Cauvery College for Women (Autonomous)

Nationally Accredited (III Cycle) with 'A' Grade by NAAC

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Department	:	Mathematics
Programme	:	M.sc Mathematics
Batch	:	2018 Onwards
Semester	:	IV
Course	:	Advanced numerical analysis
Course Code	:	P16MA43
Unit	:	I
Topics Covered	:	Transcandental and polynomial

Equations-muller method, chebyshey method , polynomial equation.discrete rule of science-hydrative method-vieta method ,bairstow method ,graffe's root squaring method.

Bijection Method:

1. Find the $x^3-4x-9=0$ correct to the using bijection method.

Soln:

Let $f(x)=x^3-4x-9$

$f(0) = -9$ (-ve)

$f(1) = 1-4(1)-9 = -12$ (-ve)

$f(2) = 8-8-9 = -9$ (-ve)

$f(3) = 27-12-9 = 6$ (+ve)

$\therefore$ The root lies between 2 and 3.

n	a	b	$x_n = \frac{a+b}{2}$	$f(x)=x^3-4x-9$
1	2	3	2.5	-3.3750
2	2.5	3	2.75	0.7968
3	2.5	2.75	2.625	-1.412
4	2.625	2.75	2.6875	-0.3391
5	2.6875	2.75	2.71875	0.2209
6	2.6875	2.71875	2.70312	-0.0610
7	2.70312	2.71875	2.71094	0.079
8	2.70312	2.71094	2.70703	0.00902
9	2.70507	2.70703	2.70507	-0.06261
10	2.70507	2.70703	2.70605	-0.0085
11	2.70605	2.70703	2.70654	0.0002
12	2.70605	2.70654	2.70629	-0.0042
13	2.70629	2.70654	2.70641	-0.0021
14.	2.70641	2.70654	2.70647	-0.0010

$\therefore$ The root is 2.706.

H.W.

2. $x^3 - x - 1 = 0$

soln:

Let $f(x) = x^3 - x - 1$

$f(0) = 0 - 0 - 1 = -1$ (-ve)

$f(1) = 1 - 1 - 1 = -1$ (-ve)

$f(2) = 2^3 - 2 - 1 = 5$ (+ve)

$\therefore$ The root lies between 1 and 2.

n	a	b	$x_n = \frac{a+b}{2}$	$f(x) = x^3 - x - 1$
1	1	2	1.5	0.875
2	1	1.5	1.25	-0.2968
3	1.25	1.5	1.375	0.2246
4	1.25	1.375	1.3125	-0.0515
5	1.3125	1.375	1.3435	0.08150
6	1.3125	1.3435	1.328	0.0140
7	1.3125	1.328	1.3202	-0.0189
8	1.3202	1.328	1.3241	-0.0026
9	1.3241	1.328	1.3261	0.0059
10	1.3241	1.3261	1.3251	0.0016
11	1.3241	1.3251	1.3246	-0.0005
12	1.3246	1.3251	1.3248	0.0003
13	1.3246	1.3248	1.3247	-0.00007
14	1.3247	1.3248	1.3247	-0.00007
15	1.3247	1.3248	1.3247	-0.00007

$\therefore$ The root is 1.3247

2. $x^3 - x^2 + x - 7 = 0$

Soln:

Let $f(x) = x^3 - x^2 + x - 7$

$f(0) = 0 - 0 + 0 - 7 = -7$ (-ve)

$f(1) = 1 - 1 + 1 - 7 =$ (-ve)

$f(2) = 2^3 - 2^2 + 2 - 7 = 8 - 4 + 2 - 7 =$ (-ve)

$f(3) = 3^3 - 3^2 + 3 - 7 = 14$ (+ve)

$\therefore$ The root lies between 2 and 3.

n	a	b	$x_n = \frac{a+b}{2}$	$f(x) = x^3 - x^2 + x - 7$
1	2	3	2.5	4.875
2	2	2.5	2.25	1.5781
3	2	2.25	2.125	0.2050
4	2	2.125	2.0625	-0.4177
5	2.0625	2.125	2.0937	-0.1119
6	2.0937	2.125	2.1093	0.0447
7	2.0937	2.1093	2.1015	-0.0339
8	2.1015	2.1093	2.1054	0.0053
9	2.1015	2.1054	2.1035	-0.0138
10	2.1035	2.1054	2.1045	-0.0037
11	2.1045	2.1054	2.1049	0.0002
12	2.1045	2.1049	2.1047	-0.0017
13	2.1047	2.1049	2.1048	-0.0007
14	2.1048	2.1049	2.1048	-0.0007
15	2.1048	2.1049	2.1048	-0.0007

$\therefore$ The root is 2.1048

Regular Falsi Method:

1. Determine the root of $xe^x - 3 = 0$ correct to four decimal places by regular falsi method.

Soln:

Let $f(x) = xe^x - 3$

$f(0) = -3 (-ve)$

$f(1) = 1e^1 - 3 = -0.2817$ (-ve)

$f(2) = 2e^2 - 3 = 11.7781$ (+ve)

$\therefore$ The root lies between 1 & 2.

n	a	f(a)	b	f(b)	$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$	f(x)
1	1	-0.2817	2	11.7781	1.02335	-0.1525
2	1.02335	-0.1525	2	11.7781	1.0358	-0.0817
3	1.0358	-0.0817	2	11.7781	1.0424	-0.0487
4	1.0424	-0.0437	2	11.7781	1.0459	-0.0234
5	1.0459	-0.0234	2	11.7781	1.0477	-0.0129
6	1.0477	-0.0129	2	11.7781	1.0487	-0.0070
7	1.0487	-0.0070	2	11.7781	1.0492	-0.0041
8	1.0492	-0.0041	2	11.7781	1.0495	-0.0234

$\therefore$ The root is 1.049.

2. $x^2 - \log_e x - 12 = 0$

Soln:

Let $f(x) = x^2 - \log_e x - 12$

$f(0) = -12 (-ve)$

$f(1) = 1 - 0 - 12 = -11$ (-ve)

$f(2) = 4 - 0.6931 - 12 = -8.6931$ (-ve)

$f(3) = 9 - 1.0986 - 12 = -4.0986$ (-ve)

$f(4) = 16 - 1.3862 - 12 = 2.6138$ (+ve)

$\therefore$ The root lies between 3 & 4

$a = 3$ $b = 4$

$f(a) = -4.0986$ $f(b) = 2.6138$

$$x^{(1)} = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x^{(1)} = 3.6106$$

$$f(x)^{(1)} = -0.2474$$

$a = 3.6106$ $b = 4$

$f(a) = -0.2474$ $f(b) = 2.6138$

$$x^{(2)} = 3.6442$$

$$f(x)^{(2)} = -0.0129$$

$a = 3.6442$ $b = 4$

$f(a) = -0.0129$ $f(b) = 2.6138$

$$x^{(3)} = 3.6459$$

$$f(x)^{(3)} = -0.0010$$

$a = 3.6459$ $b = 4$

$f(a) = -0.0010$ $f(b) = 2.6138$

$$x^{(4)} = 3.6460$$

$\therefore$ The root is 3.6460

2. $x - \cos x = 0$

Soln:

Let $f(x) = x - \cos x$

$f(0) = 0 - 1 = -1$ (-ve)

$f(1) = 1 - \cos 1 = 0.4597$

$\therefore$ The roots lies between 0 and 1.

n	a	$f(a)$	b	$f(b)$	$x = \frac{af(b)-bf(a)}{f(b)-f(a)}$	$f(x)$
1	0	-1	1	0.4597	0.6850	-0.0894
2	0.6850	-0.0894	1	0.4597	0.7361	-0.0049
3	0.7361	-0.0049	1	0.4597	0.7387	0.5461
4	0.7361	-0.0049	0.7387	0.5461	0.7361	0.5455
5	0.7361	-0.0049	0.7361	0.5455	0.7360	0.5454
6	0.7361	-0.0049	0.7360	0.5454	0.7359	0.5454
7	0.7361	-0.0049	0.7359	0.5454	0.7359	0.5454

$\therefore$ The root is 0.7359

Newton's Raphson Method:

Formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad , n = 0, 1, 2, \ldots$$

1. Using Newton's Raphson method find the root between 0 + 1 and correct to four decimal places of the equation.

$$x^3 - 6x + 4 = 0$$

Soln:

Consider, $f(x) = x^3 - 6x + 4$

$f'(x) = 3x^2 - 6$

$f(0) = 4$ (+ve)

$f(1) = 1 - 6 + 4 = -1$ (-ve)

Then $f(0.5) = 1.125$

$f(0.6) = 0.616$ (+ve)

$f(0.7) = 0.143$ (+ve)

$f(0.8) = -0.288$ (-ve)

$\therefore x_0 = 0.7$

$f(x_0) = 0.143$

$f'(x_0) = -4.53$

1st approximation:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.7 + \frac{0.143}{4.53} = 0.7 + 0.0315$$

$$= 0.7315673$$

$$\boxed{x_1 = 0.73156}$$

$f(x_1) = 0.002124$

$f'(x_1) = -4.39442$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.73156 - \frac{0.002124}{-4.39442} = 0.73156 + 0.0048$$

$$\boxed{x_2 = 0.73204}$$

$f(x_2) = 0.000047$

$f'(x_2) = -4.39235$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.73204 - \frac{0.000047}{-4.39235}$$

$$= 0.73204 + 0.00001$$

$$= 0.73205$$

$$\boxed{x_3 = 0.7321}$$

The root correct to four decimal places 0.7321.

Secant Method:

If x_{k+1} & x_k are two approximations to the root then we determine a_0 & a_1,

$$x = -\frac{a_1}{a_0} \rightarrow (1)$$

by using the conditions $f_{k-1} = a_0 x_{k-1} + a_1$

$$f_k = a_0 x_k + a_1$$

where $f_{k-1} = f(x_{k-1})$ &

$$f_k = f(x_k)$$

on solving we obtain

$$a_0 = \frac{(f_k - f_{k-1})}{(x_k - x_{k-1})}$$

$$a_1 = \frac{(x_k f_{k-1} - x_{k-1} f_k)}{(x_k - x_{k-1})} \rightarrow (2)$$

From the eqns (1) & (2) the next approximation x_{k+1} to the root is given by

$$x_{k+1} = \frac{x_{k-1} f_k - x_k f_{k-1}}{f_k - f_{k-1}} \rightarrow (3)$$

which may also be written as

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})}{f_k - f_{k-1}} \rightarrow (4) \quad k = 1, 2, \ldots$$

This is called Secant (or) Chord method.

1. Find the root of the eqn $f(x) = x^3 - 2x - 5 = 0$ correct to three decimal places by secant method.

Soln:

$$f(x) = x^3 - 2x - 5$$

$$f(0) = -5 \text{ (-ve)}$$

$$f(1) = 1 - 2 - 5 = 1 - 7 = -6 \text{ (-ve)}$$

$f(2) = 8-4-5 = -1 (-ve)$

$f(3) = 27-6-5 = 16 (+ve)$

$\therefore$ The root lies between 2 & 3

$x_0 = 2$ & $x_1 = 3$

I	x_0	x_1	x_2	$f(x_0)$	$f(x_1)$	$f(x_2)$
1	2	3	2.059	-1	16	-0.3910
2	3	2.059	2.0814	16	-0.3910	-0.145
3	2.059	2.0814	2.095	0.381	-0.150	0.005
4	2.081	2.095	2.0942	-0.1457	0.0050	-0.0039

$\therefore$ The root is 2.095

H-W.

2. $x \sin x + \cos x = f(x)$

Soln:

$f(x) = x \sin x + \cos x$

$f(0) = 1 (+ve)$

$f(1) = 1.3817 (+ve)$

$f(2) = 1.4024 (+ve)$

$f(3) = -0.5666 (-ve)$

$\therefore$ The root lies between 2 and 3.

n	x_0	x_1	$f(x_0)$	$f(x_1)$	$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$	$f(x_2)$
1	2	3	1.4024	-0.5666	2.71224	0.21982
2	3	2.71224	-0.5666	0.2198	2.79266	0.01505
3	2.71224	2.79266	0.21982	0.01505	2.79857	-0.00048
4	2.79266	2.79857	0.01505	-0.00048	2.79838	0.000016
5	2.79857	2.79838	-0.000421	0.000016	2.79838	0.000016

$\therefore$ The root is 2.7984

Muller Method:

We assume for the function $f(x)$ a Polynomial of degree 2 and write as,

$$f(x) = a_0 x^2 + a_1 x + a_2 = 0, \quad a_0 \neq 0 \rightarrow (1)$$

where,

a_0, a_1, a_2 are arbitrary parameters to be determined by prescribing three appropriate Conditions on $f(x)$ and its derivatives.

If x_{k-2}, x_{k-1} and x_k are three approximations to the root ξ of $f(x)$ equal to zero.

