Cauvery College for Women (Autonomous)

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Unit	:	II

Topics Covered : system of linear algebraic equation & eigen value problem –error analysis of direct method- operational count of gauss elimination- iteration method –Gauss seidal iteration method – successive over relaxation method , Jacobi method for symmetric matrices and power method.

UNIT - II

Gauss Elimination method :-

consider the 3×3 system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 = b_2^{(2)}$$

$$a_{32}^{(2)}x_2 + a_{33}^{(2)}x_3 = b_3^{(2)}$$

$$a_{22}^{(2)} = a_{22} - \frac{a_{21}}{a_{11}}a_{12}, \quad a_{23}^{(2)} = a_{23} - \frac{a_{21}}{a_{11}}a_{13}$$

$$a_{32}^{(2)} = a_{32} - \frac{a_{31}}{a_{11}}a_{12}, \quad a_{33}^{(2)} = a_{33} - \frac{a_{31}}{a_{11}}a_{13}$$

$$b_2^{(2)} = b_2 - \frac{a_{21}}{a_{11}}b_1, \quad b_3^{(2)} = b_3 - \frac{a_{31}}{a_{11}}b_1$$

$$a_{33}^{(3)}x_3 = b_3^{(3)}$$

$$a_{33}^{(3)} = a_{33}^{(2)} - \frac{a_{32}^{(2)}}{a_{22}^{(2)}}a_{23}^{(2)}, \quad b_3^{(3)} = b_3^{(2)} - \frac{a_{32}^{(2)}}{a_{22}^{(2)}}b_2^{(2)}$$

$$a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 = b_1^{(1)}$$

$$a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 = b_2^{(2)}$$

$$a_{33}^{(3)}x_3 = b_3^{(3)}$$

where $a_{ij}^{(1)} = a_{ij}, \quad b_i^{(1)} = b_i, \quad i,j = 1,2,3$

$$[A|b] \xrightarrow[\text{Elimination}]{\text{Gauss}} [U|c]$$

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1. Solve the equations

$10x_1 - x_2 + 2x_3 = 4$

$x_1 + 10x_2 - x_3 = 3$

$2x_1 + 3x_2 + 20x_3 = 7$

Using Gauss Elimination method.

Soln :

After the 1st Elimination,

$10x_1 - x_2 + 2x_3 = 4$

$$\frac{101}{10}x_2 - \frac{12}{10}x_3 = \frac{26}{10}$$

$$\frac{32}{10}x_2 + \frac{196}{10}x_3 = \frac{62}{10}$$

Second elimination stage,

$10x_1 - x_2 + 2x_3 = 4$

$$\frac{101}{10}x_2 - \frac{12}{10}x_3 = 26/10$$

$$\frac{20180}{1010}x_3 = \frac{5430}{1010}.$$

using back substitution,

$x_3 = 0.269$, $x_2 = 0.289$, $x_1 = 0.375$.

Gauss Jordan Elimination method :

1) Find the inverse of the co-efficient matrix of the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

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Soln :

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & 5 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3/4 & -1/4 & 0 & 1/4 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3/4 & -1/4 & 0 & 1/4 & 0 \\ 0 & 1/4 & 5/4 & 1 & -1/4 & 0 \\ 0 & 11/4 & 15/4 & 0 & -3/4 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3/4 & -1/4 & 0 & 1/4 & 0 \\ 0 & 11/4 & 15/4 & 0 & -3/4 & 1 \\ 0 & 1/4 & 5/4 & 1 & -1/4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3/4 & -1/4 & 0 & 1/4 & 0 \\ 0 & 1 & 15/11 & 0 & -3/11 & 4/11 \\ 0 & 1/4 & 5/4 & 1 & -1/4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -14/11 & 0 & 5/11 & -3/11 \\ 0 & 1 & 15/11 & 0 & -3/11 & 4/11 \\ 0 & 0 & 10/11 & 1 & -2/11 & -1/11 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -14/11 & 0 & 5/11 & -3/11 \\ 0 & 1 & 15/11 & 0 & -3/11 & 4/11 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right]$$

∴ The solution of the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/5 & 1/5 & -2/5 \\ -3/2 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$$

Triangularization method [LU decomposition method]

1) consider the eqn

$$x_1 + x_2 + x_3 = 1$$

$$4x_1 + 3x_2 - x_3 = 6$$

$$3x_1 + 5x_2 + 3x_3 = 4$$

using the decomposition method to solve the system.

we choose

$u_{ii} = 1$ & write

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

on comparing the corresponding elements, we get

first column ; $l_{11} = 1$, $l_{21} = 4$, $l_{31} = 3$

first row ; $u_{12} = 1$, $u_{13} = 1$.

second column : $l_{22} = -1$, $l_{32} = 2$.

second row : $u_{23} = 5$, $l_{33} = -10$

Thus we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the forward substitution,

$$z = [1, -2, -\tfrac{1}{2}]^T$$

using the back substitution,

$$x = [1, \tfrac{1}{2}, -\tfrac{1}{2}]^T.$$

cholesky method :-

1. solve the system of equations

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 10 \end{bmatrix}$$

Using the cholesky method :-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{12} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11} l_{21} & l_{11} l_{31} \\ l_{21} l_{11} & l_{21}^2 + l_{22}^2 & l_{21} l_{31} + l_{22} l_{32} \\ l_{31} l_{11} & l_{31} l_{21} + l_{32} l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

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comparing the corresponding elements on both sides

first row :

$$l_{11}^2 = 1 \quad (or) \quad l_{11} = 1$$

$$l_{11} l_{21} = 2 \quad (or) \quad l_{21} = 2$$

$$l_{31} = 3 \quad (or) \quad l_{31} = 3$$

Second row :

$$l_{21}^2 + l_{22}^2 = 8 \quad (or) \quad l_{22} = 2$$

$$l_{31} l_{21} + l_{32} l_{22} = 22 \quad (or) \quad l_{32} = 8$$

Third row :

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 82 \quad (or) \quad l_{33} = 3$$

hence we get

$$A = L \cdot L^T \cdot \text{ where}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix}$$

we write the given system of equation as,

$$LL^T x = b$$

$$Ly = b \quad \text{and} \quad L^T x = y$$

from $Ly = b$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix} \quad (or) \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

from $L^T X = Y$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ -3 \end{bmatrix} \text{ (or) } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

Error analysis for Direct method:-

operational count for Gauss elimination method:-

Number of Divisions :-

1st step [1st eqn (Division by 1st pivot)] : n

2nd step [2nd eqn (Division by 2nd pivot)] : $n-1$

. . . .

nth step [nth eqn (Division by nth pivot)] : 1

The total number of divisions $\Sigma n = \frac{n}{2}(n+1)$.

Number of multiplications :-

1st step : 2nd eqn : n

3rd eqn : n

. . .

nth eqn

Total multiplication in the 1st step $n(n-1)$

hence total multiplication in the forward elimination $= \Sigma(n-1)n = \Sigma n^2 - n = \frac{n}{3}(n+1)(n-1)$

for back substitution, we have the no. of multiplication

as $(n-1)^{th}$ eqn : 1

$(n-2)^{th}$ eqn : 2

- - - -

1st eqn : $n-1$

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Total multiplication in the back substitution

$= \sum (n-1) = \frac{n}{2}(n-1)$

Total multiplications $= \frac{n}{3}(n+1)(n-1) + \frac{n}{2}(n-1)$

$= \frac{n}{6}(n-1)(2n+5)$

Total no. of division & multiplications.

Operational count $= \frac{n}{2}(n+1) + \frac{n}{6}(n-1)(2n+5)$

$= \frac{n}{3}(n^2+3n-1)$

For large n, operational count $\simeq \frac{n^3}{3}$

Total addition and subtraction $= \frac{n}{6}(n-1)(2n+5)$

We find that for large n. the

i) Gauss Jordan elimination method requires $n^3/2$ operations.

ii) LU decomposition method requires $\frac{n^3}{3}$ operations. (same as in Gauss elimination method).

(iii) Cholesky method requires $\frac{n^3}{6}$ operations

If all calculation are performed exactly, then only we can hope to find the exact soln to the system $Ax = b$. Usually during computation it will be necessary to round or chop the numbers.

This will introduce round of errors in the computation. Because of this, the methods used will produce result which will differ considerably from the exact soln. They exact soln x and the corresponding approximate soln will satisfy respectively the eqn.

$$A x = b$$

$$(A+\delta A)\hat{x} = b+\delta b \longrightarrow (1$$

where δA & δb are the changes in A & b respectively due to round of error

from eqn (1 we obtain

$$\hat{x} - x = (A+\delta A)^{-1}(b+\delta b) - A^{-1}b$$

$$= \{(A+\delta A)^{-1} - A^{-1}\} b + (A+\delta A)^{-1}\delta b \rightarrow (2$$

which may be called the error eqn. In order to estimate the error vector $\varepsilon = \hat{x} - x$, we recall the concept of a norm of a vector X and a matrix A.

Vector norm :-

The non-negative quantity $\|x\|$ is a measure of the size or length of a vector satisfying.

i) $\|X\| > 0$, for $x \neq 0$ & $\|x\| = 0$

ii) $\|cX\| = |c|\,\|x\|$, for arbitrary complex number c.

(iii) $\|x+y\| \leq \|x\| + \|y\|$

The most commoncy used norms are

(i) Absolute norm (l_1 norm)

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

(ii) Euclidean norm.

$$\|x\|_2 = (x, x)^{1/2} = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}.$$

(iii) maximum norm (l_∞ norm)

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

matrix norm :-

The matrix norm $\|A\|$ is a non-negative number which satisfies the properties.

(i) $\|A\| > 0$ if $A \neq 0$ & $\|0\| = 0$

(ii) $\|cA\| = |c|\|A\|$, for an arbitrary complex number c.

(iii) $\|A+B\| \leq \|A\| + \|B\|$.

(iv) $\|AB\| \leq \|A\| \|B\|$

The most commonly used norms are.

i) frobenius or Euclidean norm :-

$$F(A) = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}$$

(ii) maximum norm

$$\|A\| = \|A\|_\infty$$

$= \max_i \sum_k |a_{ik}|$ (maximum absolute row sum)

$\|A\| = \|A\|_1$

$= \max_k \sum_i |a_{ik}|$ (maximum absolute column sum)

(iii) Hilbert norm (or) Spectral norm :-

$\|A\|_2 = \sqrt{\lambda}$, where $\lambda = \rho(A^* A)$

If A is Hermitian or real and symmetric,

then $\lambda = \rho(A^2) = \rho^2(A)$

So that $\|A\|_2 = \rho(A)$.

The matrix norm must be consistent with the vector norm that we are using for any vector x and matrix A.

ie., $\|Ax\| = \|A\| \, \|x\|$.

