton's principle for conservative and non-conservative systems and derive Hamilton's canonical equations of motion. We also derive Lagrange's equations of motion.

- ***Hamilton's Principle (for non-conservative system) :***

Hamilton's principle for non-conservative systems states that "The motion of a dynamical system between two points at time intervals t_0 to t_1 is such that the line integral

$$I = \int_{t_0}^{t_1} (T + W)\,dt$$

is extremum for the actual path followed by the system" , where T is the kinetic energy and W is the work done by the particle.

It is equivalent to say that δ variation in the actual path followed by the system is zero. Mathematically, it means that

$$\delta I = \delta \int_{t_0}^{t_1} (T + W) dt = 0$$

for actual path.

- ***Hamilton's Principle (for conservative system) :***

“Of all possible paths between two points along which a dynamical system may move from one point to another within a given time interval from t_0 to t_1, the actual path followed by the system is the one which minimizes the line integral of Lagrangian.’’

This means that the motion of a dynamical system from t_0 to t_1 is such that the line integral $\int_{t_0}^{t_1} L dt$ is extremum for actual path. This implies that small δ variation in the actual path followed by the system is zero.

Mathematically, we express this as $\delta \int_{t_0}^{t_1} L dt = 0$ for the actual path.

Note : We will show bellow in the Theorem (2) that the Hamilton’s principle $\delta \int_{t_0}^{t_1} L dt = 0$ also holds good for the non-conservative system.

- ***Action in Mechanics :***

Let $L = L\left(q_j, \dot{q}_j, t\right)$ be the Lagrangian for the conservative system. Then the integral

$$I = \int_{t_0}^{t_1} L dt$$

is called the action of the system.

Hence we can also define the Hamilton’s principle as “Out of all possible paths of a dynamical system between the time instants t_0 and t_1, the actual path followed by the system is one for which the action has a stationary value”

$$\Rightarrow \quad \delta I = \delta \int_{t_0}^{t_1} L dt = 0$$

for the actual path.

Theorem 1 : Derive Hamilton's principle for non-conservative system from D'Alembert's principle and hence deduce from it the Hamilton's principle for conservative system.

Proof: We start with D'Alembert's principle which states that

$$\sum_i \left(\bar{F}_i - \dot{\bar{p}}_i \right) \delta r_i = 0 . \qquad \text{. . . (1)}$$

Note that in this principle the knowledge of force whether it is conservative or non-conservative and also the requirement of holonomic or non-holonomic constraints does not arise. We write the principle in the form

$$\sum_i F_i \delta r_i = \sum_i \dot{\bar{p}}_i \delta r_i .$$

$$\Rightarrow \quad \delta W = \sum_i \dot{\bar{p}}_i \delta r_i . \qquad \text{. . . (2)}$$

where $\delta W = \sum_i F_i \delta r_i$ is the virtual work.

Now consider

$$\sum_i \dot{\bar{p}}_i \delta r_i = \sum_i m_i \ddot{r}_i \delta r_i ,$$

$$= \sum_i \frac{d}{dt} \left(m_i \dot{r}_i \delta r_i \right) - \sum_i m_i \dot{r}_i \frac{d}{dt} \left(\delta r_i \right).$$

Since we have $\delta \dot{r}_i = \dfrac{d}{dt} \delta r_i$, therefore, we write

$$\sum_i \dot{\bar{p}}_i \delta r_i = \frac{d}{dt} \left[\sum_i m_i \dot{r}_i \delta r_i \right] - \delta \left(\sum_i \frac{1}{2} m_i \dot{r}_i^2 \right).$$

$$\sum_i \dot{\bar{p}}_i \delta r_i = \frac{d}{dt} \left[\sum_i m_i \dot{r}_i \delta r_i \right] - \delta T ,$$

where $T=\sum_i \frac{1}{2}m_i\dot{r}_i^2$

is the kinetic energy of the system. Substituting this in equation (2) we get

$$\delta \mathrm{W}=\frac{d}{dt}\left[\sum_i m_i\dot{r}_i\delta r_i\right]-\delta T$$

$$\Rightarrow \qquad \delta\left(W+T\right)=\frac{\mathrm{d}}{\mathrm{dt}}\left[\sum_i m_i\dot{r}_i\delta r_i\right].$$

Integrating the above equation with respect to t between t_0 to t_1 we get

$$\int_{t_0}^{t_1}\delta\left(W+T\right)dt=\left[\sum_i m_i\dot{r}_i\delta r_i\right]_{t_0}^{t_1}.$$

Since, there is no variation in co-ordinates along any paths at the end points. i.e., $\left(\delta r_i\right)_{t_0}^{t_1}=0$. Hence from above equation we have

$$\delta\int_{t_0}^{t_1}\left(W+T\right)dt=0. \qquad \ldots (3)$$

This is known as Hamilton's principle for non-conservative systems.

If however, the system is conservative, then the forces are derivable from potential. In this case the expression for virtual work becomes

$$\delta W=\sum_i F_i\delta r_i=-\sum_i\frac{\partial V}{\partial r_i}\delta r_i=-\delta V.$$

Hence equation (3) becomes

$$\delta\int_{t_0}^{t_1}\left(T-V\right)dt=0,$$

$$\Rightarrow \qquad \delta\int_{t_0}^{t_1}Ldt=0. \qquad \ldots (4)$$

This is the required Hamilton's principle for conservative system.

Theorem 2 : Show that the Hamilton's principle $\delta\int_{t_0}^{t_1} Ldt = 0$ also holds for the non- conservative system.

Proof : We know the Hamilton's principle for non-conservative system is given by

$$\delta\int_{t_0}^{t_1}(T+W)dt = 0 \qquad \text{. . . (1)}$$

for actual path. The expression for the virtual work is given by

$$\delta W = \sum_i F_i \delta r_i = \sum_i F_i \left[\sum_j \frac{\partial r_i}{\partial q_j}\right]\delta q_j$$

$$\delta W = \sum_j \left[\sum_i F_i \frac{\partial r_i}{\partial q_j}\right]\delta q_j$$

$$\delta W = \sum_j Q_j \delta q_j, \qquad \text{. . . (2)}$$

where $$Q_j = \sum_i F_i \frac{\partial r_i}{\partial q_j}$$

are the components of generalized forces. In the case of non-conservative system the potential energy is dependent on velocity called the velocity dependent potential. In this case the generalized force is given by

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right).$$

Substituting this in equation (2) and integrating it between the limits t_0 to t_1 we find

$$\int_{t_0}^{t_1}\delta W dt = \int_{t_0}^{t_1}\sum_j Q_j \delta q_j dt = \int_{t_0}^{t_1}\left\{\sum_j\left[-\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)\right]\delta q_j\right\}dt,$$

Substituting this in equation (1) we get

$$\int_{t_0}^{t_1}\delta T dt = \int_{t_0}^{t_1}\sum_j \frac{\partial U}{\partial q_j}\delta q_j dt - \int_{t_0}^{t_1}\left\{\sum_j\left[\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)\right]\delta q_j\right\}dt.$$

Integrating the second integral by parts we obtain

$$\int_{t_0}^{t_1} \delta T dt = \int_{t_0}^{t_1} \sum_j \frac{\partial U}{\partial \mathrm{q_j}} \delta q_j dt - \sum_j \left(\frac{\partial U}{\partial \dot{\mathrm{q}}_\mathrm{j}} \delta q_j \right)_{t_0}^{t_1} + \int_{t_0}^{t_1} \left\{ \sum_\mathrm{j} \left(\frac{\partial U}{\partial \dot{\mathrm{q}}_\mathrm{j}} \right) \frac{d}{dt} \left(\delta q_j \right) \right\} dt .$$

Since change in co-ordinates at the end point is zero. $\left(\delta q_j \right)_{t_0}^{t_1} = 0$

and also $$\frac{d}{dt} \left(\delta q_j \right) = \delta \dot{q}_j ,$$

then we have

$$\int_{t_0}^{t_1} \delta T dt = \int_{t_0}^{t_1} \left\{ \sum_\mathrm{j} \left(\frac{\partial U}{\partial \dot{\mathrm{q}}_\mathrm{j}} \delta \dot{q}_j + \frac{\partial U}{\partial \mathrm{q_j}} \delta q_j \right) \right\} dt .$$

Since time t is fixed along any path hence, there is no variation in time along any path therefore change in time along any path is zero. i.e., $\delta t = 0$

Hence we write above equation as

$$\int_{t_0}^{t_1} \delta T dt = \int_{t_0}^{t_1} \left\{ \sum_\mathrm{j} \left(\frac{\partial U}{\partial \dot{\mathrm{q}}_\mathrm{j}} \delta \dot{q}_j + \frac{\partial U}{\partial \mathrm{q_j}} \delta q_j \right) + \frac{\partial U}{\partial t} \delta t \right\} dt .$$

$$\int_{t_0}^{t_1} \delta T dt = \int_{t_0}^{t_1} \delta U dt .$$

$$\Rightarrow \qquad \int_{t_0}^{t_1} \delta \left(T - U \right) dt = 0,$$

$$\Rightarrow \qquad \delta \int_{t_0}^{t_1} L dt = 0.$$

This proves that the Hamilton's principle holds good even for non-conservative systems.

Theorem 3 : State Hamilton's principle for non-conservative system and hence derive Lagrange's equations of motion for non-conservative holonomic systems.

Proof: Let us consider a non-conservative holonomic dynamical system whose configuration at any instant t is specified by n generalized co-ordinates $q_1, q_2, q_3, ..., q_n$. Hamilton's principle for non-conservative system states that

$$\delta\int_{t_0}^{t_1}(T+W)dt = 0 \qquad \text{for actual path.} \qquad \ldots (1)$$

The virtual work done is given by

$$\delta W = \sum_i F_i \delta r_i = \sum_i F_i \left[\sum_j \frac{\partial r_i}{\partial q_j}\right]\delta q_j$$

$$\delta W = \sum_j \left[\sum_i F_i \frac{\partial r_i}{\partial q_j}\right]\delta q_j$$

$$\delta W = \sum_j Q_j \delta q_j, \qquad \ldots (2)$$

where $\quad Q_j = \sum_i F_i \frac{\partial r_i}{\partial q_j}$

are the components of generalized forces.

$$\int_{t_0}^{t_1} \delta W dt = \int_{t_0}^{t_1} \sum_j Q_j \delta q_j dt. \qquad \ldots (3)$$

The kinetic energy of the particle $T = T\left(q_j, \dot{q}_j, t\right)$.

$$\delta T = \sum_j \frac{\partial T}{\partial q_j}\delta q_j + \sum_j \frac{\partial T}{\partial \dot{q}_j}\delta \dot{q}_j + \frac{\partial T}{\partial t}\delta t. \qquad \ldots (4)$$

As the variation in time along any path is always zero. $\Rightarrow \delta t = 0$.

This implies that

$$\delta T = \sum_j \frac{\partial T}{\partial q_j}\delta q_j + \sum_j \frac{\partial T}{\partial \dot{q}_j}\delta \dot{q}_j \qquad \ldots (5)$$

Integrating equation (5) between the limits t_0 to t_1 we get

$$\int_{t_0}^{t_1} \delta T dt = \int_{t_0}^{t_1} \sum_j \frac{\partial T}{\partial q_j}\delta q_j dt + \int_{t_0}^{t_1} \sum_j \frac{\partial T}{\partial \dot{q}_j}\delta \dot{q}_j dt$$

Since we have

$$\frac{d}{dt}\delta q_j = \delta \dot{q}_j .$$

Therefore we write the above integral as

$$\int_{t_0}^{t_1} \delta T dt = \int_{t_0}^{t_1} \sum_j \frac{\partial T}{\partial q_j} \delta q_j dt + \int_{t_0}^{t_1} \sum_j \frac{\partial T}{\partial \dot{q}_j} \frac{d}{dt}\left(\delta q_j\right) dt .$$

Integrating the second integral by parts, we obtain

$$\int_{t_0}^{t_1} \delta T dt = \int_{t_0}^{t_1} \sum_j \frac{\partial T}{\partial q_j} \delta q_j dt + \left[\sum_j \frac{\partial T}{\partial \dot{q}_j} \delta q_j\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum_j \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) \delta q_j dt .$$

Since in δ variation there is no change in the co-ordinates at the end points $\Rightarrow \left(\delta q_j\right)_{t_0}^{t_1} = 0$. Hence

$$\int_{t_0}^{t_1} \delta T dt = \int_{t_0}^{t_1} \sum_j \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\right] \delta q_j dt . \qquad \ldots (6)$$

Using equations (3) and (5) in equation (1) we get

$$\int_{t_0}^{t_1} \sum_j \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + Q_j\right] \delta q_j dt = 0 . \qquad \ldots (7)$$

If the constraints are holonomic then δq_j are independent. (Note that if the constraints are non-holonomic, then δq_j are not all independent. In this case vanishing of the integral (7) does not imply the coefficient vanish separately) Hence the integral (7) vanishes if and only if the coefficient must vanish separately.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j . \qquad \ldots (8)$$

These are the Lagrange's equations of motion for non-conservative holonomic system.

Theorem 4 : Deduce Hamilton's principle for conservative system from D'Alembert's principle.

Proof: We start with D'Alembert's principle which states that

$$\sum_i \left(\bar{F}_i - \dot{\bar{p}}_i\right)\delta r_i = 0 . \qquad \text{. . . (1)}$$

We write the principle in the form

$$\sum_i F_i \delta r_i = \sum_i \dot{\bar{p}}_i \delta r_i , \qquad \text{. . . (2)}$$

where δr_i is the virtual displacement and occurs at a particular instant. Hence change in time δt along any path is zero.