Then, we determine a_0, a_1 and a_2 in (1) By using the Conditions,

$$\left.\begin{array}{l} \text{i) } f_{k-2} = a_0 x_{k-2}^2 + a_1 x_{k-2} + a_2 \\ \text{ii) } f_{k-1} = a_0 x_{k-1}^2 + a_1 x_{k-1} + a_2 \\ \text{iii) } f_k = a_0 x_k^2 + a_1 x_k + a_2 \end{array}\right\} \rightarrow (2)$$

Eliminating a_0, a_1 and a_2 from (1) and (2) we get,

$$\begin{vmatrix} f(x) & x^2 & x & 1 \\ f_{k-2} & x_{k-2}^2 & x_{k-2} & 1 \\ f_{k-1} & x_{k-1}^2 & x_{k-1} & 1 \\ f_k & x_k^2 & x_k & 1 \end{vmatrix} = 0 \rightarrow (3)$$

which may simplyfy and obtain,

$$f(x) = \frac{(x-x_{k-1})(x-x_k)}{(x_{k-2}-x_{k-1})(x_{k-2}-x_k)} f_{k-2} + \frac{(x-x_{k-2})(x-x_k)}{(x_{k-1}-x_{k-2})(x_{k-1}-x_k)} f_{k-1} + \frac{(x-x_{k-2})(x-x_{k-1})}{(x_k-x_{k-2})(x_k-x_{k-1})} f_k = 0 \longrightarrow ④$$

The equation ④ may also be written as,

$$\frac{h(h+h_k)}{h_{k-1}(h_{k-1}+h_k)} f_{k-2} - \frac{h(h+h_k+h_{k-1})}{h_k \, h_{k-1}} f_{k-1} + \frac{(h+h_k)(h+h_k+h_{k-1})}{h_k(h_k+h_{k-1})} f_k = 0 \longrightarrow ⑤$$

where,

$$h = x - x_k$$

$$h_k = x_k - x_{k-1}$$

$$h_{k-1} = x_{k-1} - x_{k-2}$$

Further define,

$$\lambda = \frac{h}{h_k} \; ; \; \lambda_k = \frac{h_k}{h_{k-1}} \text{ and}$$

$$f_k = 1 + \lambda_k$$

and express ⑤ in the form,

$$\lambda^2 c_k + \lambda g_k + \delta_k f_k = 0 \longrightarrow ⑥$$

$$g_k = \lambda_k^2 f_{k-2} - \delta_k^2 f_{k-1} + (\lambda_k + \delta_k) f_k$$

$$c_k = \lambda_k (\lambda_k f_{k-2} - f_{k-1} \delta_k + f_k)$$

So express (5) in the form,

$$\lambda^2 c_k + \lambda g_k + \delta_k f_k = 0 \longrightarrow ⑥$$

$$g_k = \lambda_k^2 f_{k-2} - \delta_k^2 f_{k-1} + (\lambda_k + \delta_k) f_k$$

$$c_k = \lambda_k (\lambda_k f_{k-2} - f_{k-1} \delta_k + f_k)$$

Solving (6) for λ, we obtain

$$\lambda = \frac{-g_k \pm \sqrt{g_k^2 - 4\delta_k f_k c_k}}{2c_k}$$

(or)

$$\lambda = \frac{2\delta_k f_k}{g_k \pm \sqrt{g_k^2 - 4\delta_k f_k c_k}} = \lambda_{k+1} \rightarrow (7)$$

The sign in the denominator in (7) is choosen such that,

λ_{k+1} has the smallest absolute value

then we have,

$$\lambda_{k+1} = \frac{h}{h_k} = \frac{x - x_k}{x_k - x_{k-1}}$$

(or)

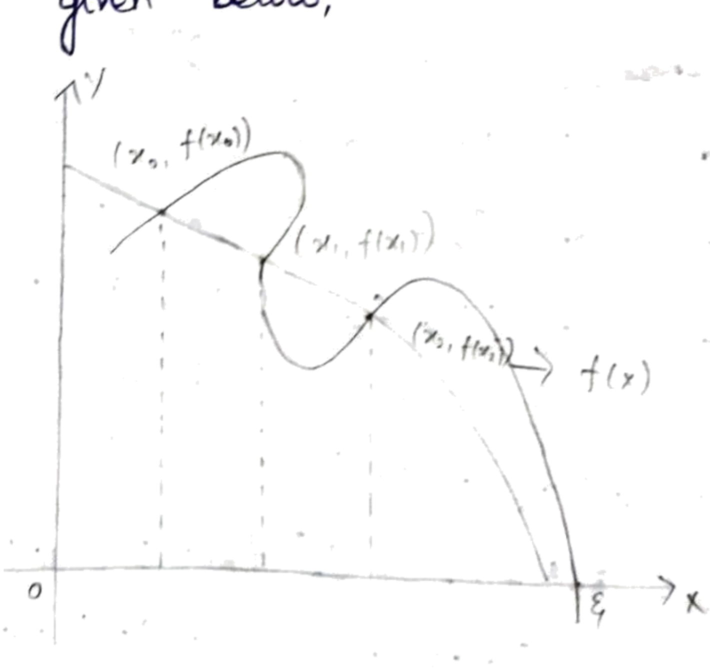
$$x = x_k + (x_k - x_{k-1})\lambda_{k+1} \rightarrow (8)$$

Replacing x on the LHS of eqn (8) by x_{k+1} we obtain the method,

$$x_{k+1} = x_k + (x_k - x_{k-1})\lambda_{k+1} \rightarrow (9)$$

which is called the Muller Method.

Now, the graphical representation of this method given below,



1. Solve $f(x) = x^3 - 3x - 5 = 0$ by using Muller's method.

soln:

Let $f(x) = x^3 - 3x - 5$

$f(0) = -5 = -ve$

$f(1) = 1-3-5 = -7 = -ve$

$f(2) = 8-6-5 = -3 = -ve$

$f(3) = 27-9 = 13$ (+ve)

Taking the initial approximations,

$x_0 = 1, \quad x_1 = 2, \quad x_2 = 3$

$f_0 = -7 \quad f_1 = -3 \quad f_2 = 13$

Formula,

$$x_{k+1} = x_k + (x_k - x_{k-1})\lambda_{k+1} \rightarrow (1), \quad k = 1, 2, \dots$$

put $k = 2$ in (1),

$$x_3 = x_2 + (x_2 - x_1)\lambda_3$$

$$= 3 + (3-2)\lambda_3$$

$$x_3 = 3 + \lambda_3 \rightarrow (2)$$

$$\lambda_{k+1} = -\frac{2\delta_k f_k}{g_k \pm \sqrt{g_k^2 - 4\delta_k f_k c_k}} \rightarrow (3)$$

$$c_k = \lambda_k [\lambda_k f_{k-2} - \delta_k f_{k-1} + f_k]$$

$$g_k = \lambda_k^2 f_{k-2} - \delta_k^2 f_{k-1} + f_k(\lambda_k + \delta_k)$$

$$\delta_k = 1 + \lambda_k \quad ; \quad \lambda_k = \frac{h_k}{h_{k-1}}$$

$$h_k = x_k - x_{k-1}$$

Put $k = 2$, $\delta_2 = 1 + \lambda_2$

$$= 1 + \frac{h_2}{h_1}$$

$$\delta_2 = 1 + \frac{(x_2 - x_1)}{(x_1 - x_2)}$$

$$= 1 + \frac{(3-2)}{2-1}$$

$$\boxed{\delta_2 = 2}$$

Sub in δ_K eqn,

$$\delta_2 = 1 + \lambda_2$$

$$2 = 1 + \lambda_2$$

$$\boxed{\lambda_2 = 1}$$

$$g_2 = \lambda_2^2 f_0 - \delta_2^2 f_1 + f_2 (\lambda_2 + \delta_2)$$

$$= 1(-7) - 4(-3) + f_2 (1+2)$$

$$= -7 + 12 + 13(3)$$

$$= -7 + 12 + 39$$

$$= -7 + 51$$

$$\boxed{g_2 = 44}$$

$$c_2 = \lambda_2 (\lambda_2 f_0 - \delta_2 f_1 + f_2)$$

$$= 1 (1(-7) - 2(-3) + 13)$$

$$= 1 (-7 + 6 + 13)$$

$$= -7 + 19$$

$$\boxed{c_2 = 12}$$

$$\Rightarrow \lambda_3 = \frac{-2\delta_2 f_2}{g_2 \pm \sqrt{g_2^2 - 4\delta_2 f_2 c_2}}$$

$$= \frac{-2(2)(13)}{44 \pm \sqrt{1936 - 1248}}$$

$$= \frac{-52}{44 \pm \sqrt{688}}$$

$$= \frac{-52}{44 + 26.229} \; ; \; \frac{-52}{44 - 26.229}$$

$$= \frac{-52}{70.229} \; ; \; \frac{-52}{17.771}$$

$\lambda_3 = -0.74043 \text{ (or) } -2.9261.$

$(2) \Rightarrow x_3 = 3 + \lambda_3$

$= 3 - 0.74043$

$= 2.25957$

$x_3 \simeq 2.26$

2^{nd} Iteration:

$x_0 = 2, \quad x_1 = 2.26, \quad x_2 = 3$

$f_0 = -3 \quad f_1 = -0.24, \quad f_2 = 13$

Put $K = 2$,

$x_3 = x_2 + (x_2 - x_1)\lambda_3$

$= 3 + (3 - 2.26)\lambda_3$

$x_3 = 3 + 0.74\lambda_3 \longrightarrow (4)$

$\delta_2 = 1 + \lambda_2 = 1 + \frac{h_2}{h_1} = 1 + \frac{(x_2 - x_1)}{(x_1 - x_0)}$

$= 1 + \frac{(3 - 2.26)}{(2.26 - 2)}$

$= 1 + \frac{0.74}{0.26}$

$= 1 + 2.846$

$\boxed{\delta_2 = 3.85}$

$\lambda_2 = \frac{x_2 - x_1}{x_1 - x_0} = \frac{3 - 2.26}{2.26 - 2}$

$= \frac{0.74}{0.26}$

$\boxed{\lambda_2 = 2.85}$

$g_2 = \lambda_2^2 f_0 - \delta_2^2 f_1 + f_2(\lambda_2 + \delta_2)$

$= (2.85)^2(-3) - (3.85)^2(-0.24) + (13)(2.85 + 3.85)$

$= -24.367 + 3.557 + 87.1$

$= 66.29$

$$C_2 = \lambda_2 [\lambda_2 f_0 - \delta_2 f_1 + f_2]$$

$$= 2.85 [2.85 (-3) - (3.85)(-0.24) + 13]$$

$$= 2.85 [-8.55 + 0.924 + 13]$$

$$= 2.85 [5.374]$$

$$C_2 = 15.32$$

$$\lambda_3 = \frac{-2 \delta_2 f_2}{g_2 \pm \sqrt{g_2^2 - 4 \delta_2 f_2 C_2}}$$

$$= \frac{-2(3.85)(13)}{66.29 \pm \sqrt{(66.29)^2 - 4(3.85)(13)(15.32)}}$$

$$= \frac{-100.1}{66.29 \pm \sqrt{4394.364 - 3067.06(1327.304)}}$$

$$= \frac{-100.1}{66.29 + 36.430} = \frac{-100.1}{66.29 - 36.430}$$

$\lambda_3 = -3.35$ and -0.97

$$x_3 = 3 + [0.74 \times 0.97]$$

$$\Rightarrow x_3 = 2.2822$$

$\therefore$ The root is 2.2822.

Chebyshev method:

We determine a_0, a_1 & a_2 using the conditions

$$f_K = a_0 x_K^2 + a_1 x_K + a_2$$

$$f_K' = 2a_0 x_K + a_1$$

$$f_K'' = 2a_0 \rightarrow (1)$$

on eliminating a_i's we obtain $f_K + (x - x_K) f_K' + \frac{1}{2}(x - x_K)^2 f_K'' = 0 \rightarrow (2)$

which is the Taylor series expansion of $f(x)$ about $x = x_K$ such that the terms of order $(x - x_K)^3$ & higher power are neglected.

The eqn (2) is a Quadratic eqn & can be solved easily only one of the 2 roots converges to the correct root. In order to get the next approximation to the correct root we write (2) as

$$x_{K+1} - x_K = -\frac{f_K}{f_K'} - \frac{1}{2}(x_{K+1} - x_K)^2 \frac{f_K''}{f_K'} \rightarrow (3)$$

We substitute for $(x_{K+1} - x_K)$ by $(-f_K/f_K')$ on the right side of (3) and obtain

$$x_{K+1} = x_K - \frac{f_K}{f_K'} - \frac{1}{2}\frac{f_K^2}{f_K'^3} f_K'' \rightarrow (4)$$

which is called the Chebyshev method. This method requires 3 evaluations for each iterations. If $(x_{K+1} - x_K)$ in the r.H.S of (3) is replaced by the secant or regula falsi method the order of the method is reduced.

Problem :

1. Solve $f(x) = x^3 - 4x - 9 = 0$ by using Chebyshev method.

Soln:

Let $f(x) = x^3 - 4x - 9$

$f'(x) = 3x^2 - 4$

$f''(x) = 6x$

$\Rightarrow f(0) = -9$ (-ve)

$f(1) = -12$ (-ve)

$f(2) = -9$ (-ve)

$f(3) = 6$ (+ve)

Let $x_0 = 3$ put $k=0$ in (1),

$f(3) = 6$

$f'(3) = 3(3)^2 - 4 = 23$

$f''(3) = 6(3) = 18$

$$x_1 = x_0 - \frac{f_0}{f_0'} - \frac{1}{2}\frac{f_0^2 f_0''}{(f_0')^3}$$

$$= 3 - \frac{6}{23} - \frac{1}{2}\frac{(6)^2(18)}{(23)^3}$$

$$x_1 = 2.7125$$

$$f(2.7125) = (2.7125)^3 - 4(2.7125) - 9$$

$$= 0.1076$$

Put $k=1$,

$$x_2 = x_1 - \frac{f_1}{f_1'} - \frac{1}{2}\frac{f_1^2 f_1''}{(f_1')^3}$$

$$= 2.7125 - \frac{0.1076}{18.073} - \frac{1}{2}\frac{(0.1076)^2(16.275)}{(18.073)^3}$$

$$= 2.7065$$

$f(2.7065) = (2.7065)^3 - 4(2.7065) - 9$

$= -0.0005$

Put $k = 2$

$$x_3 = x_2 - \frac{f_2}{f_2'} - \frac{1}{2} \cdot \frac{f_2^2 f_2''}{(f_2')^3}$$

$$= 2.7065 - \frac{(-0.0005)}{17.9754} - \frac{1}{2} \frac{(-0.0005)^2 (16.239)}{(17.9754)^3}$$

$$x_3 = 2.7065$$

I	x_k	x_{k+1}	f_{k+1}
0	3	2.7125	0.1076
1	2.7125	2.7065	-0.0005
2	2.7065	2.7065	-0.0005

$\therefore$ The root is 2.7065

2. Perform two iterations of the Chebyshev method to find an approximate value of $\frac{1}{7}$. Take the initial approximate as $x_0 = 0.1$

Soln:

Let $x = \frac{1}{7}$ we get $\frac{1}{x} = 7$

Define $f(x) = \frac{1}{x} - 7$ we get

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

using $x_0 = 0.1$ we get $f(x_0) = 3$

$f'(x_0) = -100$

$f''(x_0) = 2000$

$$x_1 = x_0 - \frac{f_0}{f_0'} - \frac{1}{2}\left(\frac{f_0^2 f_0''}{f_0'^3}\right)$$

$$= 0.1 - 0.03 - 0.5(0.0009)(-20)$$

$$= 0.139$$

$f(x_1) = 0.194245.$

$f'(x_1) = -51.757155$

$f''(x_1) = 744.707272$

$$x_2 = x_1 - \frac{f_1}{f_1'} - \frac{1}{2}\frac{f_1^2 f_2''}{(f_1')^3}$$

$$= 0.139 + 0.003753 - 0.5(0.000014)(-14.388489)$$

$$= 0.142854$$

Rate of Convergence:

Definition: Order P:

A-18 2m A-19 An iterative method is said to be of order P or has the rate of convergence P, if P is the largest positive real number for which there exists a finite constant $c \neq 0$.