It may be verified that the norm

$$\|A\| = \max_i \sum_{j=1}^{n} |a_{ij}|.$$

is consistent with maximum norm $\|x\|$.

Error Estimate :-

From (2), we obtain

$$\|\hat{x} - x\| \leq \|(A + \delta A)^{-1} - A^{-1}\| \, \|b\| + \|(A + \delta A)^{-1}\| \, \|\delta b\|$$

But $\|(A+\delta A)^{-1}\| = \|(A+\delta A)^{-1} - A^{-1} + A^{-1}\|$

$$\le \|(A+\delta A)^{-1} - A^{-1}\| + \|A^{-1}\|$$

$$\|(A+\delta A)^{-1} - A^{-1}\| = \|A^{-1} - (A+\delta A)^{-1}\|$$

$$\le \|A^{-1}\| \, \| I - (I + A^{-1}\delta A)^{-1}\|.$$

$$\le \|A^{-1}\| \, \|(I + A^{-1}\delta A)^{-1}(I + A^{-1}\delta A - I)\|.$$

$$= \|A^{-1}\| \, \|(I + A^{-1}\delta A)^{-1}(A^{-1}\delta A)\|$$

$$\le \|A^{-1}\| \, \|A^{-1}\delta A\| / (1 - \|A^{-1}\delta A\|)$$

where $\|A^{-1}\delta A\|$ is assumed to be less than 1

Hence we obtain,

$$\|\hat{x} - x\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}\delta A\|} \left[\|A^{-1}\delta A\| \, \|b\| + \|\delta b\|\right] \rightarrow (*)$$

Since $\|A^{-1}\delta A\| \le \|A^{-1}\| \, \|\delta A\|$, the eqn $(*)$ may be written as.

$$\|\hat{x} - x\| \le \frac{\|A^{-1}\|}{(1 - \|A^{-1}\| \, \|\delta A\|)} \left[\|x\| \, \|\delta A\| + \|\delta b\|\right]$$

(or)

$$\frac{\|\hat{x} - x\|}{\|x\|} \le \frac{\|A^{-1}\|}{(1 - \|A^{-1}\delta A\|} \|A\| \left[\frac{\|x\| \, \|\delta A\|}{\|A\| \, \|x\|} + \frac{\|\delta b\|}{\|A\| \, \|x\|}\right]$$

(or) $$\frac{\|\hat{x} - x\|}{\|x\|} \le \frac{K(A)}{(1 - \|A^{-1}\delta A\|} \left[\frac{\|\delta b\|}{\|b\|} + \frac{\delta A}{\|A\|}\right] \rightarrow (*$$

where the quantity $k(A) = \|A^{-1}\| \|A\|$ is called the condition no. of the matrix A and is denoted by cond (A). The left-hand side in (* *) gives the overall relative error in X.

The first term inside the brackets on the right hand is the overall relative error in b, the Second term is the relative error in A.

If there is no error in b then $\|\delta b\| = 0$

$$\frac{\|\hat{x} - x\|}{\|x\|} \leq \frac{K(A)}{(1 - \|A^{-1}\delta A\|)} \left[\frac{\|\delta A\|}{\|A\|}\right]$$

If there no errors A then $\|\delta A\| = 0$ gives

$$\frac{\|\hat{x} - x\|}{\|x\|} \leq K(A) \left[\frac{\|\delta b\|}{\|b\|}\right]$$

If K(A) is small changes in A or b produce only small change in x if K(A) is large then small relative change A or b will produce large relative changes in x and the ch system of eqn $Ax = b$ is said to be conditioned. If K(A) is near unity then the system is well conditional.

$$K(A) = \|A\|_2 \|A^{-1}\|_2 = \sqrt{\frac{\lambda}{\mu}}$$

where λ & μ are the largest and smallest eigen values in modulus of A^*A. If A is

Hermitian or real and symmetric, we have

$$K(A) = \frac{\lambda^*}{\mu^*}$$

where λ^* and μ^* are the largest and the smallest eigen values in modulus of A.

Eg., 3.18

Determine the euclidean and the maximum absolute row sum norms of the matrix.

$$A = \begin{bmatrix} 1 & 7 & 4 \\ 4 & -4 & 9 \\ 12 & -1 & 3 \end{bmatrix}$$

Soln: we have,

Euclidean norm $= F(A) = \sqrt{\sum_{i,j=1}^{3} |a_{ij}|^2}$

Therefore $[F(A)]^2 = 1+49+16+16+81+144+1+9+16$

$= 333$

$|a_{11}|+|a_{12}|+|a_{13}| = 12$

$|a_{21}|+|a_{22}|+|a_{23}| = 17$

$|a_{31}|+|a_{32}|+|a_{33}| = 16$

$[F(A)] = \sqrt{333} \simeq 18.25$

Maximum absolute row sum norm $= \max \sum_{K=1}^{3} |a_{iK}|$

$= \max\{12, 17, 16\}$

$= 17$

Find the condition number of the system

$$\begin{bmatrix} 2.1 & 1.8 \\ 6.2 & 5.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2.1 \\ 6.2 \end{bmatrix}$$

Soln: Using the spectral norm, we have

$$A^*A = A^TA = \begin{bmatrix} 42.85 & 36.64 \\ 36.64 & 31.33 \end{bmatrix} \qquad |A - \lambda I| = 0$$

$$\begin{vmatrix} 42.85\ \lambda & 36.64 \\ 36.64 & 31.33\ \lambda \end{vmatrix}$$

The eigen values of A^*A are the soln of

$\lambda^2 - 74.18\lambda + 0.0009 = 0$

we find, $\lambda_1 = 74.17998787$ & $\lambda_2 = 0.000012132$

The condition number is

$$K(A) = \sqrt{\frac{\lambda_1}{\lambda_2}} = 2472.73$$

Hence this system of eqns is highly ill conditioned and is very sensitive to round-off errors.

Example 3.20:

Determine the condition number of the matrix $A = \begin{bmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{bmatrix}$

Soln: Using the maximum absolute row sum norm we have,

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$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

$$\operatorname{adj} = \begin{vmatrix} -8 & 44 & -72 \\ 32 & -72 & 102 \\ -72 & 192 & -200 \end{vmatrix}$$

$$= -81[-8] - 44(32) - 17(-)$$

$$|A| = 64$$

$$A^{-1} = \frac{1}{8}\begin{bmatrix} -81 & 44 & -17 \\ 44 & -56 & 20 \\ -17 & 20 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 81/8 & -44/8 & 17/8 \\ -44/8 & 56/8 & -20/8 \\ 17/8 & -20/8 & -7/8 \end{bmatrix}$$

$\|A\|_\infty$ = maximum absolute row sum norm for A

$= \max\{14, 29, 50\}$

$= 50$

$$\|A^{-1}\|_\infty = \text{maximum}\left\{\left(\frac{81}{8} + \frac{44}{8} + \frac{17}{8}\right), \left(\frac{44}{8} + \frac{56}{8} + \frac{20}{8}\right), \left(\frac{17}{8} + \frac{20}{8} + \frac{7}{8}\right)\right\}$$

$$= \max\left\{\frac{92}{8}, 15, 44/8\right\}$$

$= 15$

Therefore $K(A) = \|A\|_\infty \|A^{-1}\|_\infty$

$K(A) = 750$

Iterative Improvement of the soln :-

we assume that the system $(Ax = b)$ well-conditional.

The error eqn $\hat{x} - y = \{(A + \delta A)^{-1} - A^{-1}\}$ for $\delta A = 0$ (becomes 3.62)

$A\,\delta x = \delta b$

where $\delta b = b - A\hat{x}$ is called the residual vector, The approximate soln $\hat{x}$ may be improved as follows

$$\delta x = A^{-1} \delta b$$

$$x = \hat{x} + \delta x \longrightarrow (1$$

This process of using errors eqn (1 may be carried out as far as necessary to obtain soln values to the desired actually.

Iteration Methods:-

A general linear iterative method for the soln of the system of eqns $Ax = b$ may be defined in the form

$$x^{(k+1)} = H x^{(k)} + c, \quad k = 0, 1, 2 \ldots \longrightarrow (1$$

where $x^{(k+1)}$ & $x^{(k)}$ are the approximation for x at the $(k+1)^{th}$ and k^{th} iteration respectively. H is called the iteration matrix depending on A and c is a column vector. In the limiting case when $k \to \infty$, $x^{(k)}$ converges to the exact solns.

$$x = A^{-1} b \longrightarrow (2$$

and the iteration eqn (1) becomes by substitution from (2

$$A^{-1} b = H A^{-1} b + c \longrightarrow (3$$

from (3, the column vector c is gvn by

$$c = (I - H) A^{-1} b \longrightarrow (4$$

We now determine the iteration matrix H and the column vector c for a few well known as iteration methods.

Jacobi Iteration Method :-

We assume that the quantities a_{ii} in $Ax = b$ are pivot elements. The eqns $Ax = b$ may be written as

$$a_{11}x_1 = (a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n) + b_1$$

$$a_{22}x_2 = (a_{21}x_1 + a_{23}x_3 + \dots + a_{2n}x_n) + b_2$$

$$\vdots$$

$$a_{nn}x_n = (a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn-1}x_{n-1}) + b_n$$

The Jacobi iteration method for Gauss-Jacobi iteration method may now be defined as ;

$$x_1^{(k+1)} = \frac{-1}{a_{11}}\left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \dots + a_{1n}x_n^{(k)} - b_1\right)$$

$$x_2^{(k+1)} = \frac{-1}{a_{22}}\left(a_{21}x_1^{(k)} + a_{23}x_3^{(k)} + \dots + a_{2n}x_n^{(k)} - b_2\right)$$

$$\vdots \qquad k = 0, 1, 2 \dots$$

$$x_n^{(k+1)} = \frac{-1}{a_{nn}}\left(a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \dots + a_{nn-1}x_{n-1}^{(k)} - b_n\right) \longrightarrow (2$$

Since, we replace the complete vector $x^{(k)}$ in the right side of (2 at the end of each iteration, this method is also called the method of Simultaneous displacement.

In matrix form, the method can be written

as $x^{(k+1)} = -D^{-1}(L+U)\,x^{(k)} + D^{-1}b.$

$= Hx^{(k)} + c, \quad k = 0,1,2 \ldots$

where $H = -D^{-1}(L+U)$, $c = D^{-1}bx$.

L and U are respectively lower and upper triangular matrices with zero diagonal entries, D is the diagonal matrix such that $A = L + D + U$.