Now consider

$$\sum_i \dot{\bar{p}}_i \delta r_i = \sum_i m_i \ddot{r}_i \delta r_i ,$$

$$= \sum_i \frac{d}{dt}\left(m_i \dot{r}_i \delta r_i\right) - \sum_i m_i \dot{r}_i \frac{d}{dt}\left(\delta r_i\right) .$$

Since we have $\delta \dot{r}_i = \dfrac{d}{dt}\delta r_i$,

therefore we write

$$\sum_i \dot{\bar{p}}_i \delta r_i = \frac{d}{dt}\left[\sum_i m_i \dot{r}_i \delta r_i\right] - \delta\left(\sum_i \frac{1}{2} m_i \dot{r}_i^2\right) .$$

$$\sum_i \dot{\bar{p}}_i \delta r = \frac{d}{dt}\left[\sum_i m_i \dot{r}_i \delta r_i\right] - \delta T \qquad \text{. . . (3)}$$

where

$$T = \sum_i \frac{1}{2} m_i \dot{r}_i^2$$

is the kinetic energy of the system. Substituting equation (3) in equation (2) we get

$$\sum_i F_i \delta r_i = \frac{d}{dt}\left[\sum_i m_i \dot{r}_i \delta r_i\right] - \delta T \qquad \text{. . . (4)}$$

Since the force is conservative $\Rightarrow \quad F_i = -\dfrac{\partial V}{\partial r_i}$.

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{dt}}\left[\sum_i m_i \dot{r}_i \delta r_i\right] = \delta T - \sum_i \frac{\partial V}{\partial r_i}\delta r_i .$$
$$= \delta T - \delta V$$

Integrating the above equation with respect to t between t_0 to t_1 we get

$$\left[\sum_i m_i \dot{r}_i \delta r_i\right]_{t_0}^{t_1} = \delta\int_{t_0}^{t_1} L dt .$$

Since, there is no variation in co-ordinates along any paths at the end points.

i.e. $\left(\delta r_i\right)_{t_0}^{t_1} = 0$. Hence from above equation we have

$$\delta\int_{t_0}^{t_1} L dt = 0 . \qquad \ldots (5)$$

This is the required Hamilton's principle for conservative systems.

- ***Derivation of Lagrange's equations of motion from Hamilton's Principle :***

Theorem 5 : Show that the Lagrange's equations are necessary conditions for the action to have a stationary value.

Proof: We know the action of a particle is defined by

$$I = \int_{t_0}^{t_1} L dt \qquad \ldots (1)$$

where L is the Lagrangian of the system. Consider

$$\delta I = \delta\int_{t_0}^{t_1} L dt,$$
$$= \int_{t_0}^{t_1}\left[\sum_j \frac{\partial L}{\partial q_j}\delta q_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j\right] dt$$

As there is no variation in time along any path, hence $\delta t = 0$.

$$\delta\int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial q_j}\delta q_j dt + \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j dt.$$

Since $$\delta\frac{dq_j}{dt} = \frac{d}{dt}\left(\delta q_j\right),$$

therefore, we write

$$\delta\int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial q_j}\delta q_j dt + \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}\left(\delta q_j\right) dt \qquad \text{. . . (2)}$$

Integrating the second integral on the r. h. s. of equation (2) we get

$$\delta\int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \sum_j \frac{\partial L}{\partial q_j}\delta q_j dt + \left[\sum_j \frac{\partial L}{\partial \dot{q}_j}\delta q_j\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j dt.$$

Since there is no variation in the co-ordinates along any path at the end points, hence change in the co-ordinates at the end points is zero. i.e.,

$$\left(\delta q_j\right)_{t_0}^{t_1} = 0.$$

Thus we have

$$\delta\int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]\delta q_j dt. \qquad \text{. . . (3)}$$

If the system is holonomic, then all the generalized co-ordinates are linearly independent and hence we have

$$\delta\int_{t_0}^{t_1} L dt = 0 \Leftrightarrow \int_{t_0}^{t_1} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right]\delta q_j dt = 0$$

$$\delta\int_{t_0}^{t_1} L dt = 0 \Leftrightarrow \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0. \qquad \text{. . . (4)}$$

These are the required Lagrange's equations of motion derived from the Hamilton's principle. The equation (4) also shows that the Lagrange's equations of motion for holonomic system are necessary and sufficient conditions for action to have a stationary value.

Worked Examples

Example 1 : Use Hamilton's principle to find the equations of motion of a particle of unit mass moving on a plane in a conservative force field.

Solution: Let the force $\bar{F}$ be conservative and under the action of which the particle of unit mass be moving on the xy plane. Let P (x, y) be the position of the particle. We write the force

$$\bar{F} = iF_x + jF_y .$$

Since $\bar{F}$ is conservative, we have therefore,

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y} .$$

The kinetic energy of the particle is given by

$$T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) .$$

Hence the Lagrangian of the particle becomes

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - V\left(x, y\right) . \qquad \ldots (1)$$

The Hamilton's principle states that

$$\delta\int_{t_0}^{t_1} L dt = 0, \qquad \ldots (2)$$

$$\int_{t_0}^{t_1}\left[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial \dot{y}}\delta\dot{y}\right]dt = 0 ,$$

$$\Rightarrow \qquad \int_{t_0}^{t_1}\left[\left(\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y}\right) - \frac{\partial V}{\partial x}\delta x - \frac{\partial V}{\partial y}\delta y\right]dt = 0 . \qquad \ldots (3)$$

Consider

$$\int_{t_0}^{t_1}\dot{x}\delta\dot{x}dt = \int_{t_0}^{t_1}\dot{x}\frac{d}{dt}(\delta x)dt$$

Integrating by parts we get

$$\int_{t_0}^{t_1} \dot{x}\delta\dot{x}dt = \left(\dot{x}\delta x\right)_{t_0}^{t_1} - \int_{t_0}^{t_1} \ddot{x}\left(\delta x\right)dt .$$

Since $\delta x = 0$ at both the ends t_0 and t_1 along any path, therefore,

$$\int_{t_0}^{t_1} \dot{x}\delta\dot{x}dt = -\int_{t_0}^{t_1} \ddot{x}\left(\delta x\right)dt . \qquad \ldots (4)$$

Similarly, we have

$$\int_{t_0}^{t_1} \dot{y}\delta\dot{y}dt = -\int_{t_0}^{t_1} \ddot{y}\left(\delta y\right)dt . \qquad \ldots (5)$$

Substituting these values in equation (3) we get

$$\int_{t_0}^{t_1}\left[\left(\ddot{x}+\frac{\partial V}{\partial x}\right)\delta x+\left(\ddot{y}+\frac{\partial V}{\partial y}\right)\delta y\right]dt = 0 .$$

Since δx and δy are independent and arbitrary, then we have

$$\ddot{x}+\frac{\partial V}{\partial x}=0, \qquad \ddot{y}+\frac{\partial V}{\partial y}=0 .$$

$$\begin{aligned} \ddot{x} &= -\frac{\partial V}{\partial x} = F_x, \\ \ddot{y} &= -\frac{\partial V}{\partial y} = F_y. \end{aligned} \qquad \ldots (6)$$

These are the equations of motion of a particle of unit mass moving under the action of the conservative force field.

Example 2: Use Hamilton's principle to find the equation of motion of a simple pendulum.

Solution: In case of a simple pendulum, the only generalized co-ordinate is θ, and the Lagrangian is given by (Refer Ex. 26 of Chapter I)

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\left(1-\cos\theta\right). \qquad \ldots (1)$$

The Hamilton's Principle states that "the path followed by the pendulum is one along which the line integral of Lagrangian is extremum". i.e.,

$$\delta\int_{t_0}^{t_1} \mathrm{L}dt = 0,$$

$$\int_{t_0}^{t_1} \delta\left[\frac{1}{2}ml^2\dot{\theta}^2 - mgl\left(1-\cos\theta\right)\right]dt = 0,$$

$$\int_{t_0}^{t_1} \left[ml^2\dot{\theta}\delta\dot{\theta} - mgl\sin\theta\delta\theta\right]dt = 0.$$

Since, we have $\delta\dfrac{d}{dt} = \dfrac{d}{dt}\delta$.

Therefore

$$\int_{t_0}^{t_1} \left[ml^2\dot{\theta}\frac{d}{dt}\left(\delta\theta\right) - mgl\sin\theta\delta\theta\right]dt = 0.$$

Integrating the first integral by parts we get

$$ml^2\left(\dot{\theta}\delta\theta\right)_{t_0}^{t_1} - \int_{t_0}^{t_1} m\left[l^2\ddot{\theta} + gl\sin\theta\right]\delta\theta dt = 0.$$

Since $\left(\delta\theta\right)_{t_0}^{t_1} = 0$, we have therefore,

$$\int_{t_0}^{t_1} m\left[l^2\ddot{\theta} + gl\sin\theta\right]\delta\theta dt = 0.$$

As $\delta\theta$ is arbitrary, we have

$$l^2\ddot{\theta} + gl\sin\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

This is the required equation of motion of the simple pendulum.

- ***Spherical Pendulum :***

Example 3 : A particle of mass m is moving on the surface of the sphere of radius r in the gravitational field. Use Hamilton's principle to show the equation of motion is given by

$$\ddot{\theta}-\frac{p_\phi^2\cos\theta}{m^2r^4\sin^3\theta}-\frac{g}{r}\sin\theta=0,$$

where p_ϕ is the constant of angular momentum.

Solution: Let a particle of mass m be moving on the surface of a sphere of radius r. The particle has two degrees of freedom and hence two generalized co-ordinates θ,ϕ. The Lagrangian of the motion is (Refer Ex. 28 of Chapter I) given by

$$L=\frac{1}{2}mr^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)-mgr\cos\theta. \qquad \text{. . . (1)}$$

The Hamilton's Principle states that "the path followed by the particle between two time instants t_0 and t_1 is one along which the line integral of Lagrangian is extremum". i.e.,

$$\delta\int_{t_0}^{t_1}\mathrm{L}dt=0,$$

$$\int_{t_0}^{t_1}\delta\left[\frac{1}{2}mr^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)-mgr\cos\theta\right]dt=0,$$

$$\int_{t_0}^{t_1}\left[mr^2\left(\dot{\theta}\delta\dot{\theta}+\sin^2\theta\dot{\phi}\delta\dot{\phi}+\sin\theta\cos\theta\dot{\phi}^2\delta\theta\right)+mgr\sin\theta\delta\theta\right]dt=0.$$

Since, we have $\qquad \delta\dfrac{d\theta}{dt}=\dfrac{d}{dt}\delta\theta.$

Therefore,

$$\int_{t_0}^{t_1}\left[mr^2\left(\dot{\theta}\frac{d}{dt}(\delta\theta)+\sin^2\theta\dot{\phi}\frac{d}{dt}(\delta\phi)\right)+\left(mr^2\sin\theta\cos\theta\dot{\phi}^2+mgr\sin\theta\right)\delta\theta\right]dt=0.$$

Integrating the first two integrals by parts we get

$$mr^2\left(\dot{\theta}\delta\theta\right)_{t_0}^{t_1} + mr^2\sin^2\theta\left(\dot{\phi}\delta\phi\right)_{t_0}^{t_1} - \int_{t_0}^{t_1}\left[mr^2\left(\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 - \frac{g}{r}\sin\theta\right)\delta\theta\right]dt -$$

$$-\int_{t_0}^{t_1} mr^2\frac{d}{dt}\left(\sin^2\theta\dot{\phi}\right)\delta\phi dt \ = \ 0.$$

Since $\left(\delta\theta\right)_{t_0}^{t_1} = 0 = \left(\delta\phi\right)_{t_0}^{t_1}$,

we have therefore,

$$\int_{t_0}^{t_1}\left[mr^2\left(\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 - \frac{g}{r}\sin\theta\right)\delta\theta\right]dt + \int_{t_0}^{t_1} mr^2\frac{d}{dt}\left(\sin^2\theta\dot{\phi}\right)\delta\phi dt = 0$$

Since θ and ϕ are independent and arbitrary, hence we have

$$\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 - \frac{g}{r}\sin\theta = 0,$$

$$mr^2\frac{d}{dt}\left(\sin^2\theta\dot{\phi}\right) = 0 \Rightarrow \quad mr^2\sin^2\theta\dot{\phi} = p_\phi\ (const.)$$

Eliminating $\dot{\phi}$ we obtain

$$\ddot{\theta} - \frac{p_\phi^2\cos\theta}{m^2r^4\sin^3\theta} - \frac{g}{r}\sin\theta = 0. \qquad \ldots (2)$$

as the required differential equation of motion for spherical pendulum.

Unit 2:

- ***Hamiltonian Formulation :***

Introduction:

We have developed Lagrangian formulation as a description of mechanics in terms of the generalized co-ordinates and generalized velocities with time t as a parameter in Chapter I and the equations of motion were used to solve some problems. We now introduce another powerful formulation in which the independent variables are the generalized co-ordinates and the generalized momenta known as Hamilton's formulation. This formulation is an alternative to the Lagrangian formulation but proved to be more convenient and useful, particularly in dealing with problems of modern physics. Hence all the examples solved in the Chapter I can also be solved by the Hamiltonian procedure. As an illustration few of them are solved in this Chapter by Hamilton's procedure.

- ***The Hamiltonian Function:***

The quantity $\sum_j p_j \dot{q}_j - L$ when expressed in terms of $q_1, q_2, ..., q_n, p_1, p_2, ... p_n, t$ is called Hamiltonian and it is denoted by H.

Thus $$H = H\left(q_j, p_j, t\right) = \sum_j p_j \dot{q}_j - L .$$

- ***Hamilton's Canonical Equations of Motion :***

Theorem 6 : Define the Hamiltonian and hence derive the Hamilton's canonical equations of motion.

Proof : We know the Hamiltonian H is defined as

$$H = H\left(q_j, p_j, t\right) = \sum_j p_j \dot{q}_j - L . \qquad \text{. . . (1)}$$

Consider $$H = H\left(q_j, p_j, t\right). \qquad \text{. . . (2)}$$

We find from equation (2) that

$$dH = \sum_j \frac{\partial H}{\partial p_j} dp_j + \sum_j \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial t} dt . \qquad \text{. . . (3)}$$

Now consider $H = \sum_j p_j \dot{q}_j - L$.

Similarly we find

$$dH = \sum_j \dot{q}_j dp_j + \sum_j d\dot{q}_j p_j - dL,$$

$$\Rightarrow \quad dH = \sum_j \dot{q}_j dp_j + \sum_j d\dot{q}_j p_j - \sum_j \frac{\partial L}{\partial q_j} dq_j - \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \frac{\partial L}{\partial t} dt . \qquad \text{. . . (4)}$$

We know the generalized momentum is defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}.$$

Hence equation (4) reduces to

$$dH = \sum_j \dot{q}_j dp_j - \sum_j \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial t} dt \qquad \ldots (5)$$

Now comparing the coefficients of dp_j, dq_j and dt in equations (3) and (5) we get

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}. \qquad \ldots (6)$$

However, from Lagrange's equations of motion we have

$$\dot{p}_j = \frac{\partial L}{\partial q_j}$$

Hence equations (6) reduce to

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{\mathrm{p}}_{\mathrm{j}} = -\frac{\partial H}{\partial q_j} \quad . \qquad \ldots (7)$$

These are the required Hamilton's canonical equations of motion. These are the set of 2n first order differential equations of motion and replace the n Lagrange's second order equations of motion.

