Cauvery College for Women (Autonomous)

Nationally Accredited (III Cycle) with ‘A’ Grade by NAAC

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Department	:	Mathematics
Programme	:	Msc Mathematics
Batch	:	2018 Onwards
Semester	:	IV
Course	:	Advanced numerical analysis
Course Code	:	P16MA43
Unit	:	III
Topics Covered	:	Interpolation and

approximation-hermite interpolation, piecewise &spline interpolation.bivariate interpolation-lagrange bivariate interpolation- least square approximation.

Hermite Interpolation:

An explicit expression for the interpolating polynomial satisfying

$$P(x_i) = f(x_i)$$
$$P'(x_i) = f'(x_i), \quad i = 0, 1, \ldots n.$$

Since there are $2n+2$ conditions to be satisfied $P(x)$ must be a polynomial of degree $\leq 2n+1$

$$P(x) = \sum_{i=0}^{n} A_i(x) f(x_i) + \sum_{i=0}^{n} B_i(x) f'(x_i) \rightarrow (1)$$

$A_i(x)$, $B_i(x)$ are polynomial of degree $\leq 2n+1$ satisfy

$$\left.\begin{array}{l} 1)\ A_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \\ 2)\ A_i'(x_j) = 0 \quad \forall\, i, j \\ 3)\ B_i(x_j) = 0 \quad \forall\, i, j \\ 4)\ B_i'(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \end{array}\right\} \rightarrow (2)$$

Use the lagrange fundamental polynomials $l_i(x)$

$$\left.\begin{array}{l} A_i(x) = \gamma_i(x)\, l_i^2(x) \\ B_i(x) = \delta_i(x)\, l_i^2(x) \end{array}\right\} \rightarrow (3)$$

$\because l_i^2(x)$ is of polynomial of degree $2n$

$\gamma_i(x)$ & $\delta_i(x)$ linear polynomial

Let
$$\left.\begin{array}{l} \gamma_i(x) = a_i(x) + b_i \\ \delta_i(x) = c_i(x) + d_i \end{array}\right\} \rightarrow (4)$$

we obtain Use (2)

$$\left.\begin{array}{l} a_i = -2 l_i'(x_i) \\ b_i = 1 + 2x_i l_i'(x_i) \\ c_i = 1 \ \& \ d_i = -x_i \end{array}\right\} \rightarrow (5)$$

Sub ⑤ in ④ & use ⑧, eqn ① becomes,

$$① \Rightarrow P(x) = \sum_{i=0}^{n} \left[1 - 2(x - x_i)\, l_i'(x_i) \right] l_i^2(x) f(x_i) + \sum_{i=0}^{n} (x - x_i)\, l_i^2(x) f'(x_i) \rightarrow (6)$$

This is called the Hermite interpolation polynomial.
The Truncation error associated with eqn ⑥

$$E_{2n+1}[f, x] = \frac{W^2 x}{(2n+2)!} f^{2n+2}(\xi), \quad x_0 < \xi < x_0 \rightarrow (7)$$

Determine the parameters in the formula

$$P(x) = a_0(x-a)^3 + a_1(x-a)^2 + a_2(x-a) + a_3$$

$\ni$ $P(a) = f(a)$, $P'(a) = f'(a)$

$P(b) = f(b)$, $P'(b) = f'(b)$

Soln:

$P(a) = f(a) = a_3$

$P'(a) = f'(a) = a_2$

$P(b) = f(b) = a_0(b-a)^3 + a_1(b-a)^2 + a_2(b-a) + a_3$

$P'(b) = f'(b) = 3a_0(b-a)^2 + 2a_1(b-a) + a_2$

Solving the above system of eqns

$a_3 = f(a)$

$a_2 = f'(a)$

$$a_1 = \frac{3}{(b-a)^2}[f(b) - f(a)] - \frac{1}{(b-a)}[2f'(a) + f'(b)]$$

$$a_0 = \frac{2}{(b-a)^3}[f(a) - f(b)] + \frac{1}{(b-a)^2}[f'(a) + f'(b)]$$

Given the following values of $f(x)$ & $f'(x)$

x	$f(x)$	$f'(x)$
-1	1	-5
0	1	1
1	3	7

Estimate the value of $f(+0.5)$ & $f(-0.5)$ using the Hermite interpolation. The exact values are $f(-0.5) = \frac{33}{64}$ & $f(0.5) = \frac{97}{64}$