Eqn (3 can alternatively be written as,

$x^{(k+1)} = x^{(k)} - [I + D^{-1}(L+U)]\,x^{(k)} + D^{-1}b.$

$= x^{(k)} - D^{-1}[D + L + U]\,x^{(k)} + D^{-1}b.$

$= x^{(k)} + D^{-1}[b - Ax^{(k)}]$ (or)

$v^{(k)} = D^{-1}r^{(k)} \longrightarrow (4)$

where $v^{(k)} = x^{(k+1)} - x^{(k)}$ is the error in the approximation and $r^{(k)} = b - Ax^{(k)}$ is the residual vector.

We may rewrite the above eqn as

$Dv^{(k)} = r^{(k)}$

we solve for $v^{(k)}$ & find $x^{(k+1)} = x^{(k)} + v^{(k)}$

These eqns describe the Jacobi iteration method in an error format.

Eg., 3.21

Solve the system of eqns.

$4x_1 + x_2 + x_3 = 2$

$x_1 + 5x_2 + 2x_3 = -6$

$x_1 + 2x_2 + 3x_3 = -4.$

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix}$$

Soln: Using the Jacobi iteration method given in eqn $X^{(K+1)} = HX^{(K)} + C$, $K = 0, 1, 2 \ldots$ and its error format given in eqn $V^{(K)} = D^{-1}r^{(K)}$

Take the initial approximation as, $x^{(0)} = [0.5, -0.5, -0.5]^T$ and perform three iterations in each case. The exact soln is $x_1 = 1, x_2 = -1, x_3 = -1$.

i) we have,

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}; D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}, U = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H = D^{-1}[L+U]$$

$$= -\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1/4 & -1/4 \\ -1/5 & 0 & -2/5 \\ -1/3 & -2/3 & 0 \end{bmatrix}$$

$$\therefore D^{-1}b = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -6/5 \\ -4/3 \end{bmatrix}$$

$\therefore$ Jacobi iteration method

$X^{(k+1)} = H X^{(k)} + C$, $k = 0, 1, 2 \ldots\ldots$ becomes

$$X^{(k+1)} = \begin{bmatrix} 0 & -1/4 & -1/4 \\ -1/5 & 0 & -2/5 \\ -1/3 & -2/3 & 0 \end{bmatrix} X^{(k)} + \begin{bmatrix} 1/2 \\ -6/5 \\ -4/3 \end{bmatrix}.$$

starting with $X^{(0)} = [0.5, -0.5, -0.5]$, we obtain

put $k = 0$ in above thm.

$$X^{(1)} = \begin{bmatrix} 0 & -1/4 & -1/4 \\ -1/5 & 0 & -2/5 \\ -1/3 & -2/3 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -6/5 \\ -4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 0.125 + 0.125 \\ -0.1 + 0 + 0.2 \\ -0.16667 + 0.333 + 0 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -6/5 \\ -4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 \\ -1.1 \\ -1.1667 \end{bmatrix}$$

$$X^{(2)} = \begin{bmatrix} 0 & -1/4 & -1/4 \\ -1/5 & 0 & -2/5 \\ -1/3 & -2/3 & 0 \end{bmatrix} \begin{bmatrix} 0.75 \\ -1.1 \\ -1.1667 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -6/5 \\ -4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 0.2750 + 0.2917 \\ -0.15 + 0 + 0.4667 \\ -0.25 + 0.7333 + 0 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -6/5 \\ -4/3 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} 1.0667 \\ -0.8833 \\ -0.85 \end{bmatrix}$$

$$x^{(3)} = \begin{bmatrix} 0 & -1/4 & -1/4 \\ -1/5 & 0 & -2/5 \\ -1/3 & -2/3 & 0 \end{bmatrix} \begin{bmatrix} 1.0667 \\ -0.8833 \\ -0.85 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -6/5 \\ -4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 0.2208 + 0.2125 \\ -0.2133 + 0 + 0.34 \\ -0.3556 + 0.5889 + 0 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -6/5 \\ -4/3 \end{bmatrix}$$

$$x^{(3)} = \begin{bmatrix} 0.9333 \\ -1.0733 \\ -1.1 \end{bmatrix}$$

Alternatively we may write directly

$$x_1^{(k+1)} = \frac{1}{4}\left[2 - x_2^{(k)} - x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{5}\left[-6 - x_1^{(k)} - 2x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{3}\left[-4 - x_1^{(k)} - 2x_2^{(k)}\right]$$

starting with $x_1^{(0)} = 0.5$, $x_2^{(0)} = -0.5$, $x_3^{(0)} = -0.5$

$$x^{(1)} = \begin{bmatrix} 0.75 \\ -1.1 \\ -1.1667 \end{bmatrix} \quad x^{(2)} = \begin{bmatrix} 1.0667 \\ -0.8833 \\ -0.85 \end{bmatrix} \quad x^{(3)} = \begin{bmatrix} 0.9333 \\ -1.0733 \\ -1.1 \end{bmatrix}$$

using eqn (4 in before theory, $V^{(k)} = D\,r^{(k)}$

$x^{(0)} = [0.5 \; -0.5 \; -0.5]^T$

$V^k = D^{-1} r^{(k)}$ $\qquad V^k = D\, r^k$

$r^k = b - Ax^{(k)}$ $\qquad r^k = b - Ax^{(k)}$

$x^{(k+1)} = x^{(k)} + V^{(k)}$ $\qquad x^{(k+1)} = x^k + v^k$

put $k = 0$

$$= \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \end{bmatrix} => \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$$

$$v^{(0)} = D^{-1} r^{(0)} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} => \begin{bmatrix} 1/4 \\ -3/5 \\ -2/3 \end{bmatrix}$$

$$x^{(1)} = x^{(0)} + v^{(0)} = \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \end{bmatrix} + \begin{bmatrix} 1/4 \\ -3/5 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 0.75 \\ -1.1 \\ -1.1667 \end{bmatrix}$$

put $K=1$

$$r^{(1)} = b - A x^{(1)} = \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0.75 \\ -1.1 \\ -1.667 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 0.333 \\ -7.0834 \\ -4.9501 \end{bmatrix}$$

$$r^{(1)} = \begin{bmatrix} 1.2667 \\ 1.0834 \\ 0.9501 \end{bmatrix}$$

$$v^{(1)} = D^{-1} r^{(1)} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1.2667 \\ 1.0834 \\ 0.9501 \end{bmatrix}$$

$$v^{(1)} = \begin{bmatrix} 0.3167 \\ 0.2167 \\ 0.3167 \end{bmatrix}$$

$$x^{(2)} = x^{(1)} + v^{(1)} = \begin{bmatrix} 0.75 \\ -1.1 \\ -1.1667 \end{bmatrix} + \begin{bmatrix} 0.3167 \\ 0.2167 \\ 0.3167 \end{bmatrix} = \begin{bmatrix} 1.0667 \\ -0.8833 \\ -0.850 \end{bmatrix}$$

put $K=2$

$$r^{(2)} = b - A x^{(2)} = \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1.0667 \\ -0.8833 \\ -0.85 \end{bmatrix}$$

$$r^{(2)} = \begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} - \begin{bmatrix} 2.5335 \\ -5.0498 \\ -3.2499 \end{bmatrix} \Rightarrow \begin{bmatrix} -0.5335 \\ -0.9502 \\ -0.7501 \end{bmatrix}$$

$$v^{(2)} = D^{-1} r^{(2)} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} -0.5335 \\ -0.9502 \\ -0.7501 \end{bmatrix}$$

$$= \begin{bmatrix} -0.13338 \\ -0.19004 \\ -0.25003 \end{bmatrix}$$

$$x^{(3)} = x^{(2)} + v^{(2)} = \begin{bmatrix} 1.0667 \\ 0.8833 \\ -0.85 \end{bmatrix} + \begin{bmatrix} -0.1338 \\ -0.19009 \\ -0.25003 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9333 \\ -1.0733 \\ -1.1 \end{bmatrix}$$

Note that we obtain the same result from both the techniques.

Gauss- seidal Iteration Method :-

we now use the R.H.S of eqn (2, all the available values from the present iteration we write the Gauss- seidal method as,

$$x_1^{(K+1)} = -\frac{1}{a_{11}} (a_{12} x_2^{(K)} + a_{13} x_3^{(K)} + \dots + a_{1n} x_n^{(K)}) + \frac{b_1}{a_{11}}$$

$$x_2^{(K+1)} = -\frac{1}{a_{22}} (a_{21} x_1^{(K+1)} + a_{23} x_3^{(K)} + \dots + a_{2n} x_n^{(K)}) + \frac{b_2}{a_{22}}$$

$$\vdots$$

$$x_n^{(K+1)} = -\frac{1}{a_{nn}} (a_{n1} x_1^{(K+1)} + a_{n2} x_2^{(K+1)} + \dots + a_{nn-1} x_{n-1}^{(K+1)}) + \frac{b_n}{a_{nn}}$$

Which may be rearranged in the

form,

$$a_{11} x^{(k+1)} = -\sum_{i=2}^{n} a_{1i} x_i^{(k)} + b_1$$

$$a_{21} x_1^{(k+1)} + a_{22} x_2^{(k+1)} = -\sum_{i=3}^{n} a_{2i} x_i^{(k)} + b_2 .$$

$$\vdots$$

$$a_{n1} x_1^{(k+1)} + \dots a_{nn} x_n^{(k+1)} = b_n \longrightarrow (5$$

Since we replace the vector x_3^k in the right side of eqn (2 element by element, this method is called the method of successive displacement. In matrix notation, eqn (5 becomes

$$(D+L) X^{(k+1)} = -U X^{(k)} + b.$$

$$\text{(or)} \quad X^{(k+1)} = -(D+L)^{-1} U X^{(k)} + (D+L)^{-1} b$$

$$= H X^{(k)} + c, \quad k = 0, 1, 2 \dots \rightarrow (6$$

where $H = -(D+L)^{-1} U$ & $c = (D+L)^{-1} b$.

eqn (6 can alternatively we written as

$$X^{(k+1)} = X^{(k)} - [I + (D+L)^{-1} U] X^{(k)} + (D+L)^{-1} b.$$

$$= X^{(k)} - (D+L)^{-1} (D+L+U) X^{(k)} + (D+L)^{-1} b$$

$$= X^{(k)} - (D+L)^{-1} A X^{(k)} + (D+L)^{-1} b.$$

$$= X^{(k)} + (D+L)^{-1} (b - A X^{(k)})$$

we write $v^{(k)} = (D+L)^{-1} r^{(k)}$.

where $v^{(k)} = x^{(k+1)} - x^{(k)}$ & $r^{(k)} = b - A x^{(k)}$

is the residual vector.

We may rewritten the above eqn as

$(D+L) v^{(k)} = r^{(k)}$. $\rightarrow$ (7

And solve for $v^{(k)}$ by forward substitutio

The soln is then found from

$$x^{(k+1)} = x^{(k)} + v^{(k)}.$$

These eqns described the gauss seidal method.