- ***Derivation of Hamilton's equations of motion from Hamilton's Principle :***

Theorem 7 : Obtain Hamilton's equations of motion from the Hamilton's principle.

Proof: We know the action of a particle is defined by

$$I = \int_{t_0}^{t_1} L dt \qquad \ldots (1)$$

where L is the Lagrangian of the system. If $H\left(p_j, q_j, t\right)$ is the Hamiltonian of the motion then we have by definition

$$H = \sum_j p_j \dot{q}_j - L. \qquad \ldots (2)$$

Replacing L in equation (1) by using (2) we have the action in mechanics as

$$I = \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \left[\sum_j p_j \dot{q}_j - H \right] dt . \qquad \ldots (3)$$

Now by Hamilton's principle, we have

$$\delta \int_{t_0}^{t_1} L dt = 0 \quad \Rightarrow \quad \delta \int_{t_0}^{t_1} \left[\sum_j p_j \dot{q}_j - H \right] dt = 0 . \qquad \ldots (4)$$

This is known as the modified Hamilton's principle. Thus we have

$$\delta \int_{t_0}^{t_1} L dt = \quad \delta \int_{t_0}^{t_1} \left[\sum_j p_j \dot{q}_j - H \right] dt,$$

$$\delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \left[\sum_j \delta p_j \dot{q}_j + \sum_j p_j \delta \dot{q}_j - \sum_j \frac{\partial H}{\partial q_j} \delta q_j - \sum_j \frac{\partial H}{\partial p_j} \delta p_j - \frac{\partial H}{\partial t} \delta t \right] dt.$$

Since time is fixed along any path, hence change in time along any path is zero. i.e., $\delta t = 0$ along any path. Hence above equation becomes

$$\delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \left[\sum_j \left(\dot{q}_j - \frac{\partial H}{\partial p_j} \right) \delta p_j + \sum_j p_j \delta \dot{q}_j - \sum_j \frac{\partial H}{\partial q_j} \delta q_j \right] dt \qquad \ldots (5)$$

Now consider the integral

$$\int_{t_0}^{t_1} \sum_j p_j \delta \dot{q}_j dt = \int_{t_0}^{t_1} \sum_j p_j \frac{d}{dt} \left(\delta q_j \right) dt .$$

Integrating the integral on the r. h. s. by parts we get

$$\int_{t_0}^{t_1} \sum_j p_j \delta \dot{q}_j dt = \left(\sum_j p_j \delta q_j \right)_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum_j \dot{p}_j \delta q_j dt .$$

Since $\left(\delta q_j \right)_{t_0}^{t_1} = 0$. We have therefore

$$\int_{t_0}^{t_1} \sum_j p_j \delta \dot{q}_j dt = -\int_{t_0}^{t_1} \sum_j \dot{p}_j \delta q_j dt .$$

Substituting this in equation (5) we get

$$\delta\int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1}\left[\sum_j\left(\dot{q}_j - \frac{\partial H}{\partial p_j}\right)\delta p_j + \sum_j\left(\dot{p}_j + \frac{\partial H}{\partial q_j}\right)\delta q_j\right]dt.$$

Now we see that

$$\delta\int_{t_0}^{t_1} L dt = 0 \Leftrightarrow \int_{t_0}^{t_1}\left[\sum_j\left(\dot{q}_j - \frac{\partial H}{\partial p_j}\right)\delta p_j + \sum_j\left(\dot{p}_j + \frac{\partial H}{\partial q_j}\right)\delta q_j\right]dt = 0.$$

For holonomic system we have q_j, p_j are independent, hence

$$\delta\int_{t_0}^{t_1} L dt = 0 \Leftrightarrow \dot{q}_j - \frac{\partial H}{\partial p_j} = 0, \quad \dot{p}_j + \frac{\partial H}{\partial q_j} = 0.$$

$$\Rightarrow \qquad \delta\int_{t_0}^{t_1} L dt = 0 \Leftrightarrow \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}. \qquad \text{. . . (6)}$$

These are the Hamilton's canonical equations of motion.

Remark :

We see from equation (6) that the Hamilton's canonical equations of motion are the necessary and sufficient conditions for the action to have stationary value.

Example 4 : Show that addition of the total time derivative of any function of the form $f\left(q_j, t\right)$ to the Lagrangian of a holonomic system, the generalized momentum and the Jacobi integral are respectively given by

$$p_i + \frac{\partial f}{\partial q_i} \text{ and } H - \frac{\partial f}{\partial t}.$$

Does the new Lagrangian L' unchanged the Hamilton's principle? Justify your claim.

Solution: Let the new Lagrangian function after addition of the time derivative of any function of the form $f\left(q_j, t\right)$ to the Lagrangian L be denoted by L'. Thus we have

$$L' = L + \frac{df}{dt}. \qquad \ldots (1)$$

Thus the generalized momentum corresponding to the new Lagrangian L' is defined by

$$p'_j = \frac{\partial L'}{\partial \dot{q}_j}. \qquad \ldots (2)$$

Thus $$p'_j = \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial}{\partial \dot{q}_j}\left(\frac{df}{dt}\right),$$

$$\Rightarrow \quad p'_j = p_j + \frac{\partial}{\partial \dot{q}_j}\left(\sum_k \frac{\partial f}{\partial q_k}\dot{q}_k + \frac{\partial f}{\partial t}\right),$$

$$\Rightarrow \quad p'_j = p_j + \frac{\partial f}{\partial q_j}. \qquad \ldots (3)$$

This is the required generalized momentum corresponding to the new Lagrangian L'.

Similarly, the Jacobi integral for new function L' is given by

$$H' = \sum_j p'_j \dot{q}_j - L',$$

$$H' = \sum_j \frac{\partial L'}{\partial \dot{q}_j}\dot{q}_j - \left(L + \frac{df}{dt}\right).$$

On using equation (3) we get

$$H' = \sum_j \left(p_j + \frac{\partial f}{\partial q_j}\right)\dot{q}_j - \left(L + \frac{df}{dt}\right),$$

$$H' = \sum_j \left(p_j \dot{q}_j - L\right) - \frac{\partial f}{\partial t},$$

$$\Rightarrow \quad H' = H - \frac{\partial f}{\partial t}. \qquad \ldots (4)$$

This is a required Jacobi integral for the new Lagrangian L'.

Now we show that the new Lagrangian L' also satisfies the Hamilton's principle. Therefore, consider

$$\delta\int_1^2 L'dt = \delta\int_1^2 Ldt + \delta\int_1^2 \frac{df}{dt}dt,$$

$$\delta\int_1^2 L'dt = \delta\int_1^2 Ldt + \delta\int_1^2 df,$$

$$\delta\int_1^2 L'dt = \delta\int_1^2 Ldt + (\delta f)_1^2,$$

$$\delta\int_1^2 L'dt = \delta\int_1^2 Ldt + \left[\sum_j \frac{\partial f}{\partial q_j}\delta q_j + \frac{\partial f}{\partial t}\delta t\right]_1^2.$$

But in δ variation time is held fixed along any path and hence $\delta t = 0$ along any path.

Further, co-ordinates at the end points are held fixed.

$$\Rightarrow \quad (\delta q_j)_1^2 = 0.$$

Hence we have from the above equation that

$$\delta\int_1^2 L'dt = \delta\int_1^2 Ldt.$$

Thus the Hamilton's Principle

$$\delta\int_1^2 Ldt = 0 \quad \Leftrightarrow \quad \delta\int_1^2 L'dt = 0.$$

This shows that the new Lagrangian L' satisfies the Hamilton's principle.

- ***Lagrangian from Hamiltonian and conversely :***

Example 5: Obtain Lagrangian L from Hamiltonian H and show that it satisfies Lagrange's equations of motion.

Solution: The Hamiltonian H is defined by

$$H = \sum_j p_j \dot{q}_j - L. \qquad \text{. . . (1)}$$

which satisfies the Hamilton's canonical equations of motion.

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{\mathrm{p}}_{\mathrm{j}} = -\frac{\partial H}{\partial q_j} \quad . \qquad \text{. . . (2)}$$

Now from equation (1) we find the Lagrangian

$$L = \sum_j p_j \dot{q}_j - H . \qquad \text{. . . (3)}$$

and show that it satisfies Lagrange's equations of motion. Thus from equation (3) we have

$$\frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j}, \quad and \quad \frac{\partial L}{\partial \dot{q}_j} = p_j .$$

Now consider

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = -\frac{\partial H}{\partial q_j} - \frac{d}{dt}\left(p_j\right),$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \dot{p}_j - \dot{p}_j,$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0.$$

This shows that the equation (3) gives the required Lagrangian which satisfies the Lagrange's equations of motion.

Example 6 : Obtain the Hamiltonian H from the Lagrangian and show that it satisfies the Hamilton's canonical equations of motion.

Solution: The Hamiltonian H in terms of Lagrangian L is defined as

$$H = \sum_j p_j \dot{q}_j - L . \qquad \text{. . . (1)}$$

where L satisfies the Lagrange's equations of motion viz.,

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 , \qquad \text{. . . (2)}$$

$$\Rightarrow \qquad \frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right),$$

$$= \frac{d}{dt}\left(p_j\right).$$

$$\Rightarrow \qquad \frac{\partial L}{\partial q_j} = \dot{p}_j . \qquad \ldots (3)$$

Now from equation (1) we find

$$\frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_j} . \qquad \ldots (4)$$

From equations (3) and (4) we have

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j . \qquad \ldots (5)$$

Similarly, we find from equation (1)

$$\frac{\partial H}{\partial p_j} = \dot{q}_j . \qquad \ldots (6)$$

Equations (5) and (6) are the required Hamilton's equations of motion.

- ***Physical Meaning of the Hamiltonian :***

Theorem 8 :

1. For conservative scleronomic system the Hamiltonian H represents both a constant of motion and total energy.
2. For conservative rheonomic system the Hamiltonian H may represent a constant of motion but does not represent the total energy.

Proof : The Hamiltonian H is defined by

$$H = \sum_j p_j \dot{q}_j - L . \qquad \ldots (1)$$

where L is the Lagrangian of the system and

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \qquad \ldots (2)$$

is the generalized momentum. This implies from Lagrange's equation of motion that

$$\dot{p}_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \qquad \ldots (3)$$

Differentiating equation (1) w. r. t. time t, we get

$$\frac{dH}{dt} = \sum_j \dot{p}_j \dot{q}_j + \sum_j p_j \ddot{q}_j - \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j - \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j - \frac{\partial L}{\partial t} \qquad \ldots (4)$$

On using equations (2) and (3) in equation (4) we readily obtain

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \qquad \ldots (5)$$

Now if L does not contain time t explicitly, then from equation (5), we have

$$\frac{dH}{dt} = 0$$

This shows that H represents a constant of motion.

However, the condition L does not contain time t explicitly will be satisfied by neither the kinetic energy nor the potential energy involves time t explicitly.

Now there are two cases that the kinetic energy T does not involve time t explicitly.

1. <u>For the conservative and scleronomic system :</u>

In the case of conservative system when the constraints are scleronomic, the kinetic energy T is independent of time t and the potential energy V is only function of co-ordinates. Consequently, the Lagrangian L does not involve time t explicitly and hence from equation (5) the Hamiltonian H represents a constant of motion.

Further, for scleronomic system, we know the kinetic energy is a homogeneous quadratic function of generalized velocities.

$$T = \sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k . \qquad \ldots (6)$$

Hence by using Euler's theorem for the homogeneous quadratic function of generalized velocities we have

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T . \qquad \ldots (7)$$

For conservative system we have

$$p_j = \frac{\partial \mathrm{L}}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}. \qquad \ldots (8)$$

Using (7) and (8) in the Hamiltonian H we get

$$H = 2T - (T - V),$$

$$H = T + V = E. \qquad \ldots (9)$$

where E is the total energy of the system. Equation (9) shows that for conservative scleronomic system the Hamiltonian H represents the total energy of the system.

2. <u>For conservative and rheonomic system :</u>

In the case of conservative rheonomic system, the transformation equations do involve time t explicitly, though some times the kinetic energy may not involve time t explicitly. Consequently, neither T nor V involves t, and hence L does not involve t. Hence in such cases the Hamiltonian may represent the constant of motion. However, in general if the system is conservative and rheonomic, the kinetic energy is a quadratic function of generalized velocities and is given by

$$T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k + \sum_j a_j \dot{q}_j + a \qquad \ldots (10)$$

where

$$\left.\begin{aligned} a_{jk} &= \sum_i \frac{1}{2} m_i \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial q_k}, \\ a_j &= \sum_i m_i \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial t}, \\ a &= \sum_i \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial t} \right)^2. \end{aligned}\right\} \qquad \ldots (11)$$

We see from equation (10) that each term is a homogeneous function of generalized velocities of degree two, one and zero respectively. On applying Euler's theorem for the homogeneous function to each term on the right hand side, we readily get

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T_2 + T_1. \qquad \text{. . . (12)}$$

where

$$T_2 = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k,$$

$$T_1 = \sum_j a_j \dot{q}_j,$$

$$T_0 = a$$

are homogeneous function of generalized velocities of degree two, one and zero respectively. Substituting equation (12) in the Hamiltonian (1) we obtain

$$H = T_2 - T_0 + V$$

showing that the Hamiltonian H does not represent total energy. Thus for the conservative rheonomic systems H may represent the constant of motion but does not represent total energy.

- ***Cyclic Co-ordinates In Hamiltonian :***

Theorem 9 : Prove that a co-ordinate which is cyclic in the Lagrangian is also cyclic in the Hamiltonian.

Solution: We know the co-ordinate which is absent in the Lagrangian is called cyclic co-ordinate. Thus if q_j is cyclic in L $\Rightarrow \dfrac{\partial L}{\partial q_j} = 0$.