Such that

$$|\epsilon_{k+1}| \leq c\,|\epsilon_k|^P \longrightarrow ①$$

where $\epsilon_k = x_k - \xi$ is the error in the k^{th} iterate

Defn: Asymptotic Constant: The constant c is called the asymptotic error constant and usually depends on derivatives of $f(x)$ at $x = \xi$. $x = \epsilon$

Secant Method:

We assume that ξ is a simple root of $f(x)=0$ substituting $x_k = \xi + \epsilon_k$ in

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k \quad , \quad k = 1, 2, \dots$$

we obtain,

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1}) f(\xi + \epsilon_k)}{f(\xi + \epsilon_k) - f(\xi + \epsilon_{k-1})}$$

Expanding $f(\xi + \epsilon_k)$ and $f(\xi + \epsilon_{k-1})$ in Taylor's series about the point ξ and noting that $f(\xi) = 0$ we get,

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1}) \left[f(\xi) + f'(\xi)\epsilon_k + \frac{\epsilon_k^2}{2!} f''(\xi) + \dots \right]}{\left[f(\xi) + f'(\xi)\epsilon_k + \frac{\epsilon_k^2}{2!} f''(\xi) + \dots \right] - \left[f(\xi) + f'(\xi)\epsilon_{k-1} + \frac{\epsilon_{k-1}^2}{2!} f''(\xi) + \dots \right]}$$

$$= \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1}) \left[\epsilon_k f'(\xi) + \frac{\epsilon_k^2}{2!} f''(\xi) \right]}{(\epsilon_k - \epsilon_{k-1}) f'(\xi) + \frac{1}{2}(\epsilon_k^2 - \epsilon_{k-1}^2) f''(\xi)}$$

$(\epsilon_k - \epsilon_{k-1})(\epsilon_k + \epsilon_{k-1})$

$$= \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1}) f'(\xi) \left[\epsilon_k + \frac{\epsilon_k^2}{2!} \frac{f''(\xi)}{f'(\xi)} + \dots \right]}{(\epsilon_k - \epsilon_{k-1}) f'(\xi) \left[1 + \frac{1}{2}(\epsilon_k + \epsilon_{k-1}) \frac{f''(\xi)}{f'(\xi)} + \dots \right]}$$

$$= \epsilon_k - \left[\epsilon_k + \frac{\epsilon_k^2}{2!} \frac{f''(\xi)}{f'(\xi)} + \dots \right] \left[1 + \frac{(\epsilon_k + \epsilon_{k-1})}{2} \frac{f''(\xi)}{f'(\xi)} + \dots \right]^{-1}$$

$$= \epsilon_k - \left[\epsilon_k + \frac{\epsilon_k^2}{2!} \frac{f''(\xi)}{f'(\xi)} + \dots \right] \left[1 - \left(\frac{\epsilon_k + \epsilon_{k-1}}{2} \right) \frac{f''(\xi)}{f'(\xi)} + \dots \right]$$

(or) $\epsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\epsilon_k \epsilon_{k-1} + O(\epsilon_k^2 \epsilon_{k-1} + \epsilon_k \epsilon_{k-1}^2) \rightarrow (3)$

$\epsilon_{k+1} = C\epsilon_k \epsilon_{k-1}$ (And higher powers of ϵ_k are neglected)

$C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$

The relation of the form (3) is called the error equation.

Suppose P is the order of this method

then $\epsilon_{k+1} = A\epsilon_k^P \rightarrow (4)$

where A & P are to determined from (4)

we have $\epsilon_k = A\epsilon_{k-1}^P$

(or) $\epsilon_{k-1} = A^{-1/P}\epsilon_k^{1/P}$

substitute these values of ϵ_{k+1} & ϵ_{k-1} in (3)

we obtain

$$A\epsilon_k^P = C\epsilon_k A^{-1/P}\epsilon_k^{+1/P}$$

$$\epsilon_k^P = CA^{-1/P}\epsilon_k^{1/P}\epsilon_k$$

$$\epsilon_k^P = CA^{-(1+1/P)}\epsilon_k^{1+1/P}$$

Comparing the power of ϵ_k on both sides we get

$$P = 1 + 1/P$$

$$P = \frac{P+1}{P} \Rightarrow P^2 = P+1$$

$$P^2 - P - 1 = 0$$

$$P = \frac{1}{2}(1 \pm \sqrt{5})$$

neglecting the '$-$' sign, we find the rate of convergence for the second method is $P = 1.618$

we also obtain $A = C^{\frac{P}{P+1}}$

Regula - Falsi Method :

If the function $f(x)$ in the equation $f(x)=0$ is convex in the interval (x_0, x_1) that contains the root, then one of the point x_0 or x_1 is always fixed and the other point varies with k. If the point x_0 is fixed. Then the function $f(x)$ is approximated by the straight line passing through the point.

(x_0, y_0) and (x_k, f_k) , $k = 1, 2, \dots$

The error equation ③ becomes.

$$\epsilon_{k+1} = C \epsilon_0 \epsilon_k$$

where $C = \frac{1}{2} \frac{f''(\xi)}{f'(\xi)}$ and $\epsilon_0 = x_0 - \xi$

is independent of k.

There fore, we can write

$$\epsilon_{k+1} = C^{*} \epsilon_k .$$

where $C^{*} = C \epsilon_0$ is the asymptotic error constant.

Hence the Regula - Falsi method has linear rate of covergence.

Newton - Raphson Method :

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Substitute, $x_k = \xi + \epsilon_k$ in $x_{k+1} = x_k - \frac{f_k}{f_k'}$ & expanding $f(\xi + \epsilon_k)$, $f'(\xi + \epsilon_k)$ in Taylor's series about the point ξ.

we obtain,

$$\xi + \epsilon_{k+1} = \xi + \epsilon_k - \frac{f(\xi + \epsilon_k)}{f'(\xi + \epsilon_k)}$$

$$\epsilon_{k+1} = \epsilon_k - \frac{\left[f(\xi) + \epsilon_k \cdot f'(\xi) + \frac{\epsilon_k^2}{2} f''(\xi) + \cdots\right]}{f'(\xi) + \epsilon_k f''(\xi) + \frac{\epsilon_k^2}{2} f'''(\xi) + \cdots}$$

$$= \epsilon_k - \frac{\cancel{f'(\xi)}\left[\epsilon_k + \frac{\epsilon_k^2}{2} \frac{f''(\xi)}{f'(\xi)} + \cdots\right]}{\cancel{f'(\xi)}\left[1 + \epsilon_k \frac{f''(\xi)}{f'(\xi)} + \frac{\epsilon_k^2}{2} \frac{f'''(\xi)}{f'(\xi)} + \cdots\right]}$$

$$= \epsilon_k - \left[\epsilon_k + \frac{\epsilon_k^2}{2} \frac{f''(\xi)}{f'(\xi)} + \cdots\right] \left[1 + \epsilon_k \frac{f''(\xi)}{f'(\xi)} + \cdots\right]^{-1}$$

$$= \epsilon_k - \left(\epsilon_k + \frac{\epsilon_k^2}{2} \frac{f''(\xi)}{f'(\xi)} + \cdots\right) \left[1 - \epsilon_k \frac{f''(\xi)}{f'(\xi)} + \cdots\right]$$

$$= \epsilon_k - \left[\epsilon_k - \epsilon_k^2 \frac{f''(\xi)}{f'(\xi)} + \cdots + \frac{\epsilon_k^2}{2} \frac{f''(\xi)}{f'(\xi)} - \frac{\epsilon_k^3}{2} \cdots\right]$$

$$= \epsilon_k^2 \frac{f''(\xi)}{f'(\xi)} - \frac{\epsilon_k^2}{2} \frac{f''(\xi)}{f'(\xi)} + o(\epsilon_k^3)$$

$$= \frac{1}{2} \epsilon_k^2 \frac{f''(\xi)}{f'(\xi)} + o(\epsilon_k^3)$$

$$\epsilon_{k+1} = C \epsilon_k^2$$

where $C = \frac{1}{2} \frac{f''(\xi)}{f'(\xi)}$

The Newton-Raphson method has the second order convergence (or) quadratic convergence.

Muller Method:

We assume for the function $f(x)$ a poly of degree two and write as,

$$f(x) = a_0 x^2 + a_1 x + a_2 = 0, \quad a_0 \neq 0 \rightarrow (1)$$

Where a_0, a_1 and a_2 are arbitrary parameters to be determined by prescribing 3 appropriate condition on $f(x)$ and in derivatives.

If x_{k-2}, x_{k-1} and x_k are 3 approximation to the root $f(x)=0$ then we may determine a_0, a_1 and a_2 in (1).

$$\left.\begin{aligned} f_{k-2} &= a_0 x_{k-2}^2 + a_1 x_{k-2} + a_2 \\ f_{k-1} &= a_0 x_{k-1}^2 + a_1 x_{k-1} + a_2 \\ f_k &= a_0 x_k^2 + a_1 x_k + a_2 \end{aligned}\right\} \rightarrow (2)$$

Eliminating a_0, a_1 and a_2 from (1) and (2),

$$\begin{vmatrix} f(x) & x^2 & x & 1 \\ f_{k-2} & x_{k-2}^2 & x_{k-2} & 1 \\ f_{k-1} & x_{k-1}^2 & x_{k-1} & 1 \\ f_k & x_k^2 & x_k & 1 \end{vmatrix} = 0 \rightarrow (3)$$

$$f(x) = \frac{(x-x_{k-1})(x-x_k)}{(x_{k-2}-x_{k-1})(x_{k-2}-x_k)} f_{k-2} + \frac{(x-x_{k-2})(x-x_k)}{(x_{k-1}-x_{k-2})(x_{k-1}-x_k)} f_{k-1} + \frac{(x-x_{k-2})(x-x_{k-1})}{(x_k-x_{k-2})(x_k-x_{k-1})} f_k = 0 \rightarrow (4)$$

eqn (4) may also be written as

$$\frac{h(h+h_k)}{h_{k-1}(h_{k-1}+h_k)} f_{k-2} - \frac{h(h+h_k+h_{k-1})}{h_k h_{k-1}} f_{k-1} + \frac{(h+h_k)(h+h_k+h_{k-1})}{h_k(h_k+h_{k-1})} f_k = 0 \rightarrow (5)$$

where, $h = x - x_k$

$h_k = x_k - x_{k-1}$

Define

$\lambda = h/h_k$

$\lambda_k = \frac{h_k}{h_{k-1}}$

$h_{k-1} = x_{k-1} - x_{k-2}$ $\qquad \delta_k = 1 + \lambda_k$

Alternative Method:

We assume for $f(x)$ a polynomial of degree 2 in the form $f(x) = a_0(x - x_k)^2 + a_1(x - x_k) + a_2 = 0$, $a_0 \neq 0 \rightarrow (1)$

Substituting $x = x_k, x_{k-1}, x_{k-2}$

Determine a_0, a_1, a_2 from the equation

$$\left.\begin{aligned} f_k &= a_2 \\ f_{k-1} &= a_0(x_{k-1} - x_k)^2 + a_1(x_{k-1} - x_k) + a_2 \\ f_{k-2} &= a_0(x_{k-2} - x_k)^2 + a_1(x_{k-2} - x_k) + a_2 \end{aligned}\right\} \rightarrow (1)(a)$$

We obtain $a_2 = f_k$

$$\left.\begin{aligned} a_1 &= \frac{1}{D}\left[(x_k - x_{k-2})^2(f_k - f_{k-1}) - (x_k - x_{k-1})^2(f_k - f_{k-2})\right] \\ a_0 &= \frac{1}{D}\left[(x_k - x_{k-2})(f_k - f_{k-1}) - (x_k - x_{k-1})(f_k - f_{k-2})\right] \end{aligned}\right\} \rightarrow (2)$$

where $D = (x_{k-1} - x_k)^2(x_{k-2} - x_k) - (x_{k-2} - x_k)^2(x_{k-1} - x_k)$

$= (x_{k-1} - x_k)(x_{k-2} - x_k)\left[x_{k-1} - \cancel{x_k} - x_{k-2} + \cancel{x_k}\right]$

$= (x_k - x_{k-1})(x_k - x_{k-2})(x_{k-1} - x_{k-2})$

Solving the equation (1) for $x - x_k$ and replace x by x_{k+1} (X)

We obtain,

$$x_{k+1} = x_k + \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_0}$$

$$= x_K - \frac{2a_2}{a_1 \pm \sqrt{a_1^2 - 4a_0a_2}} \qquad K, 2, 3 \ldots \longrightarrow (3)$$

The sign in the denominator in eqn (3) is choosen as that of a_1 so that the denominator has the maximum absolute value.

ie) x_K changes by a smaller value.

On substituting $x_j = \xi + \epsilon_j$, $j = K-2, K-1, K$ & expand $f(\xi + \epsilon_j)$ in Taylor's series about the point ξ in (3) & using $f(\xi) = 0$.