1. Solve the System of eqns

$2x_1 - x_2 + 0x_3 = 7$

$-x_1 + 2x_2 - x_3 = 1$

$0x_1 - x_2 + 2x_3 = 1$

Using the gauss seidal method in eqns (6 and its error format gvn in eqns (7 taking the initial approximation as $x^{(0)} = 0$ and perform 3 iterations

Soln: Given that.

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(D+L) = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$(D+L)^{-1} = \frac{1}{|D+L|} \operatorname{adj}(D+L).$$

$$\operatorname{adj}(D+L) = \begin{pmatrix} +\begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ -1 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} \\ +\begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} & +\begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix} \end{pmatrix}^T$$

$$= \cancel{5} + 2 + 1.$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ 0 & 4 & +2 \\ 0 & 0 & 4 \end{bmatrix}^T$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

$$|D+L| = 8$$

$$(D+L)^{-1} = \frac{1}{|D+L|} \operatorname{adj}(D+L)$$

$$= \frac{1}{8}\begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

$$(D+L)^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix}$$

$$-(D+L)^{-1}U = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix}$$

$$(D+L)^{-1}b = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 9/4 \\ 13/8 \end{bmatrix}$$

$\therefore$ we obtain the iteration scheme.

$$x^{(k+1)} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{pmatrix} \cdot x^{(k)} + \begin{pmatrix} 7/2 \\ 9/4 \\ 13/8 \end{pmatrix}$$

starting with zero initial vector we get,

put $k=0$

$$X^{(1)} = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 7/2 \\ 9/4 \\ 13/8 \end{bmatrix}$$

$$= \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix}$$

0.2812
0.4062

$k=2$

$$X^{(2)} = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix} \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix} + \begin{bmatrix} 7/2 \\ 9/4 \\ 13/8 \end{bmatrix}$$

$$= \begin{bmatrix} 1.125 \\ 1.375 \\ 0.687 \end{bmatrix} + \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix}$$

$$= \begin{bmatrix} 4.625 \\ 3.625 \\ 2.3125 \end{bmatrix}$$

$k=3,$

$$X^{(3)} = \begin{bmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{bmatrix} \begin{bmatrix} 4.625 \\ 3.625 \\ 2.3125 \end{bmatrix} + \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix}$$

$$= \begin{bmatrix} 1.812 \\ 2.0625 \\ 1.03125 \end{bmatrix} + \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix}$$

$r^{(K)} = b - A x^{(K)}$

$$= \begin{bmatrix} 5.3125 \\ 4.3125 \\ 2.6562 \end{bmatrix}$$

$v^{(K)} = (D+L)^{-1} r^{(K)}$

$x^{(K+1)} = x^{(K)} + v^{(K)}$

(iv) Euler Error formula,

$$r^{(K)} = b - A x^{(K)}$$

$$v^{(K)} = (D+L)^{-1} r^{(K)}$$

$$x^{(K+1)} = x^{(K)} + v^{(K)}.$$

put $K=0$ then

$$r^{(0)} = b - A x^{(0)}$$

$$= \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r^{(0)} = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$$

$$V^{(0)} = (D+L)^{-1} r^{(0)}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix}$$

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$$X^{(1)} = X^{(0)} + V^{(0)}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix}$$

put $K=1$ then

$$r^{(1)} = b - AX^{(1)}$$

$$= \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 4.75 \\ -0.625 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.25 \\ 1.625 \\ 0 \end{bmatrix}$$

$$V^{(1)} = (D+L)^{-1} r^{(1)}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 2.25 \\ 1.625 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.125 \\ 1.375 \\ 0.6875 \end{bmatrix}$$

$$X^{(2)} = \begin{bmatrix} 3.5 \\ 2.25 \\ 1.625 \end{bmatrix} + \begin{bmatrix} 1.125 \\ 1.375 \\ 0.6875 \end{bmatrix} = \begin{bmatrix} 4.625 \\ 3.625 \\ 2.312 \end{bmatrix}$$

Put $k=2$ then

$$r^{(2)} = b - Ax^{(2)}$$

$$= \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 4.625 \\ 3.625 \\ 2.3125 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 5.625 \\ 0.3125 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.375 \\ 0.6875 \\ 0 \end{bmatrix}$$

$$v^{(2)} = (D+L)^{-1} r^{(2)}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1.375 \\ 0.6875 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6875 \\ 0.6875 \\ 0.34375 \end{bmatrix}$$

$$x^{(3)} = \begin{bmatrix} 4.625 \\ 3.625 \\ 2.3125 \end{bmatrix} + \begin{bmatrix} 0.6875 \\ 0.6875 \\ 0.34375 \end{bmatrix} = \begin{bmatrix} 5.3125 \\ 4.3125 \\ 2.6563 \end{bmatrix}$$

//.

Successive over Relaxation Method :-

This is the generalization of gauss seidal method. The A system of eqn is symmetric. Define axillary vector $\hat{x}$

$$\hat{x}^{(K+1)} = D^{-1}L\,x^{(K+1)} - D^{-1}U x^{(K)} + D^{-1}b \longrightarrow (1$$

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The final soln is

$$x^{(k+1)} = x^{(k)} + w\,(\hat{x}^{(k+1)} - x^{(k)}).$$

(or) $$x^{(k+1)} = (1-w)\,x^{(k)} + w\,\hat{x}^{(k+1)} \longrightarrow (2$$

substitute (2 in (1 And simplifying we get,

$$x^{(k+1)} = (1-w)\,x^{(k)} - wD^{-1}L\,x^{(k+1)} - wD^{-1}U\,x^{(k)} + wD^{-1}b$$

$$x^{(k+1)} + wD^{-1}L\,x^{(k+1)} = (1-w)\,x^{(k)} - wD^{-1}U\,x^{(k)} + wD^{-1}b.$$

$$x^{(k+1)}(1 + wD^{-1}L) = [(1-w) - wD^{-1}U]\,x^{(k)} + wD^{-1}b.$$

$$x^{(k+1)} = (D+wL)^{-1}[(1-w)D - wU]\,x^{(k)} + w(D+wL)^{-1}b$$

$$= H\,x^{(k)} + c \quad k=0,1,2\ldots \longrightarrow (3$$

where $H = (D+wL)^{-1}((1-w)D - wU)$, $c = w(D+wL)^{-1}b$.

Eqn (3 can be written as,

$$x^{(k+1)} = x^{(k)} - (D+wL)^{-1}[(D+wL) - (1-w)D + wU]\,x^{(k)} + w(D+wL)^{-1}b.$$

$$= x^{(k)} + w(D+wL)^{-1}\gamma^{(k)}.$$

where $\gamma^{(k)} = b - A\,x^{(k)}$ is the residual

$$v^{(k)} = w(D+wL)^{-1}\gamma^{(k)}.$$

$$\downarrow$$

$$(x^{(k+1)} - x^{(k)}) = w(D+wL)^{-1}\gamma^{(k)}$$

$$(D+wL)\,v^{(k)} = w\,\gamma^{(k)} \longrightarrow (4$$

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Eqn (4 describes the SOR method in its error format

$W=1$, eqn (4 reduces to the gauss seidal method. $W \rightarrow$ Relaxation parameter. $X^{(K+1)}$ is a weighted mean of $\hat{X}^{(K+1)}$ & $X^{(K)}$.

from (2 we find the weights are non-negative for $0 \leq W \leq 1$.

If $W > 1$ the method is called an over relaxation method and if $W < 1$ then it is called an under relaxation method.

Convergence Analysis of Iterative Methods :-

The eqs of the iteration method is

$X^{(K+1)} = H X^{(K)} + c, \quad K = 0,1,2 \ldots \rightarrow (I)$

we can see the difference b/w the exact soln X and then approximation $X^{(K)}$, the exact soln X will be satisfies $X = HX + C \rightarrow (5$

Subtracting (5 from (I and sub $\varepsilon^{K} = X^{(K)} - X$ we get,

$$\varepsilon^{(K+1)} = H \varepsilon^{(K)}, \quad K = 0,1,2 \ldots \longrightarrow (6)$$

we obtain $\varepsilon^{(K)} = H^{(K)} \varepsilon^{(0)}, \quad K = 0,1,2 \ldots \rightarrow (7$

where we have assumed that the iteration matrix H remains constant for each iteration

Thm 1 :-

Let A be a square matrix then

$\lim_{m\to\infty} A^m = 0$ of $\|A\| < 1$, or iff $\rho(A) < 1$.

Proof :-

If $\|A\| < 1$ then $\|A^m\| \leq \|A\|^m$.

& $\|\lim_{m\to\infty} A^m\| \leq \|\lim_{m\to\infty} A\|^m = 0$.

Assume that all the eigen values of A are distinct then there exist a similarity transformation S such that $A = S^{-1} D S$ where D is the diagonal matrix having the eigen values of A on the diagonal. Therefore

$A^m = S^{-1} D^m S$.

$$D^m = \begin{bmatrix} \lambda_1^m & \cdots & \cdots & 0 \\ 0 & \lambda_2^m & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & \lambda_n^m \end{bmatrix}$$

Obviously $\lim_{m\to\infty} A^m = 0$ iff $|\lambda_i| < 1$ i.e., $\rho(A) < 1$.

Thm 2 :-

The infinite series $I + A + A^2 + \cdots$ cgs iff $\lim_{m\to\infty} A^m = 0$. This series cgs to $(I - A)^{-1}$.

Proof:

If $\lim_{m\to\infty} A^m = 0$

By thm 1, $\rho(A) < 1$

Hence $|I - A| \neq 0$ & $(I-A)^{-1}$ exists

consider $(I + A + A^2 + \dots + A^m)(I - A) = I - A^{m+1}$

$(I+A+A^2 \dots + A^m)(I-A)(I-A)^{-1} = (I - A^{m+1})(I-A)^{-1}$.

$\times$ by $(I-A)^{-1}$ on both sides,

$(I + A + A^2 + \dots + A^m) = (I - A^{m+1})(I-A)^{-1}$.

As $m \to \infty$ we get,

$I + A + A^2 + \dots = (I-A)^{-1}$.

Thrm 3:

no eigen values of a matrix A, x is the norm of a matrix ie., $\|A\| \geq \rho(A)$

Proof:

$Ax = \lambda x$

$\|\lambda x\| = \|Ax\| \leq \|A\| \|x\|$

(or)

$|\lambda| \|x\| = |A| \|x\| \leq \|A\| \|x\|$, $\|x\| \neq 0$.