Hence the Lagrange's equation of motion reduces to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad \Rightarrow \quad \dot{p}_j = 0, \qquad \text{. . . (1)}$$

where $p_j = \dfrac{\partial L}{\partial \dot{q}_j}$ is the generalized momentum. However, from Hamilton's canonical equations of motion we have

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}. \qquad \text{. . . (2)}$$

Equations (1) and (2) gives

$$\frac{\partial H}{\partial q_j} = 0 . \qquad \ldots (3)$$

This shows that the co-ordinate q_j is also absent in the Hamiltonian, and consequently, it is also cyclic in H. Thus a co-ordinate which is cyclic in the Lagrangian is also cyclic in the Hamiltonian.

Worked Examples

Example 7 : Describe the motion of a particle of mass m moving near the surface of the Earth under the Earth's constant gravitational field by Hamilton's procedure.

Solution: Consider a particle of mass m moving near the surface of the Earth under the Earth's constant gravitational field. Let (x, y, z) be the Cartesian co-ordinates of the projectile, z being vertical. Then the Lagrangian of the projectile is given by

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz . \qquad \ldots (1)$$

We see that the generalized co-ordinates x and y are absent in the Lagrangian, hence they are the cyclic co-ordinates. This implies that any change in these co-ordinates can not affect the Lagrangian. This implies that the corresponding generalized momentum is conserved. In this case the generalized momentum is the linear momentum and is conserved.

i.e.,
$$\begin{aligned} p_x &= m\dot{x} = const. \\ p_y &= m\dot{y} = const. \end{aligned} \qquad \ldots (2)$$

$$p_z = m\dot{z} .$$

This shows that the horizontal components of momentum are conserved.

The Hamiltonian of the particle is defined by

$$H = \sum_j p_j \dot{q}_j - L,$$

$$H = p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + mgz . \qquad ...(3)$$

Eliminating $\dot{x}, \dot{y}, \dot{z}$ between equations (2) and (3) we get

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + mgz . \qquad ...(4)$$

The Hamilton's equations of motion give

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0, \dot{p}_y = -\frac{\partial H}{\partial y} = 0, \dot{p}_z = -\frac{\partial H}{\partial z} = -mg . \qquad ...(5)$$

and $$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}, \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}, \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} . \qquad ...(6)$$

From these set of equations we obtain

$$\ddot{x} = 0,\ \ddot{y} = 0,\ \ddot{z} = -g \qquad ...(7)$$

These are the required equations of motion of the projectile near the surface of the Earth.

Example 8 : Obtain the Hamiltonian H and the Hamilton's equations of motion of a simple pendulum. Prove or disprove that H represents the constant of motion and total energy.

Solution: The Example is solved earlier by various methods. The Lagrangian of the pendulum is given by

$$L = \frac{1}{2} ml^2\dot{\theta}^2 - mgl\left(1 - \cos\theta\right), \qquad ...(1)$$

where the generalized momentum is given by

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \Rightarrow \quad \dot{\theta} = \frac{p_\theta}{ml^2} . \qquad ...(2)$$

The Hamiltonian of the system is given by

$$H = p_\theta \dot{\theta} - L,$$

$$\Rightarrow \quad H = p_\theta \dot{\theta} - \frac{1}{2} ml^2\dot{\theta}^2 + mgl\left(1 - \cos\theta\right).$$

Eliminating $\dot{\theta}$ we obtain

$$H = \frac{p_\theta^2}{2ml^2} + mgl\left(1 - \cos\theta\right). \qquad \ldots (3)$$

Hamilton's canonical equations of motion are

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}.$$

These equations give

$$\dot{\theta} = \frac{p_\theta}{ml^2}, \quad \dot{p}_\theta = -mgl\sin\theta. \qquad \ldots (4)$$

Now eliminating p_θ from these equations we get

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0. \qquad \ldots (5)$$

Now we claim that H represents the constant of motion.

Thus differentiating equation (3) with respect to t we get

$$\frac{dH}{dt} = \frac{p_\theta \dot{p}_\theta}{ml^2} + mgl\sin\theta\,\dot{\theta},$$

$$= ml^2\dot{\theta}\ddot{\theta} + mgl\sin\theta\dot{\theta},$$

$$= ml^2\dot{\theta}\left(\ddot{\theta} + \frac{g}{l}\sin\theta\right),$$

$$\frac{dH}{dt} = 0.$$

This proves that H is a constant of motion. Now to see whether H represents total energy or not, we consider

$$T + V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\left(1 - \cos\theta\right).$$

Using equation (4) we eliminate $\dot{\theta}$ from the above equation, we obtain

$$T+V=\frac{p_\theta^2}{2ml^2}+mgl\left(1-\cos\theta\right). \qquad \text{. . . (6)}$$

This is as same as the Hamiltonian H from equation (3). Thus Hamiltonian H represents the total energy of the pendulum.

Example 9: The Lagrangian for a particle moving on a surface of a sphere of radius r is given by

$$L=\frac{1}{2}mr^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)-mgr\cos\theta.$$

Find the Hamiltonian H and show that it is constant of motion. Prove or disprove that H represents the total energy. Is the energy of the particle constant? Justify your claim.

Solution: We are given the Lagrangian of a particle moving on the surface of a sphere (Spherical Pendulum) in the form

$$L=\frac{1}{2}mr^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)-mgr\cos\theta. \qquad \text{. . . (1)}$$

We see that ϕ is a cyclic co-ordinate. This implies the corresponding generalized momentum is conserved. i.e.

$$p_\phi=\frac{\partial L}{\partial\dot{\phi}}=mr^2\sin^2\theta\dot{\phi}=const. \qquad \text{. . . (2)}$$

Similarly,
$$p_\theta=\frac{\partial L}{\partial\dot{\theta}}=mr^2\dot{\theta}. \qquad \text{. . . (3)}$$

The Hamiltonian of the particle is defined as

$$H=\dot{\theta}p_\theta+\dot{\phi}p_\phi-\frac{1}{2}mr^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)+mgr\cos\theta. \qquad \text{. . . (4)}$$

Using equations (2) and (3) we obtain the Hamiltonian function

$$H=\frac{1}{2}\left(\frac{p_\theta^2}{mr^2}+\frac{p_\phi^2}{mr^2\sin^2\theta}\right)+mgr\cos\theta. \qquad \text{. . . (5)}$$

The Hamilton's canonical equations of motion give

$$\begin{aligned}
\dot{p}_\theta &= \frac{\cos\theta p_\phi^2}{mr^2\sin^3\theta} + mgr\sin\theta, \\
\dot{p}_\phi &= 0, \\
\dot{\theta} &= \frac{p_\theta}{mr^2}, \\
\dot{\phi} &= \frac{p_\phi}{mr^2\sin^2\theta}.
\end{aligned} \qquad \ldots (6)$$

Eliminating p_θ from equation (6) we get the equation of motion of spherical pendulum as

$$mr^2\ddot{\theta} - \frac{\cos\theta p_\phi^2}{mr^2\sin^3\theta} - mgr\sin\theta = 0\,. \qquad \ldots (7)$$

(i) Now we claim that H is a constant of motion, differentiate equation (5) with respect to t, we get

$$\frac{dH}{dt} = \frac{p_\theta \dot{p}_\theta}{mr^2} + \frac{p_\phi \dot{p}_\phi}{mr^2\sin^2\theta} - \frac{p_\phi^2\cos\theta}{mr^2\sin^3\theta}\dot{\theta} - mgr\sin\theta\dot{\theta},$$

Putting the values of $\dot{p}_\theta, \dot{p}_\phi$ from equation (6) we get

$$\frac{dH}{dt} = \frac{p_\theta}{mr^2}\left(\frac{\cos\theta p_\phi^2}{mr^2\sin^3\theta} + mgr\sin\theta\right) - \frac{p_\theta\cos\theta p_\phi^2}{m^2r^4\sin^3\theta} - \frac{p_\theta}{mr^2}mgr\sin\theta$$

$$\frac{dH}{dt} = 0,$$

showing that H is a constant of motion.

(ii) Now consider the sum of the kinetic and potential energy of the spherical pendulum, where

$$\begin{aligned}
T &= \frac{1}{2}mr^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right), \\
V &= mgr\cos\theta
\end{aligned}$$

Thus

$$T+V=\frac{1}{2}mr^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)+mgr\cos\theta. \qquad ...(8)$$

We eliminate $\dot{\theta},\dot{\phi}$ from equation (8) by using equation (6) to get

$$T+V=\frac{1}{2}\left(\frac{p_\theta^2}{mr^2}+\frac{p_\phi^2}{mr^2\sin^2\theta}\right)+mgr\cos\theta. \qquad ...(9)$$

We see from equations (5) and (9) that the total energy of the spherical pendulum is the Hamiltonian of motion. Now to see it is constant or not, multiply equation (7) by $\dot{\theta}$ we get

$$mr^2\ddot{\theta}\dot{\theta}-\frac{p_\phi^2\cos\theta\dot{\theta}}{mr^2\sin^3\theta}-mgr\sin\theta\dot{\theta}=0$$

$$\frac{d}{dt}\left(\frac{1}{2}mr^2\dot{\theta}^2\right)+\frac{d}{dt}\left(\frac{p_\phi^2}{2mr^2\sin^2\theta}\right)+\frac{d}{dt}\left(mgr\cos\theta\right)=0.$$

Integrating we get

$$\left(\frac{1}{2}mr^2\dot{\theta}^2\right)+\left(\frac{p_\phi^2}{2mr^2\sin^2\theta}\right)+\left(mgr\cos\theta\right)=const.$$

Eliminating $\dot{\theta}$ on using equation (6) we get

$$\frac{1}{2}\left(\frac{p_\theta^2}{mr^2}+\frac{p_\phi^2}{mr^2\sin^2\theta}\right)+mgr\cos\theta=const.. \qquad ...(10)$$

We see from equations (5), (9) and (10) that the Hamiltonian H represents the total energy and the energy of the particle is conserved.

Example 10: Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, write down the Hamiltonian for the system and hence the equations of motion. Prove or disprove that

i) Hamiltonian H represents the constant of motion.

ii) H represents total energy of the system.

Solution: The example is solved in Chapter I. (please refer to Example 24). The Lagrangian of the system is given by

$$L = \frac{1}{2} m_1 \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{1}{2} m_2 \dot{r}^2 + m_2 g \left(l - r \right) \qquad \text{. . . (1)}$$

We see that the co-ordinate θ is cyclic in the Lagrangian L and hence the corresponding generalized momentum is conserved.

$$\Rightarrow \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m_1 r^2 \dot{\theta} = const. \qquad \text{. . . (2)}$$

Similarly, we find

$$p_r = \frac{\partial L}{\partial \dot{r}} = \left(m_1 + m_2 \right) \dot{r} = const. \qquad \text{. . . (3)}$$

Now the Hamiltonian function is defined as

$$H = \dot{r} \frac{\partial L}{\partial \dot{r}} + \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} - L,$$

$$H = \frac{1}{2} \left(m_1 + m_2 \right) \dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\theta}^2 - m_2 g \left(l - r \right).$$

Eliminating $\dot{r}$ and $\dot{\theta}$ we obtain

$$H = \frac{p_r^2}{2\left(m_1 + m_2 \right)} + \frac{p_\theta^2}{2 m_1 r^2} - m_2 g \left(l - r \right). \qquad \text{. . . (4)}$$

The Hamilton canonical equations of motion viz.,

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = \frac{\partial H}{\partial p_j}$$

give

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{m_1 r^3} - m_2 g, \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0. \qquad \text{. . . (5)}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\left(m_1 + m_2 \right)}, \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m_1 r^2}. \qquad \text{. . . (6)}$$

From equations (5) and (6) we have

$$(m_1+m_2)\ddot{r}-\frac{p_\theta^2}{m_1r^3}+m_2g=0. \qquad ...(7)$$

This is the required equation of motion.

i) To prove H represents a constant of motion, we differentiate equation (4) with respect to t. Thus we have

$$\frac{dH}{dt}=\frac{p_r\dot{p}_r}{(m_1+m_2)}+\frac{p_\theta\dot{p}_\theta}{m_1r^2}-\frac{p_\theta^2\,\dot{r}}{m_1r^3}+m_2g\dot{r}$$

Using equations (5) and (6), we have

$$\frac{dH}{dt}=\frac{p_rp_\theta^2}{(m_1+m_2)m_1r^3}-\frac{p_rm_2g}{m_1+m_2}-\frac{p_\theta^2\,p_r}{(m_1+m_2)m_1r^3}+\frac{m_2gp_r}{(m_1+m_2)_r},$$

$$\frac{dH}{dt}=0.$$

This shows that The Hamiltonian H represents a constant of motion.

ii) We have the kinetic and potential energies of the system are respectively given by

$$T=\frac{1}{2}m_1\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{1}{2}m_2\dot{r}^2,$$

$$V=-m_2g(l-r).$$

Now consider

$$T+V=\frac{1}{2}m_1\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{1}{2}m_2\dot{r}^2-m_2g(l-r). \qquad ...(8)$$

Eliminating $\dot{r}$ and $\dot{\theta}$ from equation (8) on using equations (6) we obtain

$$T+V=\frac{p_r^2}{2(m_1+m_2)}+\frac{p_\theta^2}{2m_1r^2}-m_2g(l-r) \qquad ...(9)$$

From equations (4) and (9) we see that the total energy is equal to the Hamiltonian function. Thus Hamiltonian H represents total energy of the system. To prove that the total energy is conserved, multiply the equation of motion (7) by $\dot{r}$, we get

$$(m_1+m_2)\ddot{r}\dot{r}-\frac{p_\theta^2\,\dot{r}}{m_1r^3}+m_2g\dot{r}=0.$$

This we write as

$$\frac{d}{dt}\left[(m_1+m_2)\frac{\dot{r}^2}{2}\right]+\frac{d}{dt}\left(\frac{p_\theta^2}{2m_1r^2}\right)+\frac{d}{dt}(m_2gr)=0.$$

Integrating and then eliminating $\dot{r}$ we get

$$\frac{p_r^2}{2(m_1+m_2)}+\frac{p_\theta^2}{2m_1r^2}+m_2gr=const.. \qquad \text{. . . (10)}$$

Equations (9) and (10) show that the total energy of the system is conserved.