We get,

$$D = (\epsilon_K - \epsilon_{K-2})(\epsilon_K - \epsilon_{K-1})(\epsilon_{K-1} - \epsilon_{K-2})$$

$$a_2 = \epsilon_K f'(\xi) + \frac{1}{2}\epsilon_K^2 f''(\xi) + \frac{1}{6}\epsilon_K^3 f'''(\xi) + \ldots$$

$$a_1 = \frac{1}{D}\Big[(\epsilon_K - \epsilon_{K-2})^2 \Big\{(\epsilon_K - \epsilon_{K-1}) f'(\xi) + \frac{1}{2}(\epsilon_K^2 - \epsilon_{K-1}^2) f''(\xi)$$

$$+ \tfrac{1}{6}(\epsilon_K^3 - \epsilon_{K-1}^3) f'''(\xi) + \ldots\Big\}$$

$$- (\epsilon_K - \epsilon_{K-1})^2 \Big\{(\epsilon_K - \epsilon_{K-2}) f'(\xi) +$$

$$\tfrac{1}{2}(\epsilon_K^2 - \epsilon_{K-2}^2) f''(\xi) + \tfrac{1}{6}(\epsilon_K^3 - \epsilon_{K-2}^3) f'''(\xi) + \ldots\Big\}\Big]$$

$$= \frac{1}{D}\Big[(\epsilon_K - \epsilon_{K-2})(\epsilon_K - \epsilon_{K-1})(\epsilon_{K-1} - \epsilon_{K-2}) f'(\xi)$$

$$+ \frac{1}{2}(\epsilon_K - \epsilon_{K-2})(\epsilon_K - \epsilon_{K-1})\Big\{(\epsilon_K - \epsilon_{K-2})(\epsilon_K + \epsilon_{K-1})$$

$$- (\epsilon_K - \epsilon_{K-1})(\epsilon_K + \epsilon_{K-2})\Big\} f''(\xi) + \frac{1}{6}(\epsilon_K - \epsilon_{K-2}) \times$$

$$(\epsilon_K - \epsilon_{K-1})\Big\{(\epsilon_K - \epsilon_{K-2})(\epsilon_K^2 + \epsilon_K\epsilon_{K-1} + \epsilon_{K-1}^2)$$

$$- (\epsilon_K - \epsilon_{K-1})(\epsilon_K^2 + \epsilon_K\epsilon_{K-2} + \epsilon_{K-2}^2)\Big\} f'''(\xi) + \ldots\Big]$$

$$a_0 = \frac{1}{D}\Big[(\epsilon_k - \epsilon_{k-2})\{(\epsilon_k - \epsilon_{k-1})f'(\xi) + \frac{1}{2}(\epsilon_k^2 - \epsilon_{k-1}^2)f''(\xi) + \frac{1}{6}(\epsilon_k^3 - \epsilon_{k-1}^3)f'''(\xi) + \dots\} - (\epsilon_k - \epsilon_{k-1})\{(\epsilon_k - \epsilon_{k-2}) f'(\xi) + \frac{1}{2}(\epsilon_k^2 - \epsilon_{k-2}^2)f''(\xi) + \frac{1}{6}(\epsilon_k^3 - \epsilon_{k-2}^3) f'''(\xi) + \dots\}\Big]$$

$$= \frac{1}{D}\Big[\frac{1}{2}(\epsilon_k - \epsilon_{k-2})(\epsilon_k - \epsilon_{k-1})(\epsilon_{k-1} - \epsilon_{k-2})f''(\xi) + \frac{1}{6}(\epsilon_k - \epsilon_{k-2})(\epsilon_k - \epsilon_{k-1})\{\epsilon_k(\epsilon_{k-1} - \epsilon_{k-2}) + (\epsilon_{k-1}^2 - \epsilon_{k-2}^2)\} f'''(\xi) + \dots\Big]$$

$$= \frac{1}{2}f''(\xi) + \frac{1}{6}(\epsilon_k + \epsilon_{k-1} + \epsilon_{k-2})f'''(\xi) + \dots$$

$$a_1^2 - 4a_0a_2 = [f'(\xi)]^2 + 2\epsilon_k f'(\xi)f''(\xi) + \epsilon_k^2[f''(\xi)]^2 + \frac{1}{3}[2\epsilon_k^2 + \epsilon_k\epsilon_{k-1} + \epsilon_k\epsilon_{k-2} - \epsilon_{k-1}\epsilon_{k-2}] f'(\xi)f''(\xi) + \dots - 4\Big[\epsilon_k f'(\xi) + \frac{1}{2}\epsilon_k^2 f''(\xi) + \frac{1}{6}\epsilon_k^3 f'''(\xi) + \dots\Big] \times \Big[\frac{1}{2}f''(\xi) + \frac{1}{6}(\epsilon_k + \epsilon_{k-1} + \epsilon_{k-2})f'''(\xi) + \dots\Big]$$

$$= [f'(\xi)]^2 - \frac{1}{3}(\epsilon_k\epsilon_{k-1} + \epsilon_k\epsilon_{k-2} + \epsilon_{k-1}\epsilon_{k-2}) f'(\xi)f''(\xi) + \dots$$

$$\sqrt{a_1^2 - 4a_0a_2} = f'(\xi)\Big[1 - \frac{1}{6}(\epsilon_k\epsilon_{k-1} + \epsilon_k\epsilon_{k-2} + \epsilon_{k-1}\epsilon_{k-2})c_3 + \dots\Big]$$

Where $c_i = \dfrac{f^{(i)}(\xi)}{f'(\xi)}$, $i = 1, 2, \dots$

$$a_1 + \sqrt{a_1^2 - 4a_0a_2} = 2f'(\xi)\Big[1 + \tfrac{1}{2}\epsilon_k c_2 + \tfrac{1}{6}(\epsilon_k^2 - \epsilon_{k-1}\epsilon_{k-2})c_3 + \dots\Big]$$

Hence we obtain

$$x_{k+1} = x_k - \frac{2a_2}{a_1 \pm \sqrt{a_1^2 - 4a_0a_2}} \qquad k, 2, 3, \dots$$

$$\epsilon_{k+1} = \epsilon_k - \left[\epsilon_k + \frac{1}{2}\epsilon_k^2 c_2 + \frac{1}{6}\epsilon_k^3 c_3 + \dots\right] \times \left[1 + \left\{\frac{1}{2}\epsilon_k c_2 + \frac{1}{6}\left(\epsilon_k^2 - \epsilon_{k-1}\epsilon_{k-2}\right)c_3 + \dots\right\}\right]^{-1}$$

$$= \epsilon_k - \left[\epsilon_k + \frac{1}{2}\epsilon_k^2 c_2 + \frac{1}{6}\epsilon_k^3 c_3 + \dots\right] \left[1 - \frac{1}{2}\epsilon_k c_2 + \frac{1}{4}\epsilon_k^2 c_2^2 - \frac{1}{6}\left(\epsilon_k^2 - \epsilon_{k-1}\epsilon_{k-2}\right)c_3 + \dots\right]$$

$$= \frac{1}{6}\epsilon_k \epsilon_{k-1} \epsilon_{k-2} c_3 + \dots$$

$\therefore$ The error equation associated with the Muller method is given by

$$\epsilon_{k+1} = c\,\epsilon_{k-2}\,\epsilon_{k-1}\,\epsilon_k \longrightarrow (4)$$

where $c = \frac{1}{6} c_3 = \frac{1}{6} \frac{f'''(\xi)}{f'(\xi)} \longrightarrow (5)$

A relation of the form

$$\epsilon_{k+1} = A\,\epsilon_k^P \longrightarrow (6)$$ A, P are to be determined

From 6, $\epsilon_k = A\,\epsilon_{k-1}^P$

$\epsilon_{k-1} = A^{-1/P}\,\epsilon_k^{1/P}$

$\epsilon_{k-1} = A\,\epsilon_{k-1}^{1/P}$

$\epsilon_{k-2} = A^{-1/P}\,\epsilon_{k-1}^{1/P}$

$= A^{-(1/P + 1/P^2)}\,\epsilon_k^{1/P^2}$

Sustituting the values of $\epsilon_{k+1}, \epsilon_{k-1}, \epsilon_{k-2}$ in (6)

we get $\epsilon_k^P = cA^{-(1+\frac{2}{P}+\frac{1}{P^2})}\,\epsilon_k^{1+\frac{1}{P}+\frac{1}{P^2}} \longrightarrow (7)$

Comparing the power of ϵ_k on both sides we obtain $P = 1 + \frac{1}{P} + \frac{1}{P^2}$

$$F(P) = P^3 - P^2 - P - 1 = 0 \longrightarrow (8)$$

The eqn $F(P)=0$ has the smallest +ve root in $(1,2)$. We use the Newton-Raphson method to determine this root. We have

$$P_{K+1} = P_K - \frac{F(P_K)}{F'(P_K)}$$

$$= P_K - \frac{P_K^3 - P_K^2 - P_{K-1}}{3P_K^2 - 2P_{K-1}}$$

(or) $$P_{K+1} = \frac{2P_K^3 - P_K^2 + 1}{3P_K^2 - 2P_{K-1}} \quad K = 0, 1$$

Starting with $P_0 = 2$

$P_1 = 1.8571$ $\quad P_3 = 1.8393$

$P_2 = 1.8395$

∴ The root of the eqn 2.45 is $P = 1.84$

The rate of convergence of the muller method is From ①

$$C A^{-\left(1 + \frac{2}{P} + \frac{1}{P^2}\right)} = 1$$

(or)

$$A = C^{P^2/P^2+P+1} = C^{0+2}$$

where C is given by 2.42.

Chebyshev method :

Such $x_K = \xi + \epsilon_K$ & expanding $f(x_K), f'(x_K), f''(x_K)$ about the point ξ in the chebyshev method.

$$x_{K+1} = x_K - \frac{f(x_K)}{f'(x_K)} - \frac{1}{2}\left[\frac{f(x_K)}{f'(x_K)}\right]^2 \left[\frac{f''(x_K)}{f'(x_K)}\right] \quad ①$$

we obtain

$$\frac{f(x_K)}{f'(x_K)} = \frac{f(\xi + \epsilon_K)}{f'(\xi + \epsilon_K)}$$

$$= \frac{\epsilon_K f'(\xi) + \frac{1}{2}\epsilon_K^2 f''(\xi) + \frac{1}{6}\epsilon_K^3 f'''(\xi) + \dots}{f'(\xi) + \epsilon_K f''(\xi) + \frac{1}{2}\epsilon_K^2 f'''(\xi) + \dots}$$

$$= \left[\epsilon_K + \frac{1}{2} c_2 \epsilon_K^2 + \frac{1}{6} c_3 \epsilon_K^3 + \dots\right] \times$$
$$\left[1 + (c_2 \epsilon_K + \tfrac{1}{2} c_3 \epsilon_K^2 + \dots)\right]^{-1}$$

$$= \left[\epsilon_K + \frac{1}{2} c_2 \epsilon_K^2 + \frac{1}{6} c_3 \epsilon_K^3 + \dots\right] \times$$
$$\left[1 - c_2 \epsilon_K + (c_2^2 - \tfrac{1}{2} c_3)\epsilon_K^2 + \dots\right]$$

$$= \epsilon_K - \frac{1}{2} c_2 \epsilon_K^2 + \left(\frac{1}{2} c_2^2 - \frac{1}{3} c_3\right)\epsilon_K^3 + \dots$$

where $c_i = \dfrac{f^{(i)}(\xi)}{f'(\xi)}$ $i = 2, 3, \dots$

we get

$$\left(\frac{f(x_K)}{f'(x_K)}\right)^2 = \epsilon_K^2 - c_2 \epsilon_K^3 + \dots$$

$$\frac{f''(x_K)}{f'(x_K)} = \frac{f''(\xi + \epsilon_K)}{f'(\xi + \epsilon_K)} = \frac{f''(\xi) + \epsilon_K f'''(\xi) + \dots}{f'(\xi) + \epsilon_K f''(\xi) + \dots}$$

$$= \frac{f''(\xi)}{f'(\xi)}\left[1 + \frac{c_3}{c_2}\epsilon_K + \dots\right]\left[1 + (c_2 \epsilon_K + \dots)\right]^{-1}$$

$$= c_2 \left[1 + \frac{c_3}{c_2}\epsilon_K + \dots\right]\left[1 - c_2 \epsilon_K + \dots\right]$$

$$= c_2 + (c_3 - c_2)^2 \epsilon_K + \dots$$

Sub in ① we obtain the error eqn

$$\epsilon_{K+1} = \epsilon_K - \left[\epsilon_K - \tfrac{1}{2} c_2 \epsilon_K^2 + (\tfrac{1}{2} c_2^2 - \tfrac{1}{3} c_3)\epsilon_{K+1}^3\right.$$
$$\left. - \tfrac{1}{2}\left[\epsilon_K^2 - c_2 \epsilon_K^3 + \dots\right]\left[c_2 + (c_3 - c_2)^2 \epsilon_K + \dots\right]\right]$$

$$= \left[-\left(\tfrac{1}{2}C_2^2 - \tfrac{1}{3}C_3\right) - \tfrac{1}{2}\left\{(C_3-C_2)^2 - C_2^2\right\}\right]\epsilon_k^3 + O(\epsilon_k^4)$$

$$= \left(\tfrac{1}{2}C_2^2 - \tfrac{1}{6}C_3\right)\epsilon_k^3 + O(\epsilon_k^4)$$

Hence the rate of convergence of the chebyshev method (1) is 3.

Polynomial equation:

The root of a real polynomial equation of degree n

$$P_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0, \quad a_0 \neq 0 \rightarrow (1)$$

where $a_0, a_1, \dots a_n$ are real Numbers.

Definition:

When a polynomial a (+ve) sign follows a (+ve) sign and a (-ve) sign follows a (-ve) sign a continuous or permanent of signs is said to occur.

But if the positive sign follows a (-ve) sign and negative sign follows a (+ve) sign.

Ex:

$$P_5(x) = 8x^5 + 12x^4 - 10x^3 + 17x^2 - 18x + 5 = 0$$

Descarte's Rule of signs:

The number of positive real roots of $P_n(x) = 0$ can not exceed the number of sign changes in $P_n(x)$.

And the number of Negative real roots of $P_n(x) = 0$ can not exceed the number of sign changes in $P_n(-x)$.

eqn ② has maximum of 4 positive roots and 1 negative root.

The exact number of real roots of the polynomial can be found by Sturm's thm.

Let $f(x)$ be a given polynomial of degree n and let $f_1(x)$ represent its first order derivatives, denote by $f_2(x)$ is the remainder of $f_0(x)$ divided by $f_1(x)$ taken with the reverse sign and $f_3(x)$ is the remainder of $f_1(x)$ divided by $f_2(x)$ with the reverse of sign and so on. And till a constant is arrived at.

Thus obtain a sequence of function $f(x), f_1(x), \dots f_n(x)$ are called Sturm's function (or) Sturm's sequence.

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Sturm's theorem:

The number of real roots of the equation $f(x) = 0$ ~~(or)~~ on $[a, b]$ equals the difference between the number of the changes of sign in the Sturm's sequence at $x = a$ & $x = b$ provided that $f(a) \neq 0$ & $f(b) \neq 0$

Proof:

If $f(x) = 0$ has a multiple root.