$|\lambda| \|x\| \leq \|A\| \|x\|$ $\therefore |\lambda| \leq \|A\|$

(or) $\|A\| \geq \lambda$

$|\lambda| \leq \|A\|$

$|\lambda| \leq \rho(A) \Rightarrow \|A\| \geq \rho(A)$

Thm 4 :- The Iteration method of the form $x^{(k+1)} = H x^{(k)} + c \quad k = 0, 1, 2 \ldots$ for the soln of thm converges to the exact soln for any initial vector if $\|H\| < 1$.

proof :-

without loss of generality, we take initial vector $x^{(0)} = 0$

$$x^{(1)} = c$$

$$x^{(2)} = Hx^{(1)} + c = (H+I)c$$

$$x^{(3)} = H x^{(2)} + c = (H^2 + H + I)c$$

$$\vdots$$

$$x^{(k+1)} = (H^k + H^{k-1} + \ldots + H + I)c$$

$$\lim_{k\to\infty} x^{(k+1)} = \lim_{k\to\infty} (H^k + H^{k-1} + \ldots + H + I)c$$

$$= (I-H)^{-1} c \rightarrow (8)$$

If $\|H\| < 1$ (or) iff $\rho(H) < 1$. (by thm 1)

In the case of Jacobi method, we have (by thm 2)

$$(I-H)^{-1} c = [I + D^{-1}(L+U)]^{-1} D^{-1} b.$$

$$= [D^{-1}(D+L+U)]^{-1} D^{-1} b$$

$$\Rightarrow [D]^{-1}(D+L+U)^{-1} D^{-1} b]$$

$$\Rightarrow (D+L+U)^{-1} D D^{-1} b = A^{-1} b = x.$$

Thrm 5 :-

A necessary and sufficient condition for converges of an iterative method of the form $X^{(K+1)} = HX^{(K)} + C$, $K = 0, 1, \ldots$ is that the eigen values of the iteration matrix satisfy $|\lambda_i(H)| < 1$, $i = 1, \ldots n$.

Proof :

we prove that the result for the case when the iteration matrix H has n independent eigen vectors $x_1, x_2, \ldots x_n$ with eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$ respectively.

The error vector $\varepsilon^{(0)}$ can be written as

$$\varepsilon^{(0)} = c_1 x_1 + c_2 x_2 \ldots + c_n x_n.$$

using $\varepsilon^{(K)} = H^K \varepsilon^{(0)}$, $K = 0, 1, 2 \ldots$ we get

$$\varepsilon^{(K)} = c_1 \lambda_1^K x_1 + c_2 \lambda_2^K x_2 + \ldots + c_n \lambda_n^K x_n \longrightarrow (9$$

i) Necessity :-

If $\lim_{K \to \infty} \varepsilon^{(K)} = 0$ for any arbitrary initial vector x^0 and thus for arbitrary error vector $\varepsilon^{(0)}$ then by eqn (9 the magnitudes of the eigen values $|\lambda_i|$, $i = 1, 2, \ldots n$ must be < unity

ii) sufficiency :- $\because$ $\times$ 2m

for $|\lambda_i| < 1$, $i = 1, 2, \ldots n$

The convergence of $\varepsilon^{(k)}$ towards the zero vector follows from eqn (9.1).

Def: (iii) The rates of convergences of an iterative method is given by $v = -\log_{10}[\rho(H)]$ (or) $v = -\ln[\rho(H)] \longrightarrow (10$ where $\rho(H)$ is the spectral radius of H.

Thrm 6

If A is a strictly diagonally dominant matrix then the Jacobi iteration scheme converges for any initial starting vectors.

Proof :-

The Jacobi iteration scheme is given by,

$$X^{(k+1)} = -D^{-1}(L+U)X^{(k)} + D^{-1}b$$

$$= -D^{-1}(A-D)X^{(k)} + D^{-1}b.$$

$$= (I - D^{-1}A)X^{(k)} + D^{-1}b.$$

The iteration scheme will be converges

By thrm 4,

$\|I - D^{-1}A\| < 1 \longrightarrow (11$

Using absolute row sum norm, we have

eqn (11 as $\frac{1}{|a_{ii}|} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| < 1 \; \forall i,$

which is true.

Since the matrix A is strictly diagonally dominant

Thm:

If A is a strictly diagonally dominant matrix then the gauss seidal iteration scheme converges for any initial starting vector.

proof:

The Gauss seidal iteration scheme is grn by,

$$X^{(K+1)} = -(D+L)^{-1} V X^{(K)} + (D+L)^{-1} b.$$

$$= -(D+L)^{-1}[A-(D+L)] X^{(K)} + (D+L)^{-1} b.$$

$$= [I-(D+L)^{-1}A] X^{(K)} + (D+L)^{-1} b.$$

Above the iteration scheme will be convergent if $\rho[I-(D+L)^{-1}A] < 1$

Let the λ be an eigen value of $I-(D+L)^{-1}A$

$$\therefore (I-(D+L)^{-1}A)X = \lambda X$$

$$(D+L)X - AX = \lambda(D+L)X.$$

(or) $$-\sum_{j=1}^{n} a_{ij} x_j = \lambda \sum_{j=1}^{i} a_{ij} \quad 1 \le i \le n.$$

(or) $$\lambda a_{ii} X_i = -\sum_{j=i+1}^{n} a_{ij} x_j - \lambda \sum_{j=1}^{i-1} a_{ij} x_j$$

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(or) $[\lambda a_{ii} x_i] \leq \sum_{j=i+1}^{n} |a_{ij}||x_j| + |\lambda| \sum_{j=1}^{i-1} |a_{ij}||x_j|$

Since x is an eigen vector, $x \neq 0$ → (12

without loss of generality, we assume that

$\|x\|_\infty = 1$

choosen an index i such that

$|x_i| = 1$ & $|x_j| \leq 1 \quad \forall j \neq i$

we obtain from (12 $|\lambda||a_{ii}| \leq \sum_{j=i+1}^{n} |a_{ij}| + |\lambda| \sum_{j=1}^{i-1} |a_{ij}|$

(or) $|\lambda| \left[|a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}| \right] \leq \sum_{j=i+1}^{n} |a_{ij}|$

$$|\lambda| \leq \frac{\sum_{j=i+1}^{n} |a_{ij}|}{|a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}|} < 1.$$

Optimal Relaxation parameter for SOR Method

$x^{(k+1)} = H x^{(k)} + c, \quad k = 0, 1, 2, \ldots$

$x^{(k+1)} = (I + WD^{-1}L)[(1-W)I - WD^{-1}U] x^{(k)}$

$+ W(I + WD^{-1}L)^{-1}(D^{-1}b).$

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$$= (I+WR)^{-1}[(1-W)I - WC]X^{(a)} + W(I+WR)^{-1}(D^{-1}b)$$

where $R = D^{-1}L$ and $C = D^{-1}U$.

Since A has property 'A' there exist permutation matrix P such that $M = PAP^{-1}$

$$= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \rightarrow (1$$

$A_{11}, A_{22} \rightarrow$ diagonal matrices,

A and M have the same eigen values

the pbm is to determine a value of W

$W = W$ optimal such that $\rho(H)$ is minimized.

It is sufficient to consider eigen values of M.

Without loss of generality assume A is in the form (1, The Jacobi method for this new system is

$$X^{(a+1)} = D^{-1}(L+V)X^{(a)} + D^{-1}b.$$

$$= -(R+C)X^{(a)} + D^{-1}b.$$

$$= BX^{(a)} + D^{-1}b.$$

where $B = -(R+C)$

Note that $\mu I - B$ is of the form

$$\mu I - B = \begin{pmatrix} \mu I_1 & \bar{A}_{12} \\ \bar{A}_{21} & \mu I_2 \end{pmatrix}$$

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since B is of the form,

$$B = \begin{pmatrix} N_1 & -\bar{A}_{12} \\ -\bar{A}_{21} & N_2 \end{pmatrix}$$

where $\bar{A}_{12} = A_{11}^{-1} A_{12}$

& $\bar{A}_{21} = \bar{A}_{22} A_{21}$

where I_1, I_2, N_1, N_2 are identity & null matrices respectively of required orders. The characteristic eqns of B is

$$|\mu I - B| = \begin{vmatrix} \mu I_1 & \bar{A}_{12} \\ \bar{A}_{21} & \mu I_2 \end{vmatrix}$$

The following properties of this determinant are,

i) the characteristic eqn of the form

$$\mu^n + a_1 \mu^{n-2} + a_2 \mu^{n-4} + \dots = 0$$

ii) If all the elements of $\bar{A}_{12}$ are multiplied by a factor K and all elements of $\bar{A}_{21}$ are divided by the same factor K then the value of the determinant is unchanged.

Let μ be an eigen value of B and λ be an eigen value of $H = H_{SOR}$

Then $|H - \lambda I| = 0$

$$|(I-WR)^{-1}[(I-W)I - WC] - \lambda I| = 0.$$

$$|(I-WR)^{-1}|\,|(I-W)I - WC - \lambda(I+WR)| = 0$$

$$|(1-W)I - WC - \lambda(I+WR)| = 0$$

Since $|(I-WR)^{-1}| \neq 0$ we have,

$$|W(C+\lambda R) + (\lambda+W-1)I| = 0$$

If we divide R by $\lambda^{1/2}$ & multiply C by $\lambda^{1/2}$ then the value of the above determinant is unchanged.

$$\left|(C\lambda^{1/2} + \lambda^{1/2} R) + \frac{\lambda+W-1}{W} I\right| = 0$$

$$\left|(C+R) + \frac{\lambda+W-1}{\lambda^{1/2}W} I\right| = 0$$

$$\left|B - \frac{\lambda+W-1}{\lambda^{1/2}W} I\right| = 0.$$

Since μ is an eigen value of B,

$$\mu = \frac{\lambda+W-1}{\lambda^{1/2}W} \rightarrow (2$$

$$\lambda - \mu W \lambda^{1/2} + (W-1) = 0$$

$$\lambda^{1/2} = \frac{1}{2}\left[\mu W \pm \sqrt{\mu^2 W^2 - 4(W-1)}\right]$$

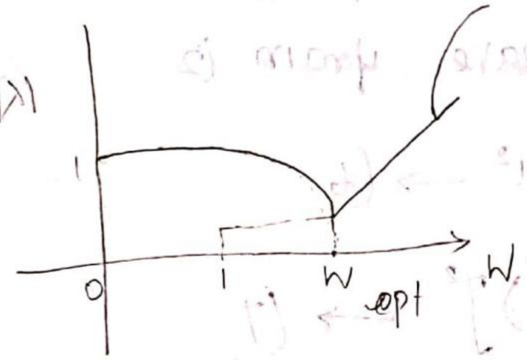
$= \frac{1}{2}\mu w \pm \frac{\mu}{2}\left[\sqrt{(w-w_1)(w-w_2)}\right]$

$w_{1,2} = \frac{2}{\mu^2}\left[1 \pm \sqrt{1-\mu^2}\right]$

For $w < w_1$, or $w > w_2$, λ is real

for $w_1 < w < w_2$, λ is complete &

$$\left|\lambda^{1/2}\right|^2 = |\lambda| = w-1$$



when $w = w_1$, $\rho(H)$ is smallest. Hence the optimal relaxation factor is gvn by.

$$w_{opt} = \frac{2}{\mu^2}\left[1-\sqrt{1-\mu^2}\right] = \frac{2}{\left[1+\sqrt{1-\mu^2}\right]} \rightarrow (3$$

when $w = w_1$, we get

$$\lambda = \frac{1}{4}\mu^2 w^2 = w-1 \longrightarrow (4$$

For congruence,

$|\lambda| < 1$.