Note : Equation (10) is the first integral of equation of motion. Its physical significance is that the Hamiltonian H represents the constant of total energy.

Example 11: A particle of mass m is moving on a xy plane which is rotating about z axis with angular velocity ω. The Lagrangian is given by

$$L=\frac{1}{2}m\left[(\dot{x}-\omega y)^2+(\dot{y}+\omega x)^2\right]-V(x,y).$$

Show that the Hamiltonian H is given by

$$H=\frac{1}{2m}\left(p_x^2+p_y^2\right)+p_x\omega y-p_y\omega x+V.$$

Find the equations of motion and hence prove or disprove that

i) H represents a constant of motion and

ii) H represents the total energy.

Solution: The Lagrangian of the particle is given by

$$L=\frac{1}{2}m\left[(\dot{x}-\omega y)^2+(\dot{y}+\omega x)^2\right]-V(x,y). \qquad \text{. . . (1)}$$

where the generalized momentum p_x and p_y are given by

$$p_x=\frac{\partial L}{\partial \dot{x}}=m(\dot{x}-\omega y)\Rightarrow \quad \dot{x}=\frac{p_x}{m}+\omega y, \qquad \text{. . . (2)}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m(\dot{y} - \omega x) \Rightarrow \quad \dot{y} = \frac{p_y}{m} - \omega x . \qquad \ldots (3)$$

The Hamiltonian H is defined by $\quad H = \dot{x}p_x + \dot{y}p_y - L$

$$H = p_x\dot{x} + p_y\dot{y} - \frac{1}{2}m\left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2\right] + V(x, y) .$$

Using equations (2) and (3) we eliminate $\dot{x}$ and $\dot{y}$ from the above equation to get the Hamiltonian of the system

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \omega\left(p_x y - p_y x\right) + V . \qquad \ldots (4)$$

The Hamilton's canonical equations of motions give

$$\begin{aligned} \dot{p}_x &= -\frac{\partial H}{\partial x} = p_y\omega - \frac{\partial V}{\partial x}, \quad & \dot{p}_y &= -\frac{\partial H}{\partial y} = -p_x\omega - \frac{\partial V}{\partial y}, \\ \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{m} + \omega y, & \dot{y} &= \frac{\partial H}{\partial p_x} = \frac{p_y}{m} - \omega x. \end{aligned} \qquad \ldots (5)$$

Solving these equations we obtain the equations which describe the motion

$$\begin{aligned} m\left(\ddot{x} - 2\omega\dot{y} - \omega^2 x\right) &= -\frac{\partial V}{\partial x}, \\ m\left(\ddot{y} + 2\omega\dot{x} - \omega^2 y\right) &= -\frac{\partial V}{\partial y} \end{aligned} \qquad \ldots (6)$$

Now to prove whether H is a constant of motion or not, differentiate equation (4) w. r. t. t to get

$$\frac{dH}{dt} = \frac{1}{m}\left(p_x\dot{p}_x + p_y\dot{p}_y\right) + \omega\left(\dot{p}_x y + p_x\dot{y} - \dot{p}_y x - p_y y\dot{x}\right) + \dot{x}\frac{\partial V}{\partial x} + \dot{y}\frac{\partial V}{\partial y} .$$

Using (5) we have

$$\frac{dH}{dt} = 0 .$$

This shows that H represents the constant of motion. Now to show whether H represents total energy or not, we have the total energy of the system

$$E = T + V ,$$

$$E = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + V(x, y). \qquad \ldots (7)$$

We see from equations (4) and (7) that the Hamiltonian H does not represent total energy.

Example 12 : A bead slides on a wire in the shape of a cycloid described by equations

$$x = a(\theta - \sin\theta), \quad y = a(1 + \cos\theta), \quad 0 \le \theta \le 2\pi.$$

Find the Hamiltonian H, hence the equations of motion. Also prove or disprove that

i) H represents a constant of motion

ii) H represents a total energy.

Solution: A particle describes a cycloid whose equations are

$$x = a(\theta - \sin\theta), \quad y = a(1 + \cos\theta), \quad 0 \le \theta \le 2\pi. \qquad \ldots (1)$$

The kinetic energy of the particle is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right),$$

where

$$\dot{x} = a\dot{\theta}(1 - \cos\theta),$$
$$\dot{y} = -a\sin\theta\dot{\theta}.$$

Hence the kinetic energy of the particle becomes

$$T = ma^2\dot{\theta}^2(1 - \cos\theta),$$

The potential energy of the particle is given by

$$V = mga(1 + \cos\theta).$$

Hence the Lagrangian of the particle becomes

$$L = ma^2\dot{\theta}^2(1 - \cos\theta) - mga(1 + \cos\theta). \qquad \ldots (2)$$

The Hamiltonian H of the particle is

$$H = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L, \qquad \ldots (3)$$

where from equation (2) we have

$$\frac{\partial L}{\partial \dot{\theta}} = p_\theta = 2ma^2\dot{\theta}\left(1-\cos\theta\right). \qquad ...(4)$$

Using equations (2) and (4) in (3) we obtain the expression for Hamiltonian as

$$H = ma^2\dot{\theta}^2\left(1-\cos\theta\right)+mga\left(1+\cos\theta\right). \qquad ...(5)$$

Using equation (4) we eliminate $\dot{\theta}$ from equation (5) to get the required Hamiltonian H as

$$H = \frac{p_\theta^2}{4ma^2\left(1-\cos\theta\right)}+mga\left(1+\cos\theta\right). \qquad ...(6)$$

The Hamilton's canonical equations of motion are

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\theta^2}{4ma^2}\frac{\sin\theta}{\left(1-\cos\theta\right)^2}+mga\sin\theta, \qquad ...(7)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{2ma^2\left(1-\cos\theta\right)}. \qquad ...(8)$$

From equations (7) and (8) we obtain the equation of motion of the particle

$$\ddot{\theta}\left(1-\cos\theta\right)+\frac{p_\theta^2}{8m^2a^4}\frac{\sin\theta}{\left(1-\cos\theta\right)^3}-\frac{g}{2a}\sin\theta = 0 \qquad ...(9)$$

Eliminating p_θ from equation (9) we obtain the equation which describes the motion of the particle in the form

$$2ma^2\left(1-\cos\theta\right)\ddot{\theta}+ma^2\sin\theta\dot{\theta}^2-mga\sin\theta = 0. \qquad ...(10)$$

Now to prove

i) H is a constant of motion, differentiate equation (6) with respect to time t we get

$$\frac{dH}{dt} = \frac{2p_\theta\dot{p}_\theta}{4ma^2\left(1-\cos\theta\right)}-\frac{p_\theta^2}{4ma^2}\frac{\sin\theta\dot{\theta}}{\left(1-\cos\theta\right)^2}-mga\sin\theta\dot{\theta}.$$

Using equations (7) and (8) we readily get

$$\frac{dH}{dt}=0.$$

This shows that the Hamiltonian H is a constant of motion.

ii) H represents the total energy

We find from the expressions for kinetic energy and potential energy that

$$T+V=ma^2\dot{\theta}^2\left(1-\cos\theta\right)+mga\left(1+\cos\theta\right). \qquad ...(11)$$

Eliminating $\dot{\theta}$ from equation (11) we get equation (6) that gives the required expression for the Hamiltonian. Now multiply equation (10) by $\dot{\theta}$ we get

$$2ma^2\left(1-\cos\theta\right)\ddot{\theta}\dot{\theta}+ma^2\sin\theta\dot{\theta}^3-mga\sin\theta\dot{\theta}=0.$$

This can be written as

$$\frac{d}{dt}\left[ma^2\left(1-\cos\theta\right)\dot{\theta}^2+mga\left(1+\cos\theta\right)\right]=0.$$

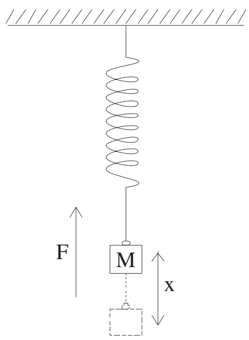
Integrating we get

$$H=T+V=ma^2\dot{\theta}^2\left(1-\cos\theta\right)+mga\left(1+\cos\theta\right)=const.$$

This shows that the Hamiltonian H represents the constant of total energy.

Example 13 : Obtain the Hamilton's equation of motion for a one dimensional harmonic oscillator.

Solution: The one dimensional harmonic oscillator consists of a mass attached to one end of a spring and other end of the spring is fixed. If the spring is pressed and released then on account of the elastic property of the spring, the spring exerts a force F on the body in the opposite direction. This is called restoring force. It is found that this force is proportional to the displacement of the body from its equilibrium position.



$$F\propto x$$
$$F=-kx$$

where k is the spring constant and negative sign indicates the force is opposite to the displacement. Hence the potential energy of the particle is given by

$$V = -\int F dx,$$
$$V = \int kx dx + c,$$
$$V = \frac{kx^2}{2} + c,$$

where c is the constant of integration. By choosing the horizontal plane passing through the position of equilibrium as the reference level, then V=0 at x=0. This gives c=0. Hence potential energy of the particle is

$$V = \frac{1}{2}kx^2. \qquad \ldots (1)$$

The kinetic energy of the one dimensional harmonic oscillator is

$$T = \frac{1}{2}m\dot{x}^2. \qquad \ldots (2)$$

Hence the Lagrangian of the system is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \qquad \ldots (3)$$

The Lagrange's equation motion gives

$$\ddot{x} + \omega^2 x = 0, \quad \omega^2 = \frac{k}{m}. \qquad \ldots (4)$$

This is the equation of motion. ω is the frequency of oscillation.

The Hamiltonian H of the oscillator is defined as

$$H = \dot{x}p_x - L,$$
$$H = \dot{x}p_x - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2,$$

where

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \quad \dot{x} = \frac{p_x}{m}.$$

Substituting this in the above equation we get the Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2. \qquad \ldots (5)$$

Solving the Hamilton's canonical equations of motion we readily get the equation (4) as the equation of motion.

Example 14: For a particle the kinetic energy and potential energy is given by respectively,

$$T = \frac{1}{2}m\dot{r}^2, \quad U = \frac{1}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right).$$

Find the Hamiltonian H and determine

1. whether $H = T + V$
2. whether $\frac{dH}{dt} = 0$.

Solution: The kinetic and potential energies of a particle are given by

$$T = \frac{1}{2}m\dot{r}^2, \quad U = \frac{1}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right)$$

respectively. The Lagrangian function is therefore given by

$$L = \frac{1}{2}m\dot{r}^2 - \frac{1}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right). \qquad \ldots (1)$$

We see that the particle has only one degree of freedom and hence it has only one generalized co-ordinate. The generalized momentum is defined by

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} - \frac{2\dot{r}}{rc^2},$$

$$\Rightarrow \quad \dot{r} = \frac{p_r\, r c^2}{\left(mrc^2 - 2\right)}. \qquad \ldots (2)$$

Thus the corresponding Hamiltonian function is defined by

$$H = p_r\,\dot{r} - L,$$

$$H = p_r\,\dot{r} - \frac{1}{2}m\dot{r}^2 + \frac{1}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right). \qquad \ldots (3)$$

Eliminating $\dot{r}$ between (2) and (3) we obtain the Hamiltonian H as

$$H = \frac{1}{2}\frac{p_r^2\, r\, c^2}{\left(mrc^2 - 2\right)} + \frac{1}{r}. \qquad \text{. . . (4)}$$

1. Now the sum of the kinetic and potential energies is given by

$$T + U = \frac{1}{2}m\dot{r}^2 + \frac{1}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right). \qquad \text{. . . (5)}$$

Eliminating $\dot{r}$ between (2) and (5) we get

$$T + U = \frac{1}{2}\frac{p_r^2\, r\, c^2\left(mrc^2 + 2\right)}{\left(mrc^2 - 2\right)^2} + \frac{1}{r}. \qquad \text{. . . (6)}$$

We see from equations (4) and (6) that the Hamiltonian H does not represent the total energy.

$$\Rightarrow \quad T + U \neq H.$$

2. Now differentiating equation (4) w. r. t. time t we get

$$\frac{dH}{dt} = \frac{p_r \dot{p}_r r c^2}{\left(mrc^2 - 2\right)} - \frac{p_r^2 \dot{r} c^2}{\left(mrc^2 - 2\right)^2} - \frac{\dot{r}}{r^2}, \qquad \text{. . . (7)}$$

where

$$\dot{p}_r = \ddot{r}\left(m - \frac{2}{rc^2}\right) + \frac{2\dot{r}^2}{r^2 c^2}.$$

Substituting this in equation (7) and simplifying we get

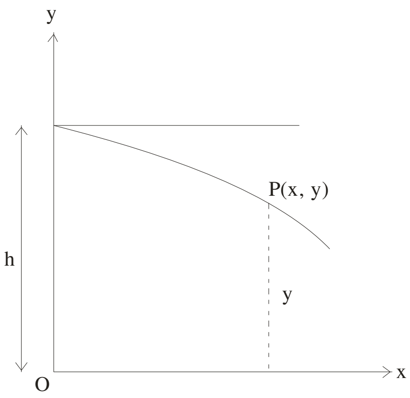
$$\frac{dH}{dt} = p_r \ddot{r} + \frac{\dot{r}^3}{r^2 c^2} - \frac{\dot{r}}{r^2}.$$

$$\Rightarrow \quad \frac{dH}{dt} \neq 0.$$

This shows that the Hamiltonian H is not a constant of motion.

Example 15 : A particle is thrown horizontally from the top of a building of height h with an initial velocity u. Write down the Hamiltonian of the problem. Show that H represents both a constant of motion and the total energy.