Before obtain the Sturm's sequence.

$f(x), f_1(x), \dots f_l(x)$, where $f_{l-1}(x)$ is exactly divisible by $f_l(x)$.

In this case $f_l(x)$ will not be a Constant. Since $f_l(x)$ gives the greatest common divisor of $f(x)$ and $f'(x)$ the multiplicity of the root of $f(x)=0$ is one more than that of $f_l(x)$.

We obtain a new Sturm's sequence by dividing all the function by

$$f(x), f_1(x) \ldots f_l(x) \text{ by } f_l(x)$$

Using this sequence we can find the number real roots of the equation

$$f(x)=0 \text{ on } [a,b]$$

In the same way without taking multiplicity

Since a polynomial of degree n has exactly n roots.

The number of complex root equal to n. where a root of multiplicity is to be counted.

If $\xi_1, \xi_2, \ldots \xi_n$ are real and distinct root of eqn ①.

$$(x-\xi_n)=0.$$

$$P_n(x) = a_0(x-\xi_1)(x-\xi_2)\ldots$$

Assume that $\xi_1, \xi_2, \ldots \xi_n$ are real and distinct roots with multiplicity $\nu_1, \nu_2, \ldots \nu_s$ then the eqn ① takes of a form

$$P_n(x) = a_0(x-\xi_1)(x-\xi_2)\ldots(x-\xi_n)=0$$

where $\nu_1+\nu_2+\ldots+\nu_s=n$.

If either ξ_1 and ξ_2 are the complex pair or ξ_1 and ξ_2 real and multiplied together and all other roots ξ_i, $i = 3, 4, \ldots n$ are real and distinct then eqn ① becomes,

$$P_n(x) = a_0(x^2 + px + q)(x - \xi_3) \ldots (x - \xi_n)$$

1. Find the no. of real and complex roots of the polynomial equations.

1) $P_3(x) = x^3 - 5x + 1 = 0$

2) $P_4(x) = 4x^4 + 2x^2 - 1 = 0$

1) soln:

we consider

$f(x) = x^3 - 5x + 1$

$f_1(x) = f'(x) = 3x^2 - 5$

$$\frac{f(x)}{f'(x)} = \frac{x^3 - 5x + 1}{3x^2 - 5}$$

$$= \frac{x}{3} + \frac{\left(-\frac{10x}{3} + 1\right)}{3x^2 - 5}$$

$$3x^2 - 5 \,\big)\, x^3 - 5x + 1 \,\big(\, \tfrac{1}{3}x$$
$$x^3 - \tfrac{5}{3}x$$
$$(-)\quad(+)$$
$$-5x + \tfrac{5}{3}x + 1$$
$$\frac{-15x + 5x}{3} + 1$$
$$= -\frac{10x}{3} + 1$$

$$= -\left[\text{remainder of } \frac{f(x)}{f'(x)}\right]$$

$$= -\left[-\frac{10x}{3} + 1\right] = \frac{10x}{3} - 1$$

$f_2(x) = 10x - 3$

$$\frac{f_1(x)}{f_2(x)} = \frac{3x^2 - 5}{10x - 3}$$

$$= \frac{3x}{10} + \frac{\left(\frac{9x}{10} - 5\right)}{10x - 3}$$

$$f_3(x) = -\left[\text{remainder of } \frac{f_1(x)}{f_2(x)}\right]$$

$$= -\left[\frac{9x}{10} - 5\right] = -\frac{9x}{10} + 3$$

$$f_3(x) = -9x + 50$$

$$\frac{f_2(x)}{f_3(x)} = \frac{10x - 3}{-9x + 50}$$

$-10/9$

$-9x+50 \,)\, 10x - 3$
$10x - \frac{500}{9}$

$-3 + \frac{500}{9} = \frac{-27+500}{9}$

$= \frac{473}{9}$

$$f_4(x) = -\left[\text{remainder of } \frac{f_2(x)}{f_3(x)}\right]$$

$$= -\left[\frac{473}{9}\right]$$

Let $V(x)$ denote the no. of sign changes in the Sturm's ~~sequ~~ sequence at $x = a$. We construct the following table of sign changes in $V(a)$.

x	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$V(x)$
$-\infty$	$-$	$+$	$-$	$+$		~~3~~ 4
-3	$-$	$+$	$-$	$+$		3 4
-2	$+$	$+$	$-$	$+$		~~2~~ 3
0	$+$	$-$	$-$	$+$		~~2~~ 3
1	$-$	$-$	$+$	$+$		~~1~~ 2
2	$-$	$+$	$+$	$+$		~~1~~ 2
3	$+$	$+$	$+$	$+$		~~0~~ 1
∞	$+$	$+$	$+$	$+$		0

We find that the polynomial has 3 real roots. The root lie in the intervals $(-3, -2), (0, 1), (2, 3)$.

The polynomial has no complex roots.

ii) Soln:

$$f(x) = 4x^4 + 2x^2 - 1$$

$$f_1(x) = f'(x) = 16x^3 + 4x$$

$$f'(x) = 4x^3 + x$$

$$\frac{f(x)}{f_1(x)} = \frac{4x^4+2x^2-1}{4x^3+x}$$

$$= x + \frac{x^2-1}{4x^3+x}$$

$$f_2(x) = -\left[\text{remainder of } \frac{f(x)}{f_1(x)}\right]$$

$$= -(x^2-1)$$

$$f_2(x) = -x^2+1$$

$$\frac{f_1(x)}{f_2(x)} = \frac{4x^3+x}{-x^2+1} = -4x + \frac{5x}{-x^2+1}$$

$$-x^2+1 \overset{-4x}{\overline{\big) 4x^3+x}}$$
$$-4x^3+4x$$
$$5x$$

$$f_3(x) = -\left[\text{remainder of } \frac{f_1(x)}{f_2(x)}\right] = -5x$$

$$f_3(x) = -x$$

$$\frac{f_2(x)}{f_3(x)} = \frac{-x^2+1}{-x} = x + \frac{1}{-x}$$

$$f_4(x) = -1$$

x	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$V(x)$
$-\infty$	+	−	−	+	−	3
−1	+	−	−	+	−	3
0	−	0(−)	+	0(+)	−	2
1	+	+	0(+)	−	−	1
∞	+	+	~~+~~ −	−	−	1

If any element in Sturm's sequence becomes zero for some value of x, we give to it the sign of the immediate preceeding element then, we find that the polynomial has 2 real roots one in the interval $(-1,0)$ and one in the interval $(0,1)$.

$\therefore$ The polynomial has the pair of complex roots.

2. Find the no. of real and complex roots of the polynomial

3 real roots

$-\infty$
-1.5
0
1.5
2.5
∞

$$P_4(x) = x^4 - 4x^3 + 3x^2 + 4x - 4$$

Soln:

Let $f(x) = x^4 - 4x^3 + 3x^2 + 4x - 4$

$$f'(x) = 4x^3 - 12x^2 + 6x + 4$$

$$\div 2 \Rightarrow f'(x) = 2x^3 - 6x^2 + 3x + 2$$

$$f_2(x) = \frac{f(x)}{f_1(x)} = \frac{x^4 - 4x^3 + 3x^2 + 4x - 4}{2x^3 - 6x^2 + 3x + 2}$$

$$f_2(x) = -\left[\text{Remainder of } \frac{f(x)}{f_1(x)}\right]$$

$$= -\left[\frac{3}{2}x^2 - \frac{9}{2}x - 3\right]$$

$$= \frac{3}{2}x^2 - \frac{9}{2}x + 3$$

$$2x \Rightarrow = 3x^2 - 9x + 6$$

$$f_3(x) = \frac{f_1(x)}{f_2(x)} = \frac{2x^3 - 6x^2 + 3x + 2}{3x^2 - 9x + 6}$$

$$= -\left(\text{Remainder of } \frac{f_1(x)}{f_2(x)}\right)$$

$$= -(-x + 2)$$

$$= x - 2$$

$$f_4(x) = \frac{f_2(x)}{f_3(x)} = \frac{3x^2-9x+6}{x-2}$$

$$= -\left(\text{Remainder of } \frac{f_2(x)}{f_3(x)}\right)$$

$$= 0$$

$$\begin{array}{r} -3x+3 \\ x-2 \overline{\smash{)}\, 3x^2-9x+6} \\ -3x^2-6x \\ \hline -3x+6 \\ 3x-6 \\ \hline 0 \end{array}$$

When we divide $\frac{f_2(x)}{f_3(x)}$ we get 0 as the remainder.

$\therefore f_3(x)$ is the last element of the strum sequence of the polynomial.

hence $x=2$ is a double root

We divide each element of the Strum sequence of by $x-2$ and obtain the new sequence as

$f^*(x) = x^3-2x^2-x+2$ $\quad \frac{f(x)}{x-2}$

$f_1^*(x) = 2x^2-2x-1$ $\quad \frac{f_1(x)}{x-2}$

$f_2^*(x) = x-1$ $\quad \frac{f_2(x)}{x-2}$

$f_3^*(x) = 1$ $\quad \frac{f_3(x)}{x-2}$

We construct the following table of sign changes in the strum sequence.

x	$f^*(x)$	$f_1^*(x)$	$f_2^*(x)$	$f_3^*(x)$	$V(b_i)$
$-\infty$	-	-	+	+	3
-1.5	-	+	-	+	3
0	+	-	-	+	2
1.5	-	+	+	+	1
2.5	+	+	+	+	0
∞	+	+	+	+	0

we find that the polynomial has 3 real roots in the interval $(-1.5, 0)$, $(0, 1.5)$ & $(1.5, 2.5)$

$x=2$ which lies in the interval $(-1.5, 2.5)$ is a double root.

hence the polynomial has 2 simple root in the interval $(-1.5, 0)$ + $(0, 1.5)$ and a double root in the interval $(-1.5, 2.5)$.

Iterative Methods:

Birge - Vieta Method:

To determine a real number t such that $(x-p)$ is a factor of the polynomial eqn

$$P_n(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0, \quad a_0 \neq 0$$

If we divide $P_n(x)$ by the factor $(x-p)$ then we get a quotient Q_{n-1} of degree $n-1$

$$Q_{n-1}(x) = b_0 x^{n-1} + b_1 x^{n-2} + \dots + b_{n-2} x + b_{n-1} \rightarrow ①$$

and a remainder R.

Thus we have

$$P_n(x) = (x-p) Q_{n-1}(x) + R \rightarrow ②$$

The value R depends on P.

Starting with an initial approximation ~~to step~~ P_0 to P we use some iterative method to improve the value of P such that

$$R_n(P) = R(P) = 0 \rightarrow ③$$

This is the single equation in one unknown and the Newton-Raphson method (or) any other iterative method can be applied to improve the assumed value P_0.

∴ The Newton-Raphson method for eqn ③ becomes,

$$P_{k+1} = P_k - \frac{P_n(P_k)}{P_n'(P_k)}, \quad k = 0, 1, 2, \dots \rightarrow ④$$

for obtaining a multiple root we use the modified Newton-Raphson method.

For polynomial eqn the computation of $P_n(P_0)$ & $P_n'(P_0)$ can be computed of like Power of x on both sides of ② we get.

with the help of synthetic division.

On comparing the co-efficients of like powers of x on both sides of (2) we get,

$$\left.\begin{array}{ll} a_0 = b_0 & b_0 = a_0 \\ a_1 = b_1 - bP_0 & b_1 = a_1 + Pb_0 \\ a_2 = b_2 - Pb_1 & b_2 = a_2 + Pb_1 \\ \vdots & \vdots \\ a_K = b_K - Pb_{K-1} & b_K = a_K + Pb_{K-1} \\ \vdots & \vdots \\ a_n = R - Pb_{n-1} & R = a_n + Pb_{n-1} \end{array}\right\} \rightarrow (5)$$

Let us introduce a quantity b_n & define the following recurrence relation

$$b_K = a_K + P b_{K-1} \quad , \quad K = 1, 2, \ldots n \rightarrow (6)$$

with $b_0 = a_0$

eqn (2) we have

$$P_n(p) = R = b_n \longrightarrow (7)$$

To determine $P_n'(p)$, we differentiate (6) w.r.to p & obtain

$$\frac{db_K}{dp} = b_{K-1} + p\frac{db_{K-1}}{dp} \longrightarrow (8)$$

If we put $\frac{db_K}{dp} = C_{K-1} \longrightarrow (9)$

(8) becomes $C_{K-1} = b_{K-1} + PC_{K-2}$

Which can be written as

$$C_K = b_K + PC_{K-1} \quad , \quad K = 1, 2, \ldots n-1 \longrightarrow (10)$$

$$\& \ C_0 = \frac{db_1}{dp} = \frac{d}{dP}(a_1 + Pb_0) = b_0$$

Diff (7) & using (9) we get

$$P_n'(p) = \frac{dR}{dp} = \frac{db_n}{dp} = c_{n-1} \rightarrow (11)$$

The Newton-Raphson method in the above notation becomes

$$p_{k+1} = p_k - \frac{b_n}{c_{n-1}}, \quad k = 0, 1, 2, \ldots$$

This method is often is called the Birge-vieta method. The co-efficients c_k are determined from b_k in a similar way as b_k are obtained from a_k. The calculation of the co-efficients b_k & c_k can be carried out as given below.

p	a_0	a_1	a_2	$\ldots$	a_{n-2}	a_{n-1}	a_n
		pb_0	pb_1	$\ldots$	pb_{n-3}	pb_{n-2}	pb_{n-1}
p	b_0	b_1	b_2	$\ldots$	b_{n-2}	b_{n-1}	$b_n = R$
		pc_0	pc_1	$\ldots$	pc_{n-3}	pc_{n-2}	
	c_0	c_1	c_2	$\ldots$	c_{n-2}	$c_{n-1} = \frac{dR}{dp}$	

The polynomial $P_n(x)$ must be complete

(ie) it has $(n+1)$ terms.

If some term is not present. we introduce it at the proper place with zero co-efficient when p has been obtained to the determined accuring the polynomal

$$Q_{n-1}(x) = b_0 x^{n-1} + b_1 x^{n-2} + \ldots + b_{n-2} x + b_{n-1}$$

is called the deflated polynomial, we repeat the 1st step of one Birge-vieta method using the last value of p to obtain The deflated polynomial. The next real root is obtained using this deflated polynomial.