$|w-1| < 1$ (or) $0 < w < 2$. $\rightarrow$ (5

For $0 < w < 1$ it is called over-relaxation.

The rate of convergence of the SOR scheme is $-\log(w-1)$

The relaxation on factor w_{opt}, should be rounded to the next digit for when $w \to w_{opt}^{-}$, the slope is infinite.

when $w = 1$, we have from (2

$$\mu = \lambda^{1/2} \text{ or } \lambda = \mu^2 \to (6$$

$$\rho(H_G) = [\rho(H_J)]^2 \to (7$$

$\therefore$ The rate of convergence of Gauss seidal scheme.

from (3, we find the w_{opt} is real if $|\mu| < 1$

$\therefore$ the necessary condition for the SOR method to converge is that the corresponding Jacobi iteration is convergent.

Eigen values and Eigen vectors.

consider the eigen value problem

$$AX = \lambda X \to (1$$

The eigen values of a matrix A are gen by the root of the characteristic eqn.

Let $(A-\lambda I)=0 \rightarrow$ (2

which gives the polynomial eqn $P(\lambda)=$

$P(\lambda)=(-1)^n\lambda^n+a_1\lambda^{n-1}+\dots+a_n=0 \rightarrow$ (3

$(-1)^n$ is used to give terms of the polynomial, same sign that they would have. If the polynomial was generated by expanding the determinant. the co-efficient of polynomial of eqn (3 is determined by Faddeev - Leverrier method.

i.e., $B_1 = A$ & $a_1 = t_r B_1$

$B_2 = A(B_1 - a_1 I)$ & $a_2 = \frac{1}{2} t_r B_2$

$B_3 = A(B_2 - a_2 I)$ & $a_3 = \frac{1}{3} t_r B_3$

$\vdots$

$B_K = A(B_{K-1} - a_{K-1} I)$ & $a_K = \frac{1}{K} t_r B_K$

$\vdots$

$B_n = A(B_{n-1} - a_{n-1} I)$ & $a_n = \frac{1}{n} t_r B_n$ $\rightarrow$ (4

where $t_r A = a_{11} + a_{22} + \dots + a_{nn}$.

The roots of the polynomial eqn (3.

determined by unit 1.

A non-zero vector x_i such that

$A x_i = \lambda_i x_i \rightarrow (5$ is called eigen vector

(or) characteristic vector corresponding to λ_i.

multiply eqn (5 by arbitrary constant c

and put $y_i = c x_i$

$$A y_i = \lambda_i y_i \rightarrow (6$$

pre-multiply eqn (1 by m times by A.

$$A^m X = \lambda^m X \rightarrow (7$$

λ^m is an eigen value of A^m and

x is the corresponding eigen vector

substitute (7 into (3.

$$P(A) = 0 \longrightarrow (8$$

A square matrix A satisfies its own

eqn this is known as cayley Hamilton thm

Replace the matrix A into eq 6 by the transpose

matrix A^T.

Define

$$\det(A^T - \lambda I) = \det(A - \lambda I) = 0 \rightarrow (9$$

A & A^T have the same eigen values for distinct eigen values if $u_1, u_2, u_3 \ldots u_n$ are the eigen vector of A and $v_1, v_2, \ldots v_n$ are the eigen vectors of A^T, then we have

$AU_i = \lambda_i U_i \longrightarrow (10)$

$A^T V_j = \lambda_j V_j \longrightarrow (11)$

we obtain, $V_j^T A U_i = \lambda_i V_j^T U_i \longrightarrow (12)$

taking transpose of eqn (11 and post multiply by U_i we get,

$$V_j^T A U_i = \lambda_j V_j^T U_i \longrightarrow (13)$$

subtracting (13 from (12

$$(\lambda_i - \lambda_j) V_j^T U_i = 0 \longrightarrow (14)$$

If $i \neq j$ then $\lambda_i \neq \lambda_j$ and we have

$V_j^T U_i = 0 \longrightarrow (15)$

If $i = j$ then $V_i^T U_i \neq 0$

since the length of eign vector is arbitrary we normalize them such that

$V_i^T U_i = 1 \longrightarrow (16)$

we have $V_j^T U_i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Pre-multiply by u_i^T we get

$$\lambda_i = \frac{u_i^T A u_i}{u_i^T u_i} \longrightarrow (18$$

which gives the eigen values interms of the eigen vectors.

For arbitrary u, eqn (18 is called the Rayleigh quotient.

Let A and B be two square matrices of same order.

If a non-singular matrix S can be determined such that

$$B = S^{-1} A S \longrightarrow (19$$

Then the matrices A, B are said to be similar. And eqn (19 is called a similarity transformation. The matrix S is called the similarity matrix.

From eqn (19

$$A = S B S^{-1} \longrightarrow (20$$

If λ_i is a eigen value of A and u_i is a corresponding eigen vector.

Then, $A U_i = \lambda_i U_i$

(or) $S^{-1} A U_i = \lambda S^{-1} U_i \longrightarrow (21)$

Substituting $u_i = S V_i$ in eqn (2 and use eqn (19

$$B V_i = \lambda_i V_i \longrightarrow (22)$$

$S^{-1} A S$ as same eigen values as A and its eigen vector V_i obtained from

$$V_i = S^{-1} U_i$$

where S is the matrix of eigen vectors reduces a matrix A to its diagonal form. The eigen values of A are located on the leading diagonal of this diagonal matrix.

If the eigen values of A are linearly independent then S^{-1} exist.

Suppose that the matrix A has a eigen values λ_i with eigen vectors u_i and A has an inverse A^{-1}. Then:

$$A U_i = \lambda_i U_i$$

$$A^{-1} U_i = \frac{1}{\lambda_i} U_i \longrightarrow (23)$$

The inverse matrix A^{-1} has a same

eigen vectors as A but the eigen values $1/\lambda_i$

Consider the system of eqn $\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

where a is a real constant.

i) for which values of a, the Jacobi and gauss seidal method convergences.

ii) for $a = 0.5$ find the value of w which minimizes the spectral radius of the SOR iteration matrix.

Sol^n:

The Jacobi method becomes

$$x^{(k+1)} = H\, x^{(k)} + c, \quad H = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$$

The eigen values of the Jacobi iteration method, H is given by

$$|H - \lambda I| = \left| \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} -\lambda & a \\ a & -\lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \lambda^2 - a^2 = 0$$

$$\lambda^2 = a^2$$

$$\lambda = \pm a$$

The Spectral radius of the Jacobi iteration matrix becomes,

$$\rho(H_j) = |a|.$$

$\therefore$ The condition for the convergence of the Jacobi iteration method is $|a| < 1$.

The Gauss seidal iteration method becomes,

$$\begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} X^{(K+1)} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} X^{(k)} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$X^{(K+1)} = \begin{bmatrix} 0 & a \\ 0 & a^2 \end{bmatrix} X^{(k)} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

The eigen values of the gauss seidal iteration matrix is given by,

where $H = \begin{vmatrix} 0 & a \\ 0 & a^2 \end{vmatrix}$

$$|H - \lambda I| = \begin{vmatrix} 0 & a \\ 0 & a^2 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & a \\ 0 & a^2 - \lambda \end{vmatrix} = 0$$

$$-\lambda (a^2 - \lambda) = 0$$

$$\lambda = 0, \quad \lambda = a^2$$

$\lambda = 0, a^2$

The spectral radius of the gauss seidal iteration matrix becomes,

$\rho(H) = |a^2|$

$\Rightarrow |a| < 1.$

The optimal relaxation parameter for the SOR is given by $W_{opt} = \dfrac{2(1-\sqrt{1-\mu^2})}{\mu^2}$

put $\mu = a$, $W_{opt} = \dfrac{2(1-\sqrt{1-a^2})}{a^2}$

put $a = 0.5$,

$W_{opt} = \dfrac{2(1-\sqrt{1-(0.5)^2})}{(0.5)^2}$

$= 1.0718$

2) The system of eqns $Ax = b$ is to be solved iteratively by $x_{n+1} = Mx_n + b$. Suppose $A = \begin{bmatrix} 1 & k \\ 2k & 1 \end{bmatrix}$, $k \neq \sqrt{2}/2$, k is real.

a) find the necessary & sufficient condition of k of

(ii) for $k = 0.25$ determine the optimal relaxation factor w_i, in the system is to be solved with relaxation method.

Soln.:

The Jacobi for the given system is

$$X^{(n+1)} = \begin{bmatrix} 0 & k \\ 2k & 0 \end{bmatrix} X^{(n)} + b.$$

$$= M X^{(n)} + b.$$

The necessary and sufficient condition for convergence of Jacobi method is $\rho(M) < 1$.

The eigen values of m are given by

$$|M - \lambda I| = \begin{vmatrix} 0 & k \\ 2k & 0 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} -\lambda & k \\ 2k & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 2k^2 = 0$$

$$\lambda^2 = 2 \cdot k^2$$

$$\lambda = \sqrt{2k^2} = \pm\sqrt{2}\,k.$$

$$|\sqrt{2}\,k| < 1$$

$$|k| < 1/\sqrt{2}$$

$$\rho(M) < 1.$$

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(ii) The optimal relaxation parameter for the SOR method is given by

$$W_{opt} = \frac{2(1-\sqrt{1-\mu^2})}{\mu^2}$$

$$= \frac{2(1-\sqrt{1-(\sqrt{2}K)^2})}{(\sqrt{2}K)^2}$$

$$= \frac{\cancel{2}(1-\sqrt{1-2K^2})}{\cancel{2}K^2}$$

put $K = 0.25$

$$W_{opt} = \frac{1-\sqrt{1-2(0.25)^2}}{(0.25)^2}$$

$$= \frac{1-\sqrt{1-2(0.0625)}}{(0.0625)}$$

$$= \frac{1-0.9354}{0.0625}$$

$$W_{opt} = 1.0834$$

For the soln of the system of eqn

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

: setup the SOR method. find the optimal relaxation factor and the rate of convergence. perform 3 iteration of the SOR method given in the eqns

i) $X^{(K+1)} = H X^{(K)} + C$

ii) $(D+WL) V^{(K)} = W r^{(K)}$.