Solution: Let the particle be thrown horizontally from the top of a building of height h with an initial velocity u. The motion of the particle is in a plane. If P (x, y) are the position co-ordinates of the particle at any instant, then its kinetic energy and the potential energy are respectively given by



$$T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right), \qquad \ldots (1)$$

$$V = -mg\left(h - y\right). \qquad \ldots (2)$$

Hence the Lagrangian of the particle becomes

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + mg\left(h - y\right). \qquad \ldots (3)$$

The particle has two degrees of freedom and hence two generalized co-ordinates. We see that the generalized co-ordinate x is cyclic in L, hence the corresponding generalized momentum is conserved.

$$\begin{aligned} \Rightarrow \quad p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} = const. \\ p_y &= \frac{\partial L}{\partial \dot{y}} = m\dot{y}. \end{aligned} \qquad \ldots (4)$$

The Hamiltonian function H is defined as

$$H = \sum_j p_j \dot{q}_j - L$$

$$H = p_x \dot{x} + p_y \dot{y} - \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - mh\left(h - y\right) \qquad \ldots (5)$$

Eliminating the velocities from equations (4) and (5) we obtain the Hamiltonian of motion as

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) - mg\left(h - y\right). \qquad \text{. . . (6)}$$

The corresponding Hamilton's canonical equations of motion are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}, \quad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m},$$

$$\textit{and} \quad \dot{p}_x = -\frac{\partial H}{\partial x} = 0, \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -mg.$$

Solving these equations we get the equations of motion as

$$\ddot{x} = 0, \quad \ddot{y} = -g . \qquad \text{. . . (7)}$$

Now differentiating equation (6) with respect to t we get

$$\frac{dH}{dt} = \frac{1}{m}\left(p_x \dot{p}_x + p_y \dot{p}_y\right) + mg\dot{y}$$

$$\Rightarrow \quad \frac{dH}{dt} = 0,$$

This proves that H is a constant of motion.

Now to see whether H represents total energy or not, we consider

$$T + V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mg\left(h - y\right)$$

Putting the values of $\dot{x}$ and $\dot{y}$ we obtain the expression for the Hamiltonian as

$$T + V = \frac{1}{2m}\left(p_x^2 + p_y^2\right) - mg\left(h - y\right). \qquad \text{. . . (8)}$$

This represents the Hamiltonian H, proving that H represents the total energy of the particle.

Example 16: A particle is constrained to move on the arc of a parabola $x^2 = 4ay$ under the action of gravity. Show that the Hamiltonian of the system is

$$H = \frac{2a^2 p_x^2}{m\left(4a^2 + x^2\right)} + \frac{mg}{4a}x^2.$$

Is the Hamiltonian of the particle representing total energy? Is it a constant of motion?

Solution: Given that a particle is constrained to move on the arc of the parabola

$$x^2 = 4ay \qquad \ldots (1)$$

where y is vertical axis, under the action of gravity. The kinetic energy of the particle is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \qquad \ldots (2)$$

and the potential energy is given by

$$V = mgy\,. \qquad \ldots (3)$$

However, x and y are not the generalized co-ordinates, because they are related by the constraint equation (1). Eliminating y from equations (2) and (3) on using (1) we obtain

$$T = \frac{1}{2}m\dot{x}^2\left(1 + \frac{x^2}{4a^2}\right), \quad V = \frac{x^2}{4a}mg\,.$$

Hence the Lagrangian function is

$$L = \frac{1}{2}m\dot{x}^2\left(1 + \frac{x^2}{4a^2}\right) - \frac{x^2}{4a}mg\,. \qquad \ldots (4)$$

Now we see that the system has one degree of freedom and only one generalized co-ordinate x.

$$\Rightarrow \quad p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}\left(1 + \frac{x^2}{4a^2}\right),$$

$$\Rightarrow \quad \dot{x} = \frac{4a^2 p_x}{m\left(4a^2 + x^2\right)}\,. \qquad \ldots (5)$$

Now the Hamiltonian H becomes

$$H = \dot{x}p_x - L$$

$$H = \dot{x}p_x - \frac{1}{2}m\dot{x}^2\left(1+\frac{x^2}{4a^2}\right)+\frac{x^2}{4a}mg\,. \qquad \ldots (6)$$

On using (5) we write equation (6) as

$$H = \frac{2a^2 p_x^2}{m\left(4a^2+x^2\right)}+\frac{mgx^2}{4a}. \qquad \ldots (7)$$

This is the required Hamiltonian function. Now to see whether this H represents total energy or not, we consider

$$\mathrm{T+V} = \frac{1}{2}m\dot{x}^2\left(1+\frac{x^2}{4a^2}\right)+\frac{x^2}{4a}mg\,. \qquad \ldots (8)$$

Using equation (5) we obtain

$$T+V = \frac{2a^2 p_x^2}{m\left(4a^2+x^2\right)}+\frac{mgx^2}{4a}. \qquad \ldots (9)$$

Which is the Hamiltonian of the motion, showing that it represent the total energy of the particle. Now to show that the Hamiltonian H represents constant of motion, we first find the equation of motion. From equation (4) we have

$$\frac{\partial L}{\partial x} = \frac{m}{4a^2}x\dot{x}^2 - \frac{mgx}{2a},$$
$$\frac{\partial L}{\partial \dot{x}} = \left(1+\frac{x^2}{4a^2}\right)m\dot{x}.$$

Hence the equation of motion becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0 \;\Rightarrow\; \frac{d}{dt}\left[\left(1+\frac{x^2}{4a^2}\right)m\dot{x}\right]-\frac{m}{4a^2}x\dot{x}^2+\frac{mgx}{2a}=0,$$

$$\Rightarrow \quad \left(4a^2+x^2\right)\ddot{x}+x\dot{x}^2+2agx=0\,. \qquad \ldots (10)$$

Now differentiating equation (7) with respect to t we get

$$\frac{dH}{dt} = \frac{4a^2}{m}\left[\frac{p_x\dot{p}_x}{\left(4a^2+x^2\right)}-\frac{x\dot{x}p_x^2}{\left(4a^2+x^2\right)^2}\right]+\frac{mg}{2a}x\dot{x}\,.$$

Eliminating $p_x, \dot{p}_x$ we obtain

$$\frac{dH}{dt} = \frac{m}{4a^2}\left[\left(4a^2 + x^2\right)\ddot{x} + x\dot{x}^2 + 2agx\right]\dot{x}.$$

This implies from equation (10) that $\frac{dH}{dt} = 0$.

This shows that the Hamiltonian H is a constant of motion.

Example 17 : Set up the Hamiltonian for the Lagrangian

$$L(q,\dot{q},t) = \frac{m}{2}\left[\dot{q}^2 \sin^2 \omega t + q\dot{q}\sin 2\omega t + q^2\omega^2\right].$$

Derive the Hamilton's equations of motion. Reduce the equations in to a single second order differential equation.

Solution: The Lagrangian of the system is given by

$$L(q,\dot{q},t) = \frac{m}{2}\left[\dot{q}^2 \sin^2 \omega t + q\dot{q}\sin 2\omega t + q^2\omega^2\right] \qquad \text{. . . (1)}$$

The system has only one degree of freedom and hence only one generalized co-ordinate q. The generalized momentum is given by

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{m}{2}\left(2\dot{q}\sin^2 \omega t + q\omega \sin 2\omega t\right) \qquad \text{. . . (2)}$$

$$\Rightarrow \quad \dot{q} = \frac{1}{\sin^2 \omega t}\left[\frac{p}{m} - \frac{q}{2}\omega \sin 2\omega t\right]. \qquad \text{. . . (3)}$$

Now the Hamiltonian function H is defined as

$$H = p\dot{q} - \frac{m}{2}\left(\dot{q}^2 \sin^2 \omega t + q\dot{q}\omega \sin 2\omega t + q^2\omega^2\right). \qquad \text{. . . (4)}$$

Substituting the value of $\dot{q}$ from equation (3) in (4) and simplifying we get

$$H = \frac{p^2}{2m\sin^2 \omega t} - pq\omega \cot \omega t + \frac{q^2 m\omega^2}{2}\cos^2 \omega t - \frac{m}{2}q^2\omega^2. \qquad \text{. . . (5)}$$

This is the Hamiltonian of the system. The Hamilton's canonical equations of motion give

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m\sin^2 \omega t} - q\omega\cot\omega t \; . \qquad \ldots (6)$$

and $\quad \dot{p} = p\omega\cot\omega t - q\omega^2 m\cos^2 \omega t + mq\omega^2 . \qquad \ldots (7)$

From equation (6) we find

$$p = \frac{m}{2}\left[2\dot{q}\sin^2 \omega t + q\omega\sin 2\omega t\right] \qquad \ldots (8)$$

Differentiating equation (8) w. r. t. t we get

$$\dot{p} = \frac{m}{2}\left[2\ddot{q}\sin^2 \omega t + 4\dot{q}\omega\sin\omega t\cos\omega t + \dot{q}\omega\sin 2\omega t + 2q\omega^2\cos 2\omega t\right] \quad \ldots (9)$$

Equating equations (7) and (9) we get

$$\ddot{q} + 2\omega\dot{q}\cot\omega t - 2q\omega^2 = 0 \, . \qquad \ldots (10)$$

This equation determines the motion of the particle.

Example 18 : A Lagrangian of a system is given by

$$L(x, y, \dot{x}, \dot{y}) = \frac{m}{2}\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{k}{2}\left(ax^2 + 2bxy + cy^2\right),$$

where a, b, c, k, m are constants and $b^2 - ac \neq 0$. Find the Hamiltonian and equations of motion. Examine the particular cases $a = 0, c = 0$ and $b = 0, c = -a$.

Solution: Given that

$$L(x, y, \dot{x}, \dot{y}) = \frac{m}{2}\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{k}{2}\left(ax^2 + 2bxy + cy^2\right), \qquad \ldots (1)$$

where a, b, c, k, m are constants and $b^2 - ac \neq 0$. We see that the system has two generalized co-ordinates x and y. Hence the corresponding generalized momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\left(a\dot{x} + b\dot{y}\right), \qquad \ldots (2)$$

and $$p_y = \frac{\partial L}{\partial \dot{y}} = m\left(b\dot{x} + c\dot{y}\right). \qquad \ldots (3)$$

Solving these equations for $\dot{x}$ and $\dot{y}$ we get

$$\dot{x} = -\frac{cp_x - bp_y}{m\left(b^2 - ac\right)}, \quad \dot{y} = \frac{bp_x - ap_y}{m\left(b^2 - ac\right)}. \qquad \text{. . . (4)}$$

The Hamiltonian H is defined by

$$H = \sum_j p_j \dot{q}_j - L,$$

$$H = p_x \dot{x} + p_y \dot{y} - \frac{m}{2}\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) + \frac{k}{2}\left(ax^2 + 2bxy + cy^2\right). \quad ...(5)$$

Using equations (4) in (5) we obtain after simplifying

$$H = \frac{1}{m\left(b^2 - ac\right)}\left[bp_x p_y - \frac{a}{2}p_y^2 - \frac{c}{2}p_x^2\right] + \frac{k}{2}\left(ax + 2bxy + cy^2\right) .. \text{. . (6)}$$

This is the required Hamiltonian of the system. The Hamilton's equations of motion corresponding to two generalized co-ordinates x, y are

$$\begin{aligned} \dot{p}_x &= -\frac{\partial H}{\partial x} = -k\left(ax + by\right), \\ \dot{p}_y &= -\frac{\partial H}{\partial y} = -k\left(bx + cy\right). \end{aligned} \qquad \text{. . . (7)}$$

and

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{1}{m\left(b^2 - ac\right)}\left(bp_y - cp_x\right) \\ \dot{y} &= \frac{\partial H}{\partial p_y} = \frac{1}{m\left(b^2 - ac\right)}\left(bp_x - cp_y\right) \end{aligned} \qquad \text{. . . (8)}$$

From equations (2), (3) and (7) we have

$$\begin{aligned} m\left(a\ddot{x} + b\ddot{y}\right) + k\left(ax + by\right) &= 0, \\ m\left(b\ddot{x} + c\ddot{y}\right) + k\left(bx + cy\right) &= 0. \end{aligned} \qquad \text{. . . (9)}$$

These are the required equations of motion. Solving these equations for $\ddot{x}$ and $\ddot{y}$ we obtain respectively

$$m\ddot{x} + kx = 0. \qquad \text{. . . (10)}$$

$$m\ddot{y} + ky = 0. \qquad \text{. . . (11)}$$

The solutions of these equations are

$$x = c_1 \cos\left(\frac{k}{m}\right)t + c_2 \sin\left(\frac{k}{m}\right)t, \qquad \text{. . . (12)}$$

and $$y = d_1 \cos\left(\frac{k}{m}\right)t + d_2 \sin\left(\frac{k}{m}\right)t. \qquad \text{. . . (13)}$$

Now the cases $a = 0, c = 0$ and $b = 0, c = -a$ yield from equations (9) the same set of equations (10) and (11).

Example 19 : The Lagrangian for a system can be written as

$$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{z}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y} - k\sqrt{x^2 + y^2} \ ,$$

where a, b, c, f, g and k are constants. What is Hamiltonian? What quantities are conserved ?

Solution: The Lagrangian of the system is

$$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{z}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y} - k\sqrt{x^2 + y^2} \ , \qquad \text{. . . (1)}$$

where a, b, c, f, g and k are constants. The system has three degrees of freedom and has three generalized co-ordinates (x, y, z), of which z is cyclic. This implies the corresponding generalized momentum p_z is conserved.

$$\Rightarrow \quad p_z = \frac{\partial L}{\partial \dot{z}} = c\dot{y} + fy^2\dot{x} = const. \qquad \text{. . . (2)}$$

Similarly, we find

$$p_x = \frac{\partial L}{\partial \dot{x}} = 2a\dot{x} + fy^2\dot{z} \ , \qquad \text{. . . (3)}$$

and $$p_y = \frac{\partial L}{\partial \dot{y}} = \frac{b}{x} + c\dot{z} + g \ . \qquad \text{. . . (4)}$$

Solving these equations for $\dot{x}, \dot{y}, \dot{z}$ we get

$$\dot{x} = \frac{1}{2a}\left[p_x - \frac{fy^2}{c}\left(p_y - \frac{b}{x} - g\right)\right],$$

$$\dot{y} = \frac{1}{c}\left[p_z - \frac{fy^2}{2a}\left(p_x - \frac{fy^2}{c}\left(p_y - \frac{b}{x} - g\right)\right)\right],$$

$$\dot{z} = \frac{1}{c}\left(p_y - \frac{b}{x} - g\right). \qquad \ldots (5)$$

The Hamiltonian of the system is defined as

$$H = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L$$

$$H = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - a\dot{x}^2 - b\frac{\dot{y}}{x} - c\dot{x}\dot{y} - fy^2\dot{x}\dot{z} - g\dot{y} + k\sqrt{x^2 + y^2}\,. \qquad \ldots (6)$$

The required Hamiltonian is obtained by eliminating $\dot{x}, \dot{y}, \dot{z}$ from equation (6).