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Use Synthetic division and perform 2 iterations of the Birge-vieta method to find the smallest +ve root of the polynomial

$$P_3(x) = 2x^3 - 5x + 1 = 0$$

Use the initial approximation $P_0 = 0.5$

Also obtain the deflated polynomial.

Soln:

We write

$$P_3(x) = 2x^3 + 0x^2 - 5x + 1$$

Use the initial approximation $P_0 = 0.5$

0.5	2	0	-5	1
	0	1	0.5	-2.25
	2	1	-4.5	1.25 $= b_3$
	0	1	1	
	2	2	-3.5 $= C_2$	

$$P_1 = P_0 - \frac{b_3}{C_2} = 0.5 - \frac{1.25}{3.5} = 0.142857$$

0.142857	2	0	-5	1
	0	0.285714	0.040816	-0.708454
	2	0.285714	-4.959184	0.291546 $= b_3$
	0	0.285714	-0.081632	
	2	0.571428	-4.877552 $= C_2$	

$$P_2 = P_1 - \frac{b_3}{C_2} = 0.142857 + \frac{0.291546}{4.877552}$$

$$P_2 = 0.202630$$

To obtain the deflated polynomial we repeat the ~~the~~ first part of the Birge-vieta method . we get

0.202630	2	0	-5	1	
	0	0.405260	0.082118	-0.995610	
	2	0.405260	-4.917882	$\boxed{0.003490}$	

$2(0.00349) + 0 - 5(0.00349) + 1$

Since $b_3 = P_3(P_2) = 0.00349$

This gives the error in satisfying the given equation $P_3(x) = 0$.

The deflated polynomial after two iteration is given by

$$Q_2(x) = b_0 x^2 + b_1 x + b_2$$
$$= 2x^2 + 0.405260x - 4.917882$$

2. Use synthetic divison and perform 2 iteration of the Birge - vieta method to find the smaller positive root of the equation

$$x^4 - 3x^3 + 3x^2 - 3x + 2 = 0$$

Use the initial approximation.

Soln:

We write

$$x^4 - 3x^3 + 3x^2 - 3x + 2 = 0$$

Use the initial approximation $P_0 = 0.5$

0.5	1	-3	3	-3	2
	0	0.5	-1.25	0.875	-1.0625
	1	-2.5	1.75	-2.125	$\boxed{0.9375} = b_4$
	0	0.5	-1	0.375	
	1	-2	0.75	$\boxed{-1.750} = C_3$	

$$P_1 = P_0 - \frac{b_4}{C_3} = 0.5 + \frac{0.9375}{1.750} = 1.035714$$

	1	-3	3	-3	2
1.035714	0	1.0356	-2.0343	1.0001	-2.0711
1.035714	1	-1.9644	0.9657	-1.9999	-0.0711 $= b_4$
	0	1.0356	-0.9619	0.0039	
	1	-0.9287	0.0038	-1.9960 $= c_3$	

$$P_2 = P_1 - \frac{b_4}{c_3}$$

$$= 1.035714 - \frac{(-0.0711)}{(-1.9960)}$$

$$P_2 = 1.0$$

The exact root is 1.0.

Bairstow Method:

It extracts a quadratic factor of the form x^2+px+q from the polynomial (I) which may give a pair of complex roots or a pair of real roots.

If we divide the polynomial (I).

By the quadratic factor x^2+px+q,

We obtain a quotient polynomial $Q_{n-2}(x)$ of degree $n-2$ and a remainder term which a polynomial of degree one. ie) $Rx+S$

$$P_n(x) = (x^2+px+q)Q_{n-2}(x) + Rx + S \longrightarrow (1)$$

where

$$Q_{n-2}(x) = b_0 x^{n-2} + b_1 x^{n-3} + \dots + b_{n-3}x + b_{n-2}$$

To find p and q such that

$$R(p,q) = 0 \qquad S(p,q) = 0 \longrightarrow (2)$$

The above eqns are 2 simultaneous equations in 2 unknowns p and q.

Suppose that (P_0, q_0) is an initial approximation and that $(P_0 + \Delta P, q_0 + \Delta q)$ is the true solution.

Following the newton - Raphson method.

$$\Delta P = \frac{-RS_q - SR_q}{R_pS_q - R_qS_p}$$

$$\Delta q = -\frac{RP_s - RS_p}{R_pS_q - R_qS_p} \rightarrow (3)$$

where R_p, R_q, S_p, S_q are the partial derivaties of R and S with respect to p and q respectively.

These quantities and R, S are evaluated at P_0, q_0.

The co-efficients b_i, R and S can be determined by comparing the like power of x in (1) we obtain

$$a_0 = b_0 \qquad\qquad b_0 = a_0$$

$$a_1 = b_1 + Pb_0 \qquad\qquad b_1 = a_1 - Pb_0$$

$$a_2 = b_2 + Pb_1 + qb_0 \qquad\qquad b_2 = a_2 - Pb_1 - qb_0$$

$$\vdots \qquad\qquad \vdots$$

$$a_K = b_K + Pb_{K-1} + qb_{K-2} \qquad\qquad b_K = a_K - Pb_{K-1} - qb_{K-2}$$

$$a_{k-1} = R + Pb_{n-2} + qb_{n-3} \qquad\qquad R = a_{n-1} - Pb_{n-2} - qb_{n-3}$$

$$a_n = S + qb_{n-2} \qquad\qquad S = a_n - qb_{n-2} \rightarrow (4)$$

we now introduce the recursion formula,

$$b_K = a_K - Pb_{K-1} - qb_{K-2} \quad , K = 1, 2 \ldots n \rightarrow (5)$$

where $b_0 = a_0$, $b_{K-1} = 0$

Comparing the last 2 eqns with those of eqn (4) we get,

$R = b_{n-1}$

$S = b_n + p b_{n-1}$

The partial derivatives R_p, R_q, S_p and S_q can be determined by differentiating (5) w.r.to p and q.

we have,

$$-\frac{\partial b_k}{\partial p} = b_{k-1} + p\frac{\partial b_{k-1}}{\partial p} + q\frac{\partial b_{k-2}}{\partial p}$$

$$\frac{\partial b_0}{\partial p} = \frac{\partial b_{k-1}}{\partial p} = 0$$

$$-\frac{\partial b_k}{\partial q} = b_{k-2} + p\frac{\partial b_{k-1}}{\partial q} + q\frac{\partial b_{k-2}}{\partial q}$$

$$\frac{\partial b_0}{\partial q} = \frac{\partial b_{k-1}}{\partial q} = 0 \rightarrow (6)$$

Put $\frac{\partial b_k}{\partial p} = -c_{k-1}$, $k = 1, 2, \dots n$

in the 1st eqn of (6) we find

$$c_{k-1} = b_{k-1} - p c_{k-2} - q c_{k-3} \rightarrow (7)$$

Further more if we write

$$c_{k-2} = -\frac{\partial b_k}{\partial q}$$

then the 2nd eqn (6) gives

$$c_{k-2} = b_{k-2} - p c_{k-3} - q c_{k-4}$$

we get a reccurrence relation for the determination of c_k from b_k as

$$c_k = b_k - p c_{k-1} - q c_{k-2}, \quad k = 1, 2 \dots n-1$$

where $c_{k-1} = 0$ and $c_0 = -\frac{\partial b_1}{\partial p} = -\frac{\partial}{\partial p}(a_1 - pb_0$

$= b$

we obtain $R_p = -c_{n-2}$

$S_p = b_{n-1} - c_{n-1} - p c_{n-2}$

$Rq = -C_{n-3}$ $\quad\quad$ $Sq = -(C_{n-2} + pC_{n-3})$

Sub the above values in eqn ③ and simplifying we get

$$\Delta p = -\frac{(b_n C_{n-3} - b_{n-1} C_{n-2})}{C_{n-2}^2 - C_{n-3}(C_{n-1} - b_{n-1})}$$

$$\Delta q = -\frac{[b_{n-1}(C_{n-1} - b_{n-1}) - b_n C_{n-2}]}{C_{n-2}^2 - C_{n-3}(C_{n-1} - b_{n-1})}$$

The improved values of p_0 and q_0 are

$$p_1 = p_0 + \Delta p, \quad q_1 = q_0 + \Delta q.$$

For computing b_k's and C_k's we use the following scheme.

$-p$	a_0	a_1	a_2	...	a_{n-2}	a_{n-1}	a_n
		$-pb_0$	$-pb_1$	...	$-pb_{n-3}$	$-pb_{n-2}$	$-pb_{n-1}$
			$-qb_0$	...	$-qb_{n-4}$	$-qb_{n-3}$	$-qb_{n-2}$
	b_0	b_1	b_2	...	b_{n-2}	b_{n-1}	b_n
$-p$		$-pC_0$	$-pC_1$	...	$-pC_{n-3}$	$-pC_{n-2}$	
			$-qC_0$	...	$-qC_{n-4}$	$-qC_{n-3}$	
	C_0	C_1	C_2	...	C_{n-2}	C_{n-1}	

Note that the polynomial $P_n(x)$ is complete. When p and q have been obtained to the desired accuracy, the polynomial.

$$Q_{n-2}(x) = \frac{P_n(x)}{(x^2+px+q)}$$

$$= b_0 x^{n-2} + b_1 x^{n-4} + \dots b_{n-2}$$

is called the deflated polynomial. The co-efficient

b_i, $i = 0, 1, 2, \ldots n-2$ are known from the synthetic division procedure. The next quadratic factor is obtained using deflated polynomial.

Perform 2 iteration of the Bairstom method to extract a quadratic factor $x^2 + px + q$ from the polynomial

(1) $$P_3(x) = x^3 + x^2 - x + 2 = 0.$$

Use the initial approximation $P_0 = -0.9$ $q = 0.$

Soln:

-0.9	1	1	-1	2
	0	0.9	1.71	-0.171
			-0.9	-1.71
	$1 = b_0$	1.9	$-0.19 = b_2$	$\boxed{0.119 = b_3}$
	0	0.9	2.52	
			-0.9	
	$1 = c_0$	$2.8 = c_1$	$\boxed{1.43 = c_2}$	

$$\Delta p = -\frac{(b_3 c_0 - b_2 c_1)}{c_1^2 - c_0(c_2 - b_2)}$$

$$= -\frac{[(0.119)(1) - (-0.19)(2.8)]}{(2.8)^2 - (1)(1.43 + 0.19)}$$

$$= -0.1047$$

$$\Delta q = \frac{-b_2(c_2 - b_2) - b_3(c_1)}{c_1^2 - c_0(c_2 - b_2)}$$

$$= 0.1031$$

$$P_1 = P_0 + \Delta P = -0.9 - 0.1047 = -1.0047$$

$$q_1 = q_0 + \Delta q = 0.9 + 0.1031 = 1.0031$$

1.0047	1	1	-1	2
-1.0031	0	1.0047	2.0141	0.0111
			-1.0031	-2.0141
	$1 = b_0$	$2.0047 = b_1$	$0.0110 = b_2$	$\boxed{0.0002 = b_3}$
	0	1.0047	3.0235	
			-1.0031	
	$1 = c_0$	$3.0094 = c_1$	$\boxed{2.0314 = c_2}$	

$$\Delta p = -\frac{(b_3 c_0 - b_2 c_1)}{c_1^2 - c_0(c_2 - b_2)}$$

$\Delta p = 0.0047, \quad \Delta q = -0.031$

$p_2 = p_1 + \Delta p = -1.000$

$q_2 = q_1 + \Delta q = 1.000$

The extracted quadratic factors

$x^2 + p_2 x + q_2 = x^2 - x + 1$ - exact factors.

1. Perform one iteration of the Bairstow method to extract a quadractic factor x^2+px+q from the

A-19 5m Polynomial $x^4+x^3+2x^2+x+1=0$. Use the initial approximation $P_0=0.5$, $q_0=0.5$ starting with $P_0=0.5$ $q_0=0.5$ we obtain.

Sol'n:

-0.5	1	1	2	1	1
-0.5	0	-0.5	-0.25	-0.625	-0.0625
			-0.5	-0.25	-0.625
	1	$0.5=b_1$	$1.25=b_2$	$0.125=b_3$	$\boxed{0.3125=b_4}$
	0	-0.5	0.0	-0.375	
			-0.5	0	
	1	$0.0=c_1$	$0.75=c_2$	$\boxed{-0.25=c_3}$	

$$\Delta p = \frac{-b_4 c_1 - b_3 c_2}{c_2^2 - c_1(c_3-b_3)}$$

$$= \frac{(-0.3125)(0.0) - (0.125)(0.75)}{(0.75)^2 - (0.0)(-0.25-0.125)}$$

$$= 0.1667$$

$$\Delta q = \frac{-b_3(c_3-b_3) - b_4 c_2}{c_2^2 - c_1(c_3-b_3)}$$

$$= \frac{-(0.125)(-0.25-0.125) - (0.3125)(0.75)}{(0.75)^2 - (0.0)(-0.25-0.125)}$$

$$= 0.5$$

$$P_1 = P_0 + \Delta p = 0.6667$$

$$q_1 = q_0 + \Delta q = 1.0$$

$x^2 + p_1 x + q_1 = x^2 + 0.6667x + 1$

Direct Method:

Graeffe's Root Squaring method:

It is used to find the roots of a polynomial with real co-efficients. The roots may be real and distinct real and equal or complex.

Distinct Roots:

We separate the roots of the equation

$$P_n(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0, \quad a_0 \neq 0 \rightarrow (1)$$

by forming another equation with the help of root squaring process, whose roots are very high powers of the roots of the eqn (1).

Separating the even and odd powers of x in (1) and squaring we get

$$(a_0 x^n + a_2 x^{n-2} + a_4 x^{n-4} + \dots)^2$$

$$= (a_1 x^{n-1} + a_3 x^{n-3} + \dots)^2$$

Simplifying we obtain.

$$a_0^2 x^{2n} - (a_1^2 - 2a_0 a_2) x^{2n-2} + (a_2^2 - 2a_1 a_3 + 2a_0 a_4)$$

$$x^{2n-4} - \dots + (-1)^n a_n^2 = 0$$

Substituting z for $-x^2$ we have

$$b_0 z^n + b_1 z^{n-1} + \dots + b_{n-1} + b_n = 0 \rightarrow (2)$$

Where

$$\left.\begin{aligned} b_0 &= a_0^2 \\ b_1 &= a_1^2 - 2a_0 a_2 \\ b_2 &= a_2^2 - 2a_1 a_3 + 2a_0 a_4 \\ &\vdots \\ b_n &= a_n^2 \end{aligned}\right\} \rightarrow (3)$$

Thus all the b_k's are known interms of a_n's.