Take the initial approximation as $X^{(0)} = 0$

The exact soln is $x_1 = 6$, $x_2 = 5$, $x_3 = 7$.

Soln:

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The SOR scheme is obtained by

$$X^{(K+1)} = H_{SOR} X^{(K)} + C$$

where $H_{SOR} = (D+WL)^{-1} [(1-W) D - WU]$

$$C = W(D+WL)^{-1} b.$$

$$D+WL = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + W \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -W & 0 & 0 \\ 0 & -W & 0 \end{bmatrix}$$

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$$(D+WL) = \begin{pmatrix} 2 & 0 & 0 \\ -W & 2 & 0 \\ 0 & -W & 2 \end{pmatrix}$$

$$(D+WL)^{-1} = \frac{1}{|(D+WL)|} \operatorname{adj}(D+WL)$$

$$|D+WL| = 2[4] = 8$$

$$\operatorname{adj}(D+WL) = \begin{bmatrix} \begin{vmatrix} 2 & 0 \\ -W & 2 \end{vmatrix} & -\begin{vmatrix} -W & 0 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} -W & 2 \\ 0 & -W \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ -W & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & -W \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ -W & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -W & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{pmatrix} 4 & 2W & W^2 \\ 0 & 4 & 2W \\ 0 & 0 & 4 \end{pmatrix}^T$$

$$\operatorname{adj}(D+WL) = \begin{pmatrix} 4 & 0 & 0 \\ 2W & 4 & 0 \\ W^2 & 2W & 4 \end{pmatrix}.$$

$$(D+WL)^{-1} = \frac{1}{8} \begin{pmatrix} 4 & 0 & 0 \\ 2W & 4 & 0 \\ W^2 & 2W & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 & 0 \\ W/4 & 1/2 & 0 \\ W^2/8 & W/4 & 1/2 \end{pmatrix}$$

$$[(1-w)D - wU] = \begin{bmatrix} 2-2w & 0 & 0 \\ 0 & 2-2w & 0 \\ 0 & 0 & 2-2w \end{bmatrix} - \begin{bmatrix} 0 & -w & 0 \\ 0 & 0 & -w \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-2w & w & 0 \\ 0 & 2-2w & w \\ 0 & 0 & 2-2w \end{bmatrix}.$$

$$H_{SOR} = (D+wL)^{-1}[(1-w)D - wU]$$

$$H_{SOR} = \begin{pmatrix} 1/2 & 0 & 0 \\ w/4 & 1/2 & 0 \\ w^2/8 & w/4 & 1/2 \end{pmatrix} \begin{pmatrix} 2-2w & w & 0 \\ 0 & 2-2w & w \\ 0 & 0 & 2-2w \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2-2w}{2} & w/2 & 0 \\ \frac{w(2-2w)}{4} & w^2/4 + 1/2(2-2w) & w/2 \\ w^2/8(2-2w) & w^3/8 + w/4(2-2w) & w^2/4 + 1/2(2-2w) \end{pmatrix}$$

$$H_{SOR} = \begin{pmatrix} 1-w & w/2 & 0 \\ w(1-w)/2 & w^2/4 + (1-w) & w/2 \\ w^2/4(1-w) & w^3/8 + w/2(1-w) & w^2/4 + (1-w) \end{pmatrix}$$

$$C = w(D+wL)^{-1}b.$$

$$C = W\begin{pmatrix} 1/2 & 0 & 0 \\ W/4 & 1/2 & 0 \\ W^2/8 & W/4 & 1/2 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}.$$

$$C = \begin{pmatrix} \frac{1}{2}W & 0 & 0 \\ W^2/4 & W/2 & 0 \\ W^3/8 & W^2/4 & W/2 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}.$$

$$C = \begin{pmatrix} 7W/2 \\ 7W^2/4 + W/2 \\ 7W^3/8 + W^2/4 + W/2 \end{pmatrix} = \begin{pmatrix} 7W/2 \\ 7W^2+2W/4 \\ \frac{7W^3+2W^2+4W}{8} \end{pmatrix}$$

$$X^{(K+1)} = \begin{pmatrix} 1-W & W/2 & 0 \\ \frac{W(1-W)}{2} & \frac{W^2}{4}+(1-W) & W/2 \\ \frac{W^2}{4}(1-W) & \frac{W^3}{8}+\frac{W}{2}(1-W) & \frac{W^2}{4}+(1-W) \end{pmatrix} X^{(K)} + \begin{pmatrix} 7W/2 \\ 7W^2+2W/4 \\ \frac{7W^3+2W^2+4W}{8} \end{pmatrix} \longrightarrow (*$$

The iteration matrix associated with Jacobi method is given by

$$H_J = -D^{-1}(L+U)$$

$$D^{-1} = \frac{1}{|D|}\,\text{adj}\,D$$

$$|D| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2\,[4] = 8.$$

$$\operatorname{adj} D = \begin{bmatrix} \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

$$D^{-1} = \frac{1}{8}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$H_j = -D^{-1}(L+U)$$

$$= \begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$H_j = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}$$

$$|\lambda I| = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$$

$$|H_J - \lambda I| = \begin{vmatrix} -\lambda & 1/2 & 0 \\ 1/2 & -\lambda & 1/2 \\ 0 & 1/2 & -\lambda \end{vmatrix}$$

$$= -\lambda[\lambda^2 - 1/4] + (-1/2)[-\lambda/2]$$

$$= -\lambda^3 + \lambda/4 + \lambda/4 = -\lambda^3 + (2\lambda/4) = -\lambda^3 + \lambda/2$$

$$= \frac{-2\lambda^3 + \lambda}{2}$$

$$\frac{-2\lambda^3 + \lambda}{2} = 0$$

$$\frac{-\lambda(2\lambda^2 + 1)}{2} = 0 \Rightarrow -\lambda(2\lambda^2 + 1) = 0$$

$$\Rightarrow \lambda = 0 \quad \Bigg| \quad \begin{aligned} &2\lambda^2 + 1 = 0 \\ &2\lambda^2 = -1 \Rightarrow \lambda^2 = -1/2 \\ &\lambda = \pm\sqrt{1/2} = \pm 1/\sqrt{2} \end{aligned}$$

The spectral radius of the Jacobi iteration matrix is $\mu = 1/2$.

The optimal relaxation if factor of the SOR scheme is

$$W_{opt} = 2/\mu^2 (1 - \sqrt{1 - \mu^2})$$

$$= \frac{2}{(1/\sqrt{2})^2}(1 - \sqrt{1 - (1/2)^2})$$

$$= \frac{2}{1/2}(1 - \sqrt{1 - (1/2)}) = \frac{2}{0.5}(1 - \sqrt{0.5})$$

$$= \frac{2}{0.5}(1-0.7071) = \frac{2}{0.5}(0.2929)$$

$W_{opt} = 1.1716$

$$\rho(H_{SOR}) = W - 1$$
$$= 1.1716 - 1$$
$$= 0.1716$$

The rate of convergence of the SOR method is $v = -\log(0.1716)$.

$$v = 0.7655.$$

i) Substitute the value of $w = 1.1716$ the SOR iteration (*) becomes

$$X^{(K+1)} = \begin{pmatrix} -0.1716 & 0.5858 & 0 \\ -0.1005 & 0.1716 & 0.5858 \\ -0.0589 & 0.1005 & 0.1716 \end{pmatrix} X^{(K)} + \begin{pmatrix} 4.1006 \\ 2.9879 \\ 2.3361 \end{pmatrix}$$

put $k = 0$

$$X^{1} = \begin{pmatrix} -0.1716 & 0.5858 & 0 \\ -0.1005 & 0.1716 & 0.5858 \\ -0.0589 & 0.1005 & 0.1716 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4.1006 \\ 2.9879 \\ 2.3361 \end{pmatrix}$$

$$X^{1} = \begin{pmatrix} 4.1006 \\ 2.9879 \\ 2.3361 \end{pmatrix}$$

$$x^{2} = \begin{pmatrix} -0.1716 & 0.5858 & 0 \\ -0.1005 & 0.1716 & 0.5858 \\ -0.0589 & 0.1005 & 0.1716 \end{pmatrix} \begin{pmatrix} 4.1006 \\ 2.9879 \\ 2.3361 \end{pmatrix} + \begin{pmatrix} 4.1006 \\ 2.9879 \\ 2.3361 \end{pmatrix}$$

$$= \begin{pmatrix} 1.0466 \\ 1.4691 \\ 0.4596 \end{pmatrix} + \begin{pmatrix} 4.1006 \\ 2.9879 \\ 2.3361 \end{pmatrix} = \begin{pmatrix} 5.1472 \\ 4.4570 \\ 2.7957 \end{pmatrix}$$

put $k=2$

$$x^{3} = \begin{pmatrix} -0.1716 & 0.5858 & 0 \\ -0.1005 & 0.1716 & 0.5858 \\ -0.0589 & 0.1005 & 0.1716 \end{pmatrix} \begin{pmatrix} 5.1472 \\ 4.4570 \\ 2.7957 \end{pmatrix} + \begin{pmatrix} 4.1006 \\ 2.9879 \\ 2.3361 \end{pmatrix}$$

$$= \begin{pmatrix} 1.7277 \\ 1.8852 \\ 0.6245 \end{pmatrix} + \begin{pmatrix} 4.1006 \\ 2.9879 \\ 2.3361 \end{pmatrix}$$

$$x^{(3)} = \begin{pmatrix} 5.8283 \\ 4.8731 \\ 2.9606 \end{pmatrix}.$$

(ii) write the SOR in the form

$$(D+WL)\, v^{k} = W\, r^{(k)}$$

$$W = W_{opt} = 1.1716 \quad \text{(or)}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1.1716 & 2 & 0 \\ 0 & 1.1716 & 2 \end{bmatrix} v^{(k)} = 1.1716\, r^{(k)}$$

where $r^{(k)} = b - A x^{(k)}$

$$x^{(k+1)} = x^{(k)} + v^{(k)}, \quad k = 0, 1, 2 \ldots$$

starting with $x^{(0)} = 0$ we obtain

put $k=0$ in $r^{(k)} = b - A x^{(k)}$

$$r^{(0)} = b - A x^{(0)}$$

$$= \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$r^{(0)} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$$

$$(D + WL) v^k = W r^{(k)}$$

$$v^k = (D + WL)^{-1} W r^{(k)}$$

$$v^{(k)} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0.2929 & 1/2 & 0 \\ 0.1716 & 0.2929 & 1/2 \end{pmatrix} (1.1716) \begin{pmatrix} r^k \end{pmatrix}$$