Unit 3: *Routh's Procedure :*

Introduction:

The presence of cyclic co-ordinates in the Lagrangian L is not much profitable because even if the co-ordinate q_j does not appear in L, the corresponding generalized momentum $\dot{q}_j$ generally does, so that one has to deal the problem with all variables and the system has n degrees of freedom. However, if q_j is cyclic in the Hamiltonian then p_j is constant and then one has to deal with the problem involving only 2n-2 variables, i.e., only n-1 degrees of freedom. Hence Hamiltonian procedure is especially adapted to the problems involving cyclic co-ordinates. The advantage of Hamiltonian formulation in handling with cyclic co-ordinates is utilized by Routh and devised a method by combining with the Lagrangian procedure and the method is known as Routh's Procedure. The Method is described in the following theorem.

Theorem 10: Describe the Routh's procedure to solve the problem involving cyclic and non-cyclic co-ordinates.

Proof: Consider a system of particles involving both cyclic and non-cyclic co-ordinates. Let $q_1, q_2, q_3, ..., q_s$ of $q_1, q_2, q_3, ..., q_n$ are cyclic co-ordinates, then a new function R, known as the Routhian is defined as

$$R\left(q_1, q_2, ..., q_n; p_1, p_2, ..., p_s; \dot{q}_{s+1}, \dot{q}_{s+2}, ..., \dot{q}_n, t\right) = \sum_{j=1}^{s} p_j \dot{q}_j - L\left(q_j, \dot{q}_j, t\right) \quad ...(1)$$

The Routhian R is obtained by modifying the Lagrangian L so that it is no longer a function of the generalized velocities corresponding to the cyclic co-ordinates, but instead involves only its conjugate momentum. The advantage in doing so is that p_j can then be considered one of the constants of integration and the remaining integrations involve only the non-cyclic co-ordinates.

Now we take $R = R\left(q_1, q_2, ..., q_n; p_1, p_2, ..., p_s; \dot{q}_{s+1}, \dot{q}_{s+2}, ..., \dot{q}_n, t\right)$, and find the total differential dR as

$$dR = \sum_{j=1}^{n} \frac{\partial R}{\partial q_j} dq_j + \sum_{j=1}^{s} \frac{\partial R}{\partial p_j} dp_j + \sum_{j=s+1}^{n} \frac{\partial R}{\partial \dot{q}_j} d\dot{q}_j + \frac{\partial R}{\partial t} dt. \quad ...(2)$$

Now we consider

$$R = \sum_{j=1}^{s} p_j \dot{q}_j - L\left(q_j, \dot{q}_j, t\right)$$

and find the total differential as

$$dR = \sum_{j=1}^{s} p_j d\dot{q}_j + \sum_{j=1}^{s} \dot{q}_j dp_j - dL,$$

$$dR = \sum_{j=1}^{s} p_j d\dot{q}_j + \sum_{j=1}^{s} \dot{q}_j dp_j - \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} dq_j - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \frac{\partial L}{\partial t} dt.$$

$$dR = \sum_{j=1}^{s} p_j d\dot{q}_j + \sum_{j=1}^{s} \dot{q}_j dp_j - \left(\sum_{j=1}^{s} \frac{\partial L}{\partial q_j} dq_j + \sum_{j=s+1}^{n} \frac{\partial L}{\partial q_j} dq_j \right) - \\ - \left(\sum_{j=1}^{s} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \sum_{j=s+1}^{n} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right) - \frac{\partial L}{\partial t} dt.$$

$$dR = \sum_{j=1}^{s} \dot{q}_j dp_j - \sum_{j=1}^{s} \frac{\partial L}{\partial q_j} dq_j - \sum_{j=s+1}^{n} \frac{\partial L}{\partial q_j} dq_j - \\ - \sum_{j=s+1}^{n} \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \frac{\partial L}{\partial t} dt. \qquad \text{. . . (3)}$$

Now equating the corresponding coefficients on both the sides of equations (2) and (3) we obtain

$$\frac{\partial R}{\partial p_j} = \dot{q}_j, \quad j = 1, 2, ..., s \qquad \text{. . . (4)}$$

$$\frac{\partial R}{\partial q_j} = -\frac{\partial L}{\partial q_j} = -\dot{p}_j, \quad j = 1, 2, ..., s \qquad \text{. . . (5)}$$

$$\frac{\partial R}{\partial q_j} = -\frac{\partial L}{\partial q_j} = -\dot{p}_j, \quad j = s+1, s+2, ..., n \qquad \text{. . . (6)}$$

and

$$\frac{\partial R}{\partial \dot{q}_j} = -\frac{\partial L}{\partial \dot{q}_j} = -p_j, \quad j = s+1, s+2, ..., n \qquad \text{. . . (7)}$$

We see that for cyclic co-ordinates $q_1, q_2, ..., q_s$ equations (4) and (5) represent Hamilton's equations of motion with R as the Hamiltonian, while equations (6) and (7) for the non-cyclic co-ordinates $q_j \quad (j = s+1, s+2, ..., n)$ represent Lagrange's equations of motion with R as the Lagrangian function. i.e., from equations (6) and (7) we obtain

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j} \right) - \frac{\partial R}{\partial q_j} = 0, \quad j = s+1, s+2, ..., n \qquad \text{. . . (8)}$$

Thus by Routhian procedure a problem involving cyclic and non-cyclic co-ordinates can be solved by solving Lagrange's equations for non-cyclic co-ordinates with

Routhian R as the Lagrangian function and solving Hamiltonian equations for the given cyclic co-ordinates with R as the Hamiltonian function. In this way The Routhian has a dual character Hamiltonian H and the Lagrangian L.

Worked Examples

Example 20 : Find Lagrangian L, Hamiltonian H and the Routhian R in spherical polar co-ordinates for a particle moving in space under the action of conservative force.

Solution: Let a particle be moving in a space. If (x, y, z) are the Cartesian co-ordinates and (r,θ,ϕ) are the spherical co-ordinates of the particle, then we have the relation between them as

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned} \qquad \ldots (1)$$

The kinetic energy $T=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$ of the particle, in spherical polar co-ordinates becomes

$$T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\right). \qquad \ldots (2)$$

Since the force is conservative, hence the potential energy of the particle is the function of position only.

$$\Rightarrow \qquad V=V\left(r,\theta,\phi\right). \qquad \ldots (3)$$

Hence the Lagrangian function of the particle becomes

$$L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\right)-V\left(r,\theta,\phi\right). \qquad \ldots (4)$$

We see that ϕ is cyclic in L, hence the corresponding generalized momentum is conserved. i.e.,

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2\theta\,\dot{\phi} = const. \qquad \ldots (5)$$

Similarly we find

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}. \end{aligned} \qquad \ldots (6)$$

Now the Hamiltonian function is defined as

$$\begin{aligned} H &= \sum_j p_j \dot{q}_j - L, \\ H &= p_r\dot{r} + p_\theta\,\dot{\theta} + p_\phi\,\dot{\phi} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) + V. \end{aligned} \qquad \ldots (7)$$

Eliminating the generalized velocities $\dot{r}, \dot{\theta}, \dot{\phi}$ between equations (5), (6) and (7) we get

$$H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\phi^2\right) + V. \qquad \ldots (8)$$

Now the Routhian R is defined by

$$R = p_\phi\,\dot{\phi} - L, \qquad \ldots (9)$$

This becomes after eliminating $\dot{r}, \dot{\theta}, \dot{\phi}$ between (5), (6) and (9) we get

$$R\left(r, \theta, \phi, \dot{r}, \dot{\theta}, t\right) = \frac{p_\phi^2}{2mr^2\sin^2\theta} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V. \qquad \ldots (10)$$

Example 21 : A planet moves under the inverse square law of attractive force, Find Lagrangian L, Hamiltonian H, and the Routhian R for the planet.

Solution: A motion of a planet is a motion in the plane. If (r, θ) are the generalized co-ordinates of the planet then it's kinetic and potential energies are respectively given by

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right), \quad V = -\frac{K}{r}.$$

Hence the Lagrangian function is defined by

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{K}{r}. \qquad \ldots (1)$$

We see that θ is the cyclic co-ordinate in L. This implies that the corresponding angular momentum of the planet is conserved.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = const. \Rightarrow \quad \dot{\theta} = \frac{p_\theta}{mr^2}. \qquad \ldots (2)$$

Also
$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \quad \dot{r} = \frac{p_r}{m} \qquad \ldots (3)$$

Now the Hamiltonian function is defined as

$$H = \sum_j p_j \dot{q}_j - L,$$

$$H = p_r\dot{r} + p_\theta\,\dot{\theta} - \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{K}{r}$$

On using equations (2) and (3) we obtain

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{K}{r}. \qquad \ldots (4)$$

This is the required Hamiltonian.

Now the Routhian is defined as

$$R\left(r,\theta,p_\theta,\dot{r},t\right) = p_\theta\dot{\theta} - L$$

$$R\left(r,\theta,p_\theta,\dot{r},t\right) = p_\theta\dot{\theta} - \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{K}{r}.$$

Eliminating $\dot{\theta}$ we get

$$R\left(r,\theta,p_\theta,\dot{r},t\right) = \frac{p_\theta^2}{2mr^2} - \frac{1}{2} m\dot{r}^2 - \frac{K}{r}. \qquad \ldots (5)$$

This can also be written as

$$R\left(r,\theta,p_\theta,\dot{r},t\right) = \frac{p_\theta^2}{2mr^2} - \frac{p_r^2}{2m} - \frac{K}{r}. \qquad \ldots (6)$$

- ***Principle of Least Action :***

Action in Mechanics :

In Mechanics the time integral of twice the kinetic energy is called the action. Thus

$$A=\int_{t_0}^{t_1} 2T dt$$

is called the action.

i.e. $$A=\int_{t_0}^{t_1}\sum_j p_j\dot{q}_j dt$$

is called action in Mechanics.

Principle of Least Action :

There is another variational principle associated with the Hamiltonian formulation and is known as the principle of least action. It involves a new type of variation which we call the Δ- variation.

In Δ- variation the co-ordinates of the end points remain fixed while the time is allowed to vary. The varied paths may terminate at different points, but still position co-ordinates are held fixed.

Mathematically, we have

$$\delta I=\frac{\partial I}{\partial \alpha}d\alpha, \quad \Delta I=\frac{dI}{d\alpha}d\alpha .$$

Thus for the family of paths represented by the equation

$$q_j=q_j(\alpha,t), \quad t=t(\alpha)$$

We have

$$\Delta q_j=\frac{dq_j}{d\alpha}d\alpha=\left(\frac{\partial q_j}{\partial \alpha}+\dot{q}_j\frac{dt}{d\alpha}\right)d\alpha .$$

$$\Delta q_j = \left(\frac{\partial q_j}{\partial \alpha} d\alpha + \dot{q}_j \frac{dt}{d\alpha} d\alpha \right).$$

$$\Delta q_j = \delta q_j + \dot{q}_j \Delta t$$

This shows that the total variation is the sum of two variations.

Worked Examples

Example 22 : If $f = f\left(q_j, \dot{q}_j, t\right)$ then show that

$$\Delta f = \delta f + \Delta t \cdot \frac{df}{dt}.$$

Solution: Consider a system of particles moving from one point to another. Let the family of paths between these two points be given by

$$q_j = q_j\left(t, \alpha\right). \qquad \text{. . . (1)}$$

In Δ variation time is not held fixed, it depends on the path. This implies that

$$t = t\left(\alpha\right) \qquad \text{. . . (2)}$$

Since $\qquad f = f\left(q_j, \dot{q}_j, t\right) \qquad$ . . . (3)

then we find Δ variation in f as

$$\Delta f = \sum_j \left(\frac{\partial f}{\partial q_j} \Delta q_j + \frac{\partial f}{\partial \dot{q}_j} \Delta \dot{q}_j \right) + \frac{\partial f}{\partial t} \Delta t. \qquad \text{. . . (4)}$$

However, we have $\qquad \Delta q_j = \delta q_j + \dot{q}_j \Delta t \qquad$ . . . (5)

Similarly we find

$$\Delta \dot{q}_j = \delta \dot{q}_j + \ddot{q}_j \Delta t, \qquad \text{. . . (6)}$$

Using equations (5) and (6) in equation (4) we get

$$\Delta f = \sum_j \left(\frac{\partial f}{\partial q_j} \left(\delta q_j + \dot{q}_j \Delta t \right) + \frac{\partial f}{\partial \dot{q}_j} \left(\delta \dot{q}_j + \ddot{q}_j \Delta t \right) \right) + \frac{\partial f}{\partial t} \Delta t.$$

$$\Delta f = \sum_j \left(\frac{\partial f}{\partial q_j} \delta q_j + \frac{\partial f}{\partial \dot{q}_j} \delta \dot{q}_j + \frac{\partial f}{\partial t} \delta t \right) + \sum_j \left(\frac{\partial f}{\partial q_j} \dot{q}_j + \frac{\partial f}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial f}{\partial t} \right) \Delta t$$

Note here that the term δt added because it is zero, since in δ variation time t is held fixed and consequently change in time t is zero. This can be written as

$$\Delta f = \delta f + \Delta t \cdot \frac{df}{dt}. \qquad \ldots (7)$$

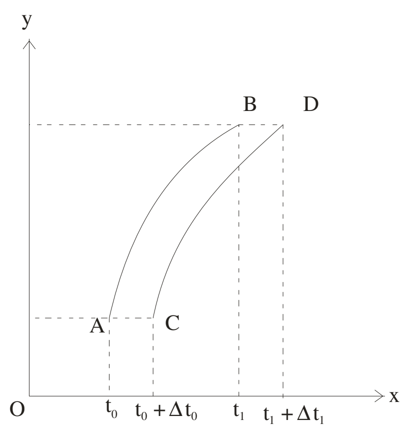
Since f is arbitrary, we can write it as

$$\Delta = \delta + \Delta t \cdot \frac{d}{dt}. \qquad \ldots (8)$$

Theorem 11 : For a conservative system for which the Hamiltonian H is conserved, the principle of least action states that

$$\Delta \int_{t_0}^{t_1} \sum_j p_j \dot{q}_j dt = 0 .$$

Proof: Consider a conservative system for which the Hamiltonian H is conserved. Let AB be the actual path and CD be the varied path. In Δ- variation the end points of the two paths are not terminated at the same point. The end points A and B after Δt take the positions C and D such that the position co-ordinates of A, C and B, D are held fixed. Now we know the action is given by



$$A = \int_{t_0}^{t_1} \sum_j p_j \dot{q}_j dt$$

$$A = \int_{t_0}^{t_1} (L + H) dt$$

$$A = \int_{t_0}^{t_1} L dt + H \left(t \right)_{t_0}^{t_1},$$

$$A = \int_{t_0}^{t_1} L dt + H\left(t_1 - t_0\right). \qquad \ldots (1)$$

Thus

$$\Delta A = \Delta \int_{t_0}^{t_1} L dt + H\left(\Delta t\right)_{t_0}^{t_1}. \qquad \ldots (2)$$