The roots of the eqn (2) are $-\xi_1^2, -\xi_2^2, \ldots \xi_n^2$ where $\xi_1, \xi_2, \ldots \xi_n$ are roots of eqn (1).

The co-efficients b_k's can be obtained as follows

$$\begin{array}{cccccc}
a_0 & a_1 & a_2 & a_3 & \ldots & a_n \\
\hline
a_0^2 & a_1^2 & a_2^2 & a_3^2 & \ldots & a_n^2 \\
 & -2a_0a_2 & -2a_1a_3 & -2a_2a_4 & \ldots & \\
 & & 2a_0a_4 & 2a_1a_5 & \ldots & \\
\hline
b_0 & b_1 & b_2 & b_3 & \ldots & b_n
\end{array}$$

The $(K+1)^{th}$ column in the above table can be obtained as follows.

The terms alternate in sign starting with a +ve sign. The 1st term is the square of the $(K+1)^{th}$ co-efficient a_K. The 2nd term is twice the product of the nearest neighbouring co-efficients a_{K-1} & a_{K+1}.

The 3rd is twice the product of the next neighbouring co-efficients a_{K-2} & a_{K+2}. This procedure is continued. Untill there are no available co-efficients to form the cross products.

This procedure can be replaced in 'm' times and we obtain the eqn.

$$B_0 x^n + B_1 x^{n-1} + B_2 x^{n-2} + \ldots + B_{n-1} x + B_n = 0 \quad (4)$$

whose roots $R_1, R_2, R_3 \ldots R_n$ are the $2m$ the power of the roots of the equation (I) ($P_n(x)$) with opposite signs.

$$R_i = -\xi_i^{2m} \quad i = 1, 2 \ldots n$$

If we assume that

$$|\xi_1| > |\xi_2| > \ldots > |\xi_n|$$

then $|R_1| >> |R_2| >> \ldots >> |R_n|$.

If the roots of ① differ in magnitude then the $2^{m\text{th}}$ power of the roots are widely seperated for large m. we have

$$-\frac{B_1}{B_0} = \sum R_i \simeq R_1$$

$$\frac{B_2}{B_1} = \sum R_i R_j \simeq R_1 R_2$$

$$-\frac{B_3}{B_2} = \sum R_i R_j R_K = R_1 R_2 R_3$$

$$(-1)^n \frac{B^n}{B^{n-1}} = R_1 R_2 \ldots R_n$$

$$R_i = -\frac{B_i}{B_{i-1}} \quad , i = 1, 2 \ldots n$$

$$|R_i| = \frac{|B_i|}{|B_{i-1}|} = |\xi_i|^{2^m}$$

(or)

$$\log|\xi_i| = 2^{-m} \left(\log|B_i| - \log|B_{i-1}| \right) \quad , i = 1, 2 \ldots m.$$

This determines the absolute values of the roots and Substitution in original equation (I) $P_n(x) =$ will give the sign of the roots.

The squaring process is stopped when another squaring process produce new Co-efficients that are almost the squares of the corresponding Co-efficients B_K's

(ii) when the cross product terms become negligible in comparison to square terms.

Thus the vanishing of all the cross product terms in the squaring process can be used as an indication that the roots have been widely separated.

Equal Roots:

After few squarings, if the magnitude of the co-efficients B_K in about half the square of the magnitude of the corresponding co-efficients in the previous eqn then it indicate that ξ_K is a double root by using we can find this double root by using the following procedure, we have,

$$R_K \simeq -\frac{B_K}{B_{K-1}} \quad \text{and} \quad R_{K+1} \simeq -\frac{B_{K+1}}{B_K}$$

$$R_K R_{K+1} \simeq R_K^2 = \left|\frac{B_{K+1}}{B_{K-1}}\right|$$

$$\therefore |R_K^2| = |\xi_K|^{2(2m)} = \left|\frac{B_{K+1}}{B_{K+1}}\right|$$

This gives the magnitude of the double root sub in the given eqn.

We can find in sign.

This double root can also be found directly since R_K and R_{K+1} converge to the same root after sufficient squarings.

Usually this convergence to the double root is slow.

By making use of the above observation we can save a number of squarings.

Complex Roots:

If ξ_K and ξ_{K+1} from a complex pair then this would case the co-efficients of x^{n-K} in the successive squarings to fluctuate both in magnitude and sign.

If $\xi_K, \xi_{K+1} = \beta_K \exp(\pm i\phi_K)$ is the complex pair then the co-efficients would fluctuate in magnitude and sign by an amount $2\beta_K^m \cos(m\phi_K)$.

A complex pair can be spotted by such an oscillation. For an sufficiently large, β_K can be determined from the relation.

$$\beta_K^{2(2m)} = \left|\frac{\beta_{K+1}}{\beta_{K-1}}\right|$$

and ϕ is suitably determined from the relation.

$$2\beta_K^m \cos(m\phi_K) = \frac{\beta_{K+1}}{\beta_{K-1}}$$

If the eqn has only one complex pair, then we can 1st determine all the real roots.

The complex pair can be written as

$\xi_K, \xi_{K+1} = p \pm iq$

The sum of the roots then gives

$\xi_1 + \xi_2 + \dots + \xi_{K-1} + 2p + \xi_{K+2} + \dots + \xi_n = -\frac{a_1}{a_0}$

This determine p,

We also have $|\beta_K|^2 = p^2 + q^2$.

$\therefore$ $|\beta_K|$ is already determined, this equation gives q.

1. Find all the roots of the polynomial $x^3-6x^2+11x-6$ using the Graeffe's root square method. 10M A-1'

Soln: The co-efficient of the successive square roots are given by,

m	2^m	a_0	a_1 ($-2a_0a_2$)	a_2 ($-2a_1a_3$)	a_3
0	1	1	-6	11	-6
		a_0^2 1	a_1^2 36	a_2^2 121	a_3^2 36
			$-2a_0a_2$ -22	$-2a_1a_3$ -72	
1	2	1	14	49	36
		1	196	2401	1296
			-98	-1008	
2	4	1	98	1393	1296
		1	9604	1940449	1679616
			-2786	-25406	
3	8	1	6818	1686433	1679616
		1	46485124	2.8440562×10^{12}	2.821699×10^{12}
			-3.372866	$+2.2903243\times10^{10}$	
4	16	1	4.3112258	2.8211530(12) $\times10^{12}$	2.8211099 (12)

Successive approximations to the roots are given in the table.

The exact roots of the equation are 3,2,1

Approximation to the roots

m	α_1	α_2	α_3
1	3.7417	1.8708	0.8571
2	3.1463	1.9417	0.9821
3	3.0144	1.9914	0.9995
4	3.0003	1.9998	1.0000

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(7) Find all the roots of the polynomial $x^3-4x^2+5x-2=0$. Using the Graeffe's root squaring method.

$x^4-x^3+3x^2+x-4=0$ A-19 10m

Soln:

The co-efficients in the root squaring by Graeffe's method.

m	2^m	a_0	a_1	a_2	a_3
0	1	1	-4	5	-2
		1	16	25	4
			-10	-16	
1	2	1	6	9	4
		1	36	81	16
			-18	-48	
2	4	1	18	33	16
		1	324	1089	256
			-66	-576	
3	8	1	258	513	256

		1	66564	263169	65536
			-1026	-132096	
4	16	1	65538	131073	65536

m	2^m	a_0	a_1	a_2	a_3
		1	0.4295×10^{10}	0.1718×10^{11}	0.42949×10^{10}
			-0.262146×10^{6}	-0.8590×10^{10}	
5	32	1	0.4297×10^{10}	0.85899×10^{10}	0.42959×10^{10}
		b_0	b_1	b_2	b_3

We notice from the table the magnitude of the co-efficient B_2 is half the square of the magnitude of the corresponding co-efficient in the previous equation.

This indicates ξ_2 is a double root.

Now we obtain the magnitude of the root.

$$|\xi_1|^{32} = \left|\frac{B_1}{B_0}\right|$$

$$= (0.4297)\, 10^{10}$$

$$|\xi_1| = 2.00$$

$$|\xi_2|^{64} = \left|\frac{B_2}{B_1}\right| = 1.999$$

$$= 1.0109$$

$$|\xi_3| = \left|\frac{B_3}{B_2}\right| = 0.5001 = 0.9892$$

All the roots the exact roots are exact 2, 1, 1 if we want to get the double root by the direct procedure it would take large number of squarings.

In this example we find directly

$$|\xi_2| = 1.0219$$

$$|\xi_3| = 0.9786$$

one more squaring procedures

$$|\xi_2| = 1.0109 \ , \ |\xi_3| = 0.9892$$

So that the convergence is slow. It would require few more squarings to stabilize the root.

after chebyshev method.

1. S.T the following 2 sequences have convergence of the 2nd order with same limit $\sqrt{a}$.

i) $x_{n+1} = \frac{1}{2} x_n \left(1 + \frac{a}{x_n^2}\right)$

ii) $x_{n+1} = \frac{1}{2} x_n \left(3 - \frac{x_n^2}{a}\right)$

If x_n is a suitably close approximation $\sqrt{a}$, show that the magnitude of the error in the 1st formula for x_{n+1} is about one - third of that in the second formula, and deduce that the formula.

(iii) $x_{n+1} = \frac{1}{8} x_n \left(6 + \frac{3a}{x_n^2} - \frac{x_n^2}{a}\right)$ gives a sequence with the third order convergences.

Soln:
Taking the limits as $n \to \infty$ and $\lim_{n \to \infty} x_n = \xi_1$,

$\lim_{n \to \infty} x_{n+1} = \xi_1$

where ξ_1 is the exact root.
we obtain from all the three methods

$\xi_1^2 = a$

Thus all the three method determine $\sqrt{a}$, where a is any, positive real number.

Substituting

$x_n = \xi_1 + \epsilon_n$,

$x_{n+1} = \xi_1 + \epsilon_{n+1}$ $\quad x_{n+1} = \frac{1}{2} x_n \left(1 + \frac{a}{x_n^2}\right)$

and $a^2 = \xi_1^2$ we get

(i) $\xi_1 + \epsilon_{n+1} = \frac{1}{2}(\xi_1 + \epsilon_n)\left[1 + \frac{\xi_1^2}{(\xi_1 + \epsilon_n)^2}\right]$

$= \frac{1}{2}(\xi_1 + \epsilon_n)\left[1 + \frac{\xi_1^2}{\xi_1^2\left(1 + \frac{\epsilon_n}{\xi_1}\right)^2}\right]$

$$= \frac{1}{2}(\xi_1 + \epsilon_n)\left[1 + \left(1 + \frac{\epsilon_n}{\xi_1}\right)^{-2}\right]$$

$$= \frac{1}{2}(\xi_1 + \epsilon_n)\left(2 - \frac{2\epsilon_n}{\xi_1} + \frac{3\epsilon_n^2}{\xi_1^2} - \cdots\right)$$

$$= \frac{1}{2}\left(2\xi_1 - \frac{2\epsilon_n}{\xi_1}(\xi_1) + \frac{3\epsilon_n^2}{\xi_1^2}\xi_1 + 2\epsilon_n - \frac{2\epsilon_n^2}{\xi_1} + \frac{3\epsilon_n^3}{\xi_1^2} + \cdots\right]$$

$$\xi_1 + \epsilon_{n+1} = \frac{1}{2}\left[2\xi_1 + (3-2)\frac{\epsilon_n^2}{\xi_1} + \cdots\right]$$

$$\xi_1 + \epsilon_{n+1} = \xi_1 + \frac{1}{2}\frac{\epsilon_n^2}{\xi_1} + O(\epsilon_n^3)$$

$$\epsilon_{n+1} = \frac{1}{2\xi_1}\epsilon_n^2 + 0 \longrightarrow (1)$$

Hence the method has second order Convergence with the error constant

$$C = \frac{1}{2\xi_1}$$

ii) $$\xi_1 + \epsilon_{n+1} = \frac{1}{2}(\xi_1 + \epsilon_n)\left[3 - \frac{1}{\xi_1^2}(\xi_1 + \epsilon_n)^2\right]$$

$$= \frac{1}{2}(\xi_1 + \epsilon_n)\left[3 - \frac{\xi_1^2}{\xi_1^2} - \frac{\epsilon_n^2}{\xi_1^2} - \frac{2\xi_1\epsilon_n}{\xi_1^2}\right]$$

$$= \frac{1}{2}(\xi_1 + \epsilon_n)\left(2 - \frac{\epsilon_n^2}{2\xi_1^2} - \frac{2\epsilon_n}{\xi_1}\right)$$

$$= (\xi_1 + \epsilon_n)\left[1 - \frac{\epsilon_n^2}{2\xi_1^2} - \frac{\epsilon_n}{\xi_1}\right]$$

$$= \xi_1 - \frac{\xi_1\epsilon_n^2}{2\xi_1^2} - \frac{\epsilon_n\xi_1}{\xi_1} + \epsilon_n - \frac{\epsilon_n^3}{2\xi_1^2} - \frac{\epsilon_n^2}{\xi_1}$$

$$\epsilon_{n+1} = \frac{-3}{2}\frac{\epsilon_n^2}{\xi_1} + O(\epsilon_n^3) \longrightarrow (2)$$

Hence, the method has second order Convergence with the error constant

$$C = \frac{-3}{2\xi_1}$$

Therefore, the magnitude of the error in the first formula is about one third of that in the second formula.

iii) If we multiply ① by 3 and add to ② we find that

$$\epsilon_{n+1} = \frac{3}{2\xi}\epsilon_n^2 + o(\epsilon_n^3) - \frac{3}{2}\frac{\epsilon_n^2}{\xi} + o(\epsilon_n^3)$$

$$\epsilon_{n+1} = o(\epsilon_n^3) \longrightarrow ③$$

It can be verified that $o(\epsilon_n^3)$ term in ③ does not vanish adding 3 times the first formula to the second formula we obtain the new formula

$$4x_{n+1} = \frac{3}{2}x_n\left(1+\frac{a}{x_n^2}\right) + \frac{1}{2}x_n\left(3-\frac{x_n^2}{a}\right)$$

$$= \frac{3}{2}x_n + \frac{3}{2}\frac{x_n}{x_n^2}a + \frac{3}{2}x_n - \frac{x_n^3}{2a}$$

$$= \frac{3}{2}x_n + \frac{3a x_n}{2x_n^2} + \frac{3x_n}{2} - \frac{x_n^3}{2a}$$

$$= 3x_n + \frac{3}{2}\frac{a x_n}{x_n^2} - \frac{x_n^3}{2a}$$

$$= \frac{1}{2}\left[6x_n + \frac{3a x_n}{x_n^2} - \frac{x_n^3}{a}\right]$$

$$x_{n+1} = \frac{1}{8}x_n\left[6 + \frac{3a^2}{x_n^2} - \frac{x_n^2}{a}\right]$$

which has third order convergence.