put $k=0$

$$v^{(0)} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0.2929 & 1/2 & 0 \\ 0.1716 & 0.2929 & 1/2 \end{pmatrix} (1.1716) \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 & 0 \\ 0.2929 & 1/2 & 0 \\ 0.1716 & 0.2929 & 1/2 \end{pmatrix} \begin{pmatrix} 8.2012 \\ 1.1716 \\ 1.1716 \end{pmatrix}$$

$$v^{(0)} = \begin{pmatrix} 4.1006 \\ 2.9879 \\ 2.3363 \end{pmatrix}$$

put $K = 0$

$$X^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4.1006 \\ 2.9879 \\ 2.3363 \end{pmatrix}$$

$$x^1 = \begin{pmatrix} 4.1006 \\ 2.9879 \\ 2.3363 \end{pmatrix}$$

$$r^{(K)} = b - AX^{(K)}$$

put $K = 1$

$$r^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 4.1006 \\ 2.9879 \\ 2.3363 \end{pmatrix}$$

$$r^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 5.2133 \\ -0.4611 \\ 1.6847 \end{pmatrix} = \begin{pmatrix} 1.7867 \\ 1.4611 \\ -0.6847 \end{pmatrix}$$

$$v^{(1)} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0.2929 & 1/2 & 0 \\ 0.1716 & 0.2929 & 1/2 \end{pmatrix} (1.1716) \begin{pmatrix} 1.7867 \\ 1.4611 \\ -0.6847 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 & 0 \\ 0.2929 & 1/2 & 0 \\ 0.1716 & 0.2929 & 1/2 \end{pmatrix} \begin{pmatrix} 2.0933 \\ 1.7118 \\ -0.8022 \end{pmatrix}$$

$$V^{(1)} = \begin{pmatrix} 1.0466 \\ 1.4690 \\ 0.4595 \end{pmatrix}$$

$$X^{(2)} = \begin{pmatrix} 4.1006 \\ 2.9879 \\ 2.3363 \end{pmatrix} + \begin{pmatrix} 1.0466 \\ 1.4690 \\ 0.4595 \end{pmatrix}$$

$$X^2 = \begin{pmatrix} 5.1472 \\ 4.4569 \\ 2.7958 \end{pmatrix}$$

put $K = 2$

$$r^2 = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 5.1472 \\ 4.4569 \\ 2.7958 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 5.8375 \\ 0.9708 \\ 1.1347 \end{pmatrix}$$

$$r^2 = \begin{pmatrix} 1.1625 \\ 0.0292 \\ -0.1347 \end{pmatrix}$$

$$V^2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0.2929 & 1/2 & 0 \\ 0.1716 & 0.2929 & 1/2 \end{pmatrix} 1.1716 \begin{pmatrix} 1.1625 \\ 0.0292 \\ -0.1347 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 & 0 \\ 0.2929 & 1/2 & 0 \\ 0.1716 & 0.2929 & 1/2 \end{pmatrix} \begin{pmatrix} 1.3620 \\ 0.0942 \\ -0.1578 \end{pmatrix}$$

$$V^2 = \begin{pmatrix} 0.6810 \\ 0.4160 \\ 0.1648 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 5.1472 \\ 4.4569 \\ 2.7958 \end{pmatrix} + \begin{pmatrix} 0.6810 \\ 0.4160 \\ 0.1648 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 5.8282 \\ 4.8729 \\ 2.9606 \end{pmatrix}$$

Q: Find all the eigen values and eigen vectors of $\begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{3} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$ by Jacobi method.

Soln:

Let $A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{3} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$

The largest of diagonal element is

$a_{13} = a_{31} = 2$ & $a_{11} = a_{33} = 1$.

consider,

$$\tan 2\theta = \frac{2a_{ik}}{a_{ii}-a_{kk}} = \frac{2a_{13}}{a_{11}-a_{33}} = \frac{2(2)}{1-1}$$

$\tan 2\theta = \infty$

$2\theta = \tan^{-1}(\infty)$

$\tan 2\theta = \frac{2a_{ik}}{a_{ii}-a_{kk}}$

$2\theta = \pi/2$

$\theta = \pi/4$

consider the matrix

$$S_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\pi/4 & 0 & -\sin\pi/4 \\ 0 & 1 & 0 \\ \sin\pi/4 & 0 & \cos\pi/4 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$\therefore S_1$ is Orthogonal.

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$\Rightarrow S_1^{-1} = S_1^{T}$

consider,

$B_1 = S_1^{-1} A S_1$

$$= \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}}+0+2/\sqrt{2} & 1+0+1 & 2/\sqrt{2}+0+1/\sqrt{2} \\ 0+\sqrt{2}+0 & 3 & \sqrt{2} \\ -1/\sqrt{2}+2/\sqrt{2} & -1+1 & -2/\sqrt{2}+0+1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{2} & 2 & 3/\sqrt{2} \\ \sqrt{2} & 3 & \sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3/2+3/2 & 2 & -3/\sqrt{2}+3/\sqrt{2} \\ 1+1 & 3 & -1+0+1 \\ 1/2-1/2 & 0 & -1/2-1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Iteration 2 :-

The largest of diagonal element is

$a_{11} = a_{22} = 3$. & $a_{12} = a_{21} = 2$.

$$\tan 2\theta = \frac{2a_{ik}}{a_{ii} - a_{kk}} = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2 \times 2}{3 - 3}$$

$$= \infty$$

$$\tan 2\theta = \infty$$

$$2\theta = \tan^{-1} \infty$$

$$2\theta = \pi/2$$

$$\theta = \pi/4$$

$S_1^{-1} = S_1^T$

consider the matrix,

$B_1 = S_1^{-1} A S_1$

$$S_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore S_2$ is orthogonal.

$\Rightarrow S_2^{-1} = S_2^T$.

onsider,

$$B_2 = S_2^{-1} B_1 S_2$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{2}+2/\sqrt{2} & 2/\sqrt{2}+3/\sqrt{2} & 0 \\ -3/\sqrt{2}+2/\sqrt{2} & -2/\sqrt{2}+3/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5/\sqrt{2} & 5/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5/2+5/2 & -5/2+5/2 & 0 \\ -1/2+1/2 & 1/2+1/2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$B_2 = S_2^{-1} B_1 S_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\therefore$ eigen values are $5, 1, -1$ eigen vector is $S = S_1 \times S_2$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{bmatrix}$$

POWER METHOD:

1. find the largest eigen value in modulus and the corresponding eigen vectors of the matrix $a \neq -15, 4, 3, 10, -12, 6, 20, -4, 2$ using the power method.

$$A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$$

soln :-

Starting with $V_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$y_1 = A V_0 = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$y_1 = AV_0$

$v_1 = \frac{y_1}{18}$

$$Y_1 = \begin{bmatrix} -15+4+7 \\ 10-12+6 \\ 20-4+2 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \\ 18 \end{bmatrix}$$

The largest magnitude in Y_1 is 18.

$$V_1 = \frac{Y_1}{18} = \cancel{\begin{bmatrix} -15/18 & 4/18 & 3/18 \end{bmatrix}}$$

$$= \begin{bmatrix} -8/18 \\ 4/18 \\ 18/18 \end{bmatrix} = \begin{bmatrix} 0.4444 \\ 0.2222 \\ 1 \end{bmatrix}$$

$$Y_2 = A V_1$$

$$= \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 0.4444 \\ 0.2222 \\ 1 \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} 10.5548 \\ -5.1104 \\ -7.7768 \end{bmatrix}$$

The largest magnitude in Y_2 is 10.5548

$$V_2 = \frac{Y_2}{10.5548} = \begin{bmatrix} 1 \\ -0.1052 \\ -0.7368 \end{bmatrix}$$

$$Y_3 = A V_2.$$

$$Y_3 = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.1052 \\ -0.7368 \end{bmatrix} = \begin{bmatrix} -17.6312 \\ 6.8416 \\ 18.9472 \end{bmatrix}$$

The largest magnitude in Y_3 is 18.9472

$$V_3 = \frac{Y_3}{18.9472} = \begin{bmatrix} \frac{-17.6312}{18.9472} \\ \frac{6.8416}{18.9472} \\ 18, 1 \end{bmatrix} = \begin{bmatrix} -0.9305 \\ 0.3611 \\ 1 \end{bmatrix}$$

$Y_4 = A V_3$

$$= \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} -0.9305 \\ 0.3611 \\ 1 \end{bmatrix}$$

$$Y_4 = \begin{bmatrix} 18.4019 \\ -7.6382 \\ -18.0544 \end{bmatrix}$$

The largest magnitude in Y_4 is 18.4019.

$$V_4 = \begin{bmatrix} 1 \\ -0.4151 \\ -0.9811 \end{bmatrix}$$

$Y_5 = A V_4$

$$= \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4151 \\ -0.9811 \end{bmatrix}$$

$$= \begin{bmatrix} -19.6087 \\ 9.0946 \\ 19.6982 \end{bmatrix}$$

The largest magnitude in Y_5 is 19.6982

$$V_5 = \begin{bmatrix} -0.9952 \\ 0.4617 \\ 1 \end{bmatrix}$$

$Y_6 = A V_5$

$$= \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} -0.9952 \\ 0.4617 \\ 1 \end{bmatrix}$$

$$Y_6 = \begin{bmatrix} 19.7748 \\ -9.4924 \\ -19.7508 \end{bmatrix}$$

The largest magnitude in Y_6 is 19.7748

$$V_6 = \begin{bmatrix} 1 \\ -0.4800 \\ -0.9988 \end{bmatrix}$$

$$Y_7 = A V_6 = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4800 \\ -0.9988 \end{bmatrix}$$

$$= \begin{bmatrix} -19.9264 \\ 9.7872 \\ 19.9224 \end{bmatrix}$$

The largest magnitude in Y_7 is 19.9224

$$V_7 = \begin{bmatrix} -0.9997 \\ -0.4903 \\ 1 \end{bmatrix}$$

$$Y_8 = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} -0.9997 \\ -0.4903 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 19.9567 \\ -9.8806 \\ -19.9552 \end{bmatrix}$$

The largest magnitude in Y_8 is 19.9567

$$V_8 = \begin{bmatrix} 1 \\ -0.4951 \\ -0.9999 \end{bmatrix}$$

$$Y_9 = A V_8 = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.4951 \\ -0.9999 \end{bmatrix}$$

$$= \begin{bmatrix} -19.9801 \\ 9.9418 \\ 19.9806 \end{bmatrix}$$

After 9^{th} iteration the ratios,

$(y_9)_r / (V_8)_r$ where $r = 1, 2, 3$ are

$|\lambda| = 19.9801$, 20.0804, 19.9826.

$|\lambda| = 20.0804$

hence, the largest eigen value 20.0804

ie., 20.

and the corresponding eigen vector is

$$\begin{pmatrix} 1 \\ -0.4951 \\ 0.9999 \end{pmatrix}.$$

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$