Since time limits are also subject to change in Δ-variation, therefore Δ can't be taken inside the integral. Let

$$\int_{t_0}^{t_1} L dt = I \quad \Rightarrow \quad \dot{I} = L.$$

Therefore

$$\Delta I = \delta I + \dot{I}\Delta t.$$

Thus we have

$$\Delta \int_{t_0}^{t_1} L dt = \delta \int_{t_0}^{t_1} L dt + L\left(\Delta t\right)_{t_0}^{t_1}.$$

$$\Delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \left[\sum_j \left(\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right) + \frac{\partial L}{\partial t} \delta t \right] dt + L\left(\Delta t\right)_{t_0}^{t_1}.$$

Since in δ variation, time is held fixed along any path, hence there is no variation in time, therefore change in time is zero. Thus we have

$$\Delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \sum_j \left(\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right) dt + L\left(\Delta t\right)_{t_0}^{t_1}.$$

Using Lagrange's equations of motion we write this equation as

$$\Delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \sum_j \left(\dot{p}_j \delta q_j + p_j \delta \dot{q}_j \right) dt + L\left(\Delta t\right)_{t_0}^{t_1}.$$

Since

$$\delta \frac{dq_j}{dt} = \frac{d}{dt} \delta q_j.$$

Hence we have

$$\Delta\int_{t_0}^{t_1} Ldt = \int_{t_0}^{t_1}\sum_j\left(\dot{p}_j\delta q_j + p_j\frac{d}{dt}\delta q_j\right)dt + L(\Delta t)_{t_0}^{t_1}.$$

$$\Rightarrow \quad \Delta\int_{t_0}^{t_1} Ldt = \int_{t_0}^{t_1}\frac{d}{dt}\left[\sum_j\left(p_j\delta q_j\right)\right]dt + L(\Delta t)_{t_0}^{t_1}.$$

Since

$$\Delta = \delta + \Delta t\frac{d}{dt}$$

Hence above integral becomes

$$\Delta\int_{t_0}^{t_1} Ldt = \int_{t_0}^{t_1} d\left[\sum_j p_j\left(\Delta - \Delta t\frac{d}{dt}\right)q_j\right]dt + L(\Delta t)_{t_0}^{t_1}.$$

$$\Delta\int_{t_0}^{t_1} Ldt = \left[\sum_j p_j\Delta q_j\right]_{t_0}^{t_1} - \left[\sum_j p_j\dot{q}_j\Delta t\right]_{t_0}^{t_1} + L(\Delta t)_{t_0}^{t_1}$$

Since in Δ variation, position co-ordinates at the end points are fixed.

$$\Rightarrow \quad \left(\Delta q_j\right)_{t_0}^{t_1} = 0.$$

Consequently above equation reduces to

$$\Delta\int_{t_0}^{t_1} Ldt = -\left[\sum_j\left(p_j\dot{q}_j - L\right)\Delta t\right]_{t_0}^{t_1}$$

$$\Delta\int_{t_0}^{t_1} Ldt = -\left(H\Delta t\right)_{t_0}^{t_1}$$

Substituting this in equation (2) we get

$$\Delta A = 0,$$

$$i.e., \quad \Delta\int_{t_0}^{t_1}\sum_j p_j\dot{q}_j dt = 0.$$

Thus the system moves in space such that Δ-variation of the line integral of twice the kinetic energy is zero. This proves the principle of least action.

Example 23 : A system of two degrees of freedom is described by the Hamiltonian

$$H = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2, \quad a, b \ \ are \ \ const.$$

Show that i) $\dfrac{p_1 - a q_1}{q_2}$, $\quad ii)\dfrac{p_2 - b q_2}{q_1}$, $\quad iii)\ q_1 q_2 \quad iv)\ H$ are constant of motion.

Solution: The Hamiltonian of a dynamical system is given by

$$H = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2, \quad a, b \ \ are \ \ const. \qquad \text{. . . (1)}$$

where we see that q_1, q_2 are the generalized co-ordinates. The Hamilton's canonical equations of motion are

$$\begin{aligned} \dot{p}_j &= -\frac{\partial H}{\partial q_j} \Rightarrow \quad \dot{p}_1 = 2 a q_1 - p_1, \\ \dot{p}_2 &= p_2 - 2 b q_2, \end{aligned} \qquad \text{. . . (2)}$$

and

$$\begin{aligned} \dot{q}_j &= \frac{\partial H}{\partial p_j} \Rightarrow \quad \dot{q}_1 = q_1 \\ \dot{q}_2 &= -q_2. \end{aligned} \qquad \text{. . . (3)}$$

Now to show

1) $\quad \dfrac{p_1 - a q_1}{q_2}$ is a constant of motion, consider

$$\frac{d}{dt}\left(\frac{p_1 - a q_1}{q_2}\right) = \frac{q_2\left(\dot{p}_1 - a\dot{q}_1\right) - \left(p_1 - a q_1\right)\dot{q}_2}{q_2^2}.$$

Using equations (2) and (3) we obtain

$$\frac{d}{dt}\left(\frac{p_1 - a q_1}{q_2}\right) = 0 \Rightarrow \quad \frac{p_1 - a q_1}{q_2} = const.$$

Similarly we prove that

$$\frac{p_1 - a q_1}{q_2} = const., \quad \frac{p_2 - b q_2}{q_1} = const., \quad q_1 q_2 = const.$$

Now to prove the Hamiltonian H is also constant, we differentiate equation (1) with respect to t to get

$$\frac{dH}{dt} = \dot{q}_1 p_1 + q_1 \dot{p}_1 - \dot{q}_2 p_2 - q_2 \dot{p}_2 - 2aq_1\dot{q}_1 + 2bq_2\dot{q}_2 .$$

Using equations (2) and (3) we see that

$$\frac{dH}{dt} = 0 \Rightarrow \quad H = const..$$

This shows that H is a constant of motion.

Example 24 : A Lagrangian for a particle of charge q moving in the electromagnetic field of force is given by

$$L = \frac{1}{2}mv^2 + q\left(\bar{v}\cdot\bar{A}\right) - q\phi .$$

Find the Hamiltonian H, the generalized momenta.

Solution: The Lagrangian of a particle moving in the electromagnetic field is given by

$$L = \frac{1}{2}mv^2 + q\left(\bar{v}\cdot\bar{A}\right) - q\phi . \qquad \text{. . . (1)}$$

We write this expression as

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + q\left(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z\right) - q\phi . \qquad \text{. . . (2)}$$

where ϕ is a scalar potential function of co-ordinates only. We see that x, y, z are the generalized co-ordinates. Hence the corresponding generalized momenta become

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad \Rightarrow \quad p_x = m\dot{x} + qA_x ,$$

$$p_y = m\dot{y} + qA_y ,$$

$$p_z = m\dot{z} + qA_z .$$

Solving these equations for velocity components we get

$$\dot{x} = \frac{1}{m}\left(p_x - qA_x\right),$$

$$\dot{y} = \frac{1}{m}\left(p_y - qA_y\right), \qquad \ldots (3)$$

$$\dot{z} = \frac{1}{m}\left(p_z - qA_z\right).$$

The Hamiltonian of the particle is given by

$$H = \sum_j p_j \dot{q}_j - L,$$

$$H = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\left(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z\right) + q\phi \quad \ldots (4)$$

Eliminating $\dot{x}, \dot{y}, \dot{z}$ from equation (4) by using equation (3) we get

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) - \frac{q}{m}\left(p_x A_x + p_y A_y + p_z A_z\right) + \frac{1}{2m}q^2\left(A_x^2 + A_y^2 + A_z^2\right) + q\phi \quad . . (5)$$

This can be written in vector notions as

$$H = \frac{1}{2m}\left(\overline{p} - q\overline{A}\right)^2 + q\phi . \qquad \ldots (6)$$

This is the required Hamiltonian of the particle moving in the electromagnetic field.

The Hamilton's equation of motion $\dot{q}_j = \frac{\partial H}{\partial p_j}$ gives the same set of equations (3),

while the equation $\dot{p}_j = -\frac{\partial H}{\partial q_j}$ gives

$$\dot{p}_x = -\frac{\partial H}{\partial x} = \frac{q}{m}\frac{\partial}{\partial x}\left(p_x A_x + p_y A_y + p_z A_z\right) - \frac{q^2}{2m}\frac{\partial}{\partial x}\left(A_x^2 + A_y^2 + A_z^2\right) - q\frac{\partial \phi}{\partial x}.$$

This can be written as

$$\dot{p}_x = q\frac{\partial}{\partial x}\left(\overline{v}\cdot\overline{A}\right) - q\frac{\partial \phi}{\partial x}.$$

Similarly, other two components are given by

$$\dot{p}_y = q\frac{\partial}{\partial y}\left(\bar{v}\cdot\bar{A}\right) - q\frac{\partial \phi}{\partial y},$$

$$\dot{p}_z = q\frac{\partial}{\partial z}\left(\bar{v}\cdot\bar{A}\right) - q\frac{\partial \phi}{\partial z}.$$

All these three equations can be put in to the single equation as

$$\dot{\bar{p}} = -q\nabla\phi + q\nabla\left(\bar{v}\cdot\bar{A}\right). \qquad \ldots (7)$$

Exercise:

1. The Lagrangian of an anharmonic oscillator of unit mass is

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - \alpha x^3 + \beta x\dot{x}, \quad \alpha, \beta \text{ are constants.}$$

Find the Hamiltonian and the equation of motion. Show also that

(i) H is a constant of motion and

(ii) $H \neq T + V$.

Ans : $H = \frac{1}{2}\left(p_x - \beta x\right)^2 + \frac{1}{2}\omega^2 x^2 + \alpha x^3$.

Equation of motion $\ddot{x} + \omega^2 x + 3\alpha x^2 = 0$.

2. Find the Hamiltonian and the equations of motion for a particle constrained to move on the surface obtained by revolving the line $x = z$ about z axis. Does it represent the constant of motion and the constant of total energy?

Hint: Surface of revolution is a cone $x^2 + y^2 = z^2$

Ans.: $H = \frac{p_r^2}{4m} + \frac{p_\phi^2}{2mr^2} + mgr$.

$$\ddot{r} - \frac{p_\phi^2}{2m^2 r^3} + \frac{g}{2} = 0, \quad p_\phi = mr^2\dot{\phi} \quad - a\ const.\ of\ motion\ .$$

3. Let a particle be moving in a field of force given by

$$F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2r\ddot{r}}{c^2}\right).$$

Find the Hamiltonian H and show that it represents the constant of motion and also total energy.

Ans. : Refer Example (25) of Chapter I; the potential energy of the particle is given by

$$V = \frac{1}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right).$$

The Hamiltonian becomes $H = \dfrac{p_r^2}{2\left(m - \dfrac{2}{rc^2}\right)} + \dfrac{1}{r}$.

4. A sphere of radius 'a' and mass m rests on the top of a fixed trough sphere of radius 'b'. The first sphere is slightly displaced so that it rolls without slipping. Obtain the Hamiltonian of the system and hence the equation of motion. Also prove that H represents a constant of motion and also total energy.

Ans. : H = $\dfrac{7}{10} m (a+b)^2 \dot{\phi}^2 + mg(a+b)\cos\phi$.

5. A particle is constrained to move on the plane curve $xy = c$, c is a constant, under gravity. Obtain the Hamiltonian H and the equations of motion. Prove that the Hamiltonian H represents the constant of motion and total energy.

Ans. : Refer Example (20) of Chapter I for the Lagrangian L and is given by

$$L = \frac{1}{2} m\dot{x}^2\left(1 + \frac{c^2}{x^4}\right) - \frac{mgc}{x}.$$

The Hamiltonian H becomes $H = \dfrac{p_x^2}{2m\left(1 + \dfrac{c^2}{x^4}\right)} + \dfrac{mgc}{x}$.

6. A body of mass m is thrown up an inclined plane which is moving horizontally with constant velocity v. Use Hamilton's procedure to find the equations of motion. Prove that the Hamiltonian H represents the constant of motion but does not represent the total energy.

Ans. : For the Lagrangian function, refer Example (28) of Chapter I. The Hamiltonian of motion is

$$H = \frac{p_r^2}{2m} - p_r v\cos\theta + mgr\sin\theta - \left(\frac{1}{2}mv^2\sin^2\theta\right).$$

7. A particle moves on the surface characterized by

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = r\cot\theta.$$

Find the Hamiltonian H and prove that it represents the constant of motion and also the constant of total energy.

Ans. : $H = \dfrac{p_r^2\sin^2\theta}{2m} + \dfrac{p_\phi^2}{2mr^2} + mgr\cot\theta.$

The equation of motion is $\ddot{r} - r\sin^2\theta\dot{\phi}^2 + g\cos\theta\sin\theta = 0$.

8. Find the Hamiltonian and the Hamilton's canonical equations of motion for the Lagrangian given by

$$L\left(r,\dot{r},\theta,\dot{\theta}\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta - \frac{1}{2}k\left(r - r_0\right)^2,$$

where k, m, g, r_0 are constants.

Ans : $H = \dfrac{p_r^2}{2m} + \dfrac{p_\theta^2}{2mr^2} - mgr\cos\theta + \dfrac{1}{2}k\left(r - r_0\right)^2$

Equations of motion:

$$m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + k\left(r - r_0\right) = 0,$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\,\dot{\theta} + \frac{g}{r}\sin\theta = 0.$$