Q. The equation $x^4 + x = \epsilon$ where ϵ is a smallest number has a root which is close to ϵ, computation of this root is done by the expression $\xi = \epsilon - \epsilon^4 + 4\epsilon^7$

(i) Find an iterative formula $x_{n+1} = F(x_n)$, for the computation. Show that we get $x_0 = 0$ for the above after 3-iteration when the expression neglecting of higher order.

(ii) Give a good estimate (of the form $N\epsilon^k$, where N & k are integers) of the maximal error. When the root is estimated by the expression above.

Soln:

We write the given eqn $x^4 + x = \epsilon$ in the form

$$x(x^3+1) = \epsilon$$

$$x = \frac{\epsilon}{x^3+1}$$

and consider the formula

$$x_{n+1} = \frac{\epsilon}{x_n^3+1}$$

Starting with $x_0 = 0$ we obtain

$$x_1 = \frac{\epsilon}{x_0^3+1}$$

$(a+b+c)^3 = a^3+b^3+c^3+3ab^2+3ac^2+3bc^2+3a^2c+3b^2c+3a^2b+6abc.$

$$x_1 = \epsilon$$

$$x_2 = \frac{\epsilon}{x_1^3+1} = \frac{\epsilon}{\epsilon^3+1} = \epsilon[1+\epsilon^3]^{-1} = \epsilon[1-\epsilon^3+(\epsilon^3)^2+\dots]$$

$$= \epsilon(1-\epsilon^3+\epsilon^6+\dots)$$

$$= \epsilon - \epsilon^4 + \epsilon^7 \text{ (neglecting hr powers of } \epsilon)$$

$$x_3 = \frac{\epsilon}{x_2^3+1} = \frac{\epsilon}{[(\epsilon-\epsilon^4+\epsilon^7)^3+1]}$$

$$= \epsilon[1+(\epsilon-\epsilon^4+\epsilon^7)^3]^{-1}$$

$$= \epsilon[1-(\epsilon-\epsilon^4+\epsilon^7)^3+(\epsilon-\epsilon^4+\epsilon^7)^6+\dots]$$

$$= \epsilon[1-(\epsilon^3-\epsilon^{12}+\epsilon^{21}+3\epsilon^9+3\epsilon^{15}-3\epsilon^{18}-6\epsilon^{12}+3\epsilon^9+3\epsilon^{15}-3\epsilon^6)+\epsilon^6)$$

$= \epsilon - \epsilon^4 + 3\epsilon^7 + \epsilon^7$

$= \epsilon - \epsilon^4 + 4\epsilon^7$

Taking $\xi_1 = \epsilon - \epsilon^4 + 4\epsilon^7$ we find that

Error $= \xi_1^4 + \xi_1 - \epsilon$

$= (\epsilon - \epsilon^4 + 4\epsilon^7)^4 + (\epsilon - \epsilon^4 + 4\epsilon^7) - \epsilon$

$= \epsilon^4 + \epsilon^{16} + 256\epsilon^{28} + 4(-\epsilon^7 - \epsilon^{13} - 4\epsilon^{19} - 64\epsilon^{25} + 64\epsilon^{22} + 4\epsilon^{10})$

$+ 6(\epsilon^{10} + 16\epsilon^{22} + 16\epsilon^{16}) + 12(-4\epsilon^{13} + 4\epsilon^{16} - 16\epsilon^{19}) + \epsilon - \epsilon^4 + 4\epsilon^7 - \epsilon$

$= \cancel{\epsilon^4} + \epsilon^{16} + 256\epsilon^{28} - \cancel{4\epsilon^7} - 4\epsilon^{13} - 16\epsilon^{19}$

$-256\epsilon^{25} + 256\epsilon^{22} + 16\epsilon^{10} + 6\epsilon^{10} +$

$96\epsilon^{22} + 96\epsilon^{16} - 48\epsilon^{16} - 192\epsilon^{19}$

$+ \cancel{\epsilon} - \cancel{\epsilon^4} + \cancel{4\epsilon^7} - \cancel{\epsilon}$

$= 22\epsilon^{10}$ + higher power of ϵ.

4. How should the constant α be choosen to ensure the fastest possible convergence with the iteration formula

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$$x_{n+1} = \frac{\alpha x_n + x_n^{-2} + 1}{\alpha + 1}$$

Soln:

W.K.T

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \xi$$

we get

$$\xi = \frac{\alpha\xi + \xi^{-2} + 1}{\alpha + 1}$$

$(\alpha+1)\xi = \alpha\xi + \xi^{-2} + 1$

$\alpha\xi + \xi = \alpha\xi + 1/\xi^2 + 1$

$\alpha\xi + \xi = \dfrac{\alpha\xi^3 + 1 + \xi^2}{\xi^2}$

$$\alpha \xi^3 + \xi^3 = \alpha \xi^3 + 1 + \xi^2$$

$$\xi^3 = 1 + \xi^2$$

$$\xi^3 - \xi^2 - 1 = 0$$

Thus the formula is being used to find the root of the equation $f(x) = x^3 - x^2 - 1 = 0$

Sub $x_n = \xi + \epsilon_n$ & $x_{n+1} = \xi + \epsilon_{n+1}$

we obtain

$$x_{n+1} = \frac{\alpha x_n + x_n^{-2} + 1}{\alpha + 1}$$

$$(\alpha+1) x_{n+1} = \alpha x_n + x_n^{-2} + 1$$

$$(1+\alpha)(\xi + \epsilon_{n+1}) = \alpha(\xi + \epsilon_n) + (\xi + \epsilon_n)^{-2} + 1$$

$$\xi + \epsilon_{n+1} + \alpha \xi + \alpha \epsilon_{n+1} = \alpha \xi + \alpha \epsilon_n + \xi^{-2}\left(1 + \frac{\epsilon_n}{\xi}\right)^{-2} +$$

$$\xi + \epsilon_{n+1} + \alpha \epsilon_{n+1} = \alpha \epsilon_n + \frac{1}{\xi^2}\left(2 - 2\frac{\epsilon_n}{\xi} + \frac{3\epsilon_n^2}{\xi^2} \ldots\right) + 1$$

$$(1+\alpha)\epsilon_{n+1} + \xi = \alpha \epsilon_n + \frac{2}{\xi^2} - \frac{2\epsilon_n}{\xi^3} + \frac{3\epsilon_n^2}{\xi^4} + \cdots + 1$$

$$(1+\alpha)\epsilon_{n+1} = -\xi + \alpha \epsilon_n + \frac{2}{\xi^2} - \frac{2\epsilon_n}{\xi^3} + 1$$

$$= \cancel{\frac{-\xi^4 + \alpha \epsilon_n \xi^3 + 2\xi - 2\epsilon_n + \xi^3}{\xi^3}}$$

$$(1+\alpha)\epsilon_{n+1} = \alpha \epsilon_n - \frac{2\epsilon_n}{\xi^3} + o(\epsilon_n^2)$$

$$(1+\alpha)\epsilon_{n+1} = \left(\alpha - \frac{2}{\xi^3}\right)\epsilon_n + o(\epsilon_n^2)$$

For fastest Convergence we must have $\alpha = \frac{2}{\xi^3}$

We can find the approximate value of ξ by using Newton Raphson method to determine a root of $x^3 - x^2 - 1 = 0$ we obtain $\xi = 1.4656$ and hence $\alpha = 0.6353 \simeq 0.64$

5. Determine α_1 and α_2 so that the order of the iterative method.

$$x_{u+1} = x_u - \alpha_1 W_1(x_u) - \alpha_2 W_2(x_u)$$

where $W_1(x_u) = f(x_u) / f'(x_u)$

$W_2(x_u) = f(x_u) / f'(x_u + \beta W_1(x_u))$. $\beta \neq 0$

soln: For finding a simple root of the eqn $f(x)=0$ becomes a high as possible. We substitute $x_u = \xi + \epsilon_K$ and $f(\xi)=0$ in $W_1(x_u)$ $W_2(x_u)$ to get,

$$W_1(\xi + \epsilon_u) = \frac{f(\xi + \epsilon_u)}{f'(\xi + \epsilon_u)}$$

$$= \epsilon_u - \frac{1}{2} \frac{f''(\xi)}{f'(\xi)} \epsilon_u^2 + o(\epsilon_u^3)$$

$$f'(x_a + \beta W_1(x_u)) = f'[\xi + \epsilon_u + \beta\{\epsilon_u + o(\epsilon_u^2)\}]$$

$$= f'[\xi + (1+\beta)\epsilon_u + o(\epsilon_u^2)]$$

$$f'(x_u + \beta W_1(x_u)) = f'(\xi) + (1+\beta)\epsilon_u f''(\xi) + o(\epsilon_u^2)$$

$$W_2(x_u) = [\epsilon_u f'(\xi) + \frac{1}{2}\epsilon_u^2 f''(\xi) + \cdots] \times \frac{1}{f'(\xi)}$$

$$[1 + (1+\beta)\frac{f''(\xi)}{f'(\xi)}\epsilon_u + o(\epsilon_u^2)]^{-1}$$

$$= [\epsilon_u + \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\epsilon_u^2 + \cdots][1 - (1+\beta)\frac{f''(\xi)}{f'(\xi)} + \epsilon_u + o(\epsilon_u^2)]$$

$$= \epsilon_u - \frac{1}{2}(1+2\beta)\frac{f''(\xi)}{f'(\xi)}\epsilon_u^2 + o(\epsilon_u^3)$$

Using these expressions in the iteration method

$$\epsilon_{u+1} = \epsilon_u - \alpha_1[\epsilon_u - \frac{1}{2} u'' [\frac{f''(\xi)}{f'(\xi)}\epsilon_u^2 + \cdots] -$$

$$\alpha_2[\epsilon_u - \frac{1}{2}(1+2\beta)\frac{f''(\xi)}{f'(\xi)}\epsilon_u^2 + \cdots$$

$$= (1-\alpha_1-\alpha_2)\epsilon_u + \frac{1}{2}(\alpha_1 + \alpha_2(1+2\beta))\frac{f''(\xi)}{f'(\xi)}\epsilon_u^2 + \cdots$$

Equating the co-efficient of ϵ_u and ϵ_u^2 the zero we obtain $\alpha_1 + \alpha_2 = 1$

$$\alpha_1 + \alpha_2(1+2\beta) = 0$$

If $\beta \neq 0$ then we have $\alpha_1 = \frac{1+2\beta}{2\beta}$, $\alpha_2 = -\frac{1}{2\beta}$. The order of the iterative method is three for arbitrary $\beta \neq 0$ and for $\beta = -1/2$.

2. Let the function $f(x)$ be four times continuously differentiable and hence a simple zero ξ - successive approximation x_n, $n=1,2\ldots$ to ξ are computed from

$$x_{n+1} = \frac{1}{2}(x'_{n+1}, x''_{n+1})$$

where $x'_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $x''_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$

$g(x) = \frac{f(x)}{f'(x)}$. P.T if the sequence $\{x_n\}$ converges to ξ then the convergence is cubic.

Soln:

We have,

$$g(x) = \frac{f(x)}{f'(x)}, \quad g'(x) = \frac{f'(x)^2 - f(x) f''(x)}{(f'(x))^2}$$

$$x'_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x''_{n+1} = x_n - \frac{f(x_n)/f'(x_n)}{\left[1 + \frac{f(x_n) f''(x_n)}{(f'(x_n))^2}\right]}$$

$$= x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{f(x_n) f''(x_n)}{(f'(x_n))^2} + \left[\frac{f(x_n) f''(x_n)}{(f'(x_n))^2}\right] + \cdots\right]$$

from the formula,

$$x_{n+1} = \frac{1}{2}(x'_{n+1} + x''_{n+1})$$

We obtain,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\left[\frac{f(x_n)}{f'(x_n)}\right]^2 \frac{f''(x_n)}{f'(x_n)} - \frac{1}{2}\left[\frac{f(x_n)}{f'(x_n)}\right]^3 \left[\frac{f''(x_n)}{f'(x_n)}\right]^2 + \cdots \quad (1)$$

Using $x_n = \xi + \epsilon_n$ and $c_i = \frac{f^{(i)}(\xi)}{f'(\xi)}$

we get,

$$\frac{f(x_n)}{f'(x_n)} = \epsilon_n - \frac{1}{2} c_2 \epsilon_n^2 + \left(\tfrac{1}{2} c_2^2 - \frac{1}{3} c_3 \right) \epsilon_n^3 + \cdots$$

$$\frac{f''(x_n)}{f'(x_n)} = c_2 + (c_3 - c_2)^2 \epsilon_n + \cdots$$

Using these expression in ① we obtain the error eqn on simplification as

$$\epsilon_{n+1} = \epsilon_n - \left[\epsilon_n - \tfrac{1}{2} c_2 \epsilon_n^2 + \left(\frac{1}{2} c_2^2 - \frac{1}{3} c_3 \right) \epsilon_n^2 \right]$$

$$- \frac{1}{2} \left[\epsilon_n^2 - c_2 \epsilon_n^3 + \cdots \right] \left[c_2 + (c_3 - c_2)^2 \epsilon_n^2 + \cdots \right]$$

$$- \frac{1}{2} \left[\epsilon_n^3 + \cdots \right] \left[c_2^2 + 2 c_2 (c_3 - c_2^2) \epsilon_n + \cdots \right]$$

$$= \frac{1}{6} c_3 \epsilon_n^3 + O(\epsilon_n^4)$$

Hence the method has cubic convergences.