Soln:

Here $n=2,\ x_0=-1,\ x_1=1\ \&\ x_2=1$

$$p(x) = \sum_{i=0}^{2} A_i(x) f(x_i) + \sum_{i=0}^{2} B_i(x) f'(x_i) \qquad 3$$

$$A_0(x) = [1-2(x-x_0)l_0'(x_0)]\,l_0^2(x)$$

$$A_1(x) = [1-2(x-x_1)l_1'(x_1)]\,l_1^2(x)$$

$$A_2(x) = [1-2(x-x_2)l_2'(x_2)]\,l_2^2(x)$$

$$B_0(x) = (x-x_0)\,l_0^2(x)$$

$$B_1(x) = (x-x_1)\,l_1^2(x)$$

$$B_2(x) = (x-x_1)\,l_2^2(x)$$

$$l_0(x) = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x(x-1)}{2},\quad l_0'(-1) = -\frac{3}{2}$$

$$l_1(x) = \frac{(x+1)(x-1)}{(0+1)(0-1)} = -(x^2-1),\quad l_1'(0) = 0$$

$$l_2(x) = \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{x(x+1)}{2},\quad l_2'(1) = \frac{3}{2}$$

$$A_0(x) = [1+3(x+1)]\,\frac{x^2(x-1)^2}{4}$$

$$= \frac{1}{4}(3x^5-2x^4-5x^3+4x^2)$$

$$A_1(x) = [1-2(x-0)(0)]\,(x^2-1)^2$$

$$= x^4-2x^2+1$$

$$A_2(x) = [1-3(x-1)]\,\frac{x^2(x+1)^2}{4}$$

$$= \frac{1}{4}(-3x^5-2x^4+5x^3+4x^2)$$

$$B_0(x) = \frac{(x+1)x^2(x-1)^2}{4} = \frac{1}{4}(x^5-x^4-x^3+x^2)$$

$$B_1(x) = x(x^2-1)^2 = x^5-2x^3+x$$

$$B_2(x) = \frac{(x-1)x^2(x+1)^2}{4} = \frac{1}{4}(x^5+x^4-x^3-x^2)$$

$$p(x) = \frac{1}{4}(3x^5 - 2x^4 - 5x^3 + 4x^2)(1) + (x^4 - 2x^2 ...)(1) + \frac{1}{4}(-3x^5 - 2x^4 + 5x^3 + 4x^2)(...)$$

(4)

$$+ \frac{1}{4}(x^5 - x^4 - x^3 - x^2)(-5) + \frac{1}{4}(x^5 + x^4 - x^3 - x^2)(7)$$

$$= 2x^4 - x^2 + x + 1$$

Sub $x = -0.5$ & 0.5

$f(-0.5) = \frac{3}{8}$ exact value is $\frac{33}{64}$

$f(0.5) = \frac{11}{8}$ exact value is $\frac{97}{64}$

Piecewise & Spline interpolation:

To keep the degree of interpolating polynomial small and also to achieve accurate results, we use piecewise interpolation.

We sub divide the given interval $[a,b]$ into a number of subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots n$ & approximate the function by some ~~linear~~ lower degree polynomials in each subintervals.

We subdivide the interval $[a,b]$ where $a = x_0 < x_1 < x_2 \ldots < x_n = b$ into a number of non-over lapping subintervals each containing 2 (or) 3 (or) 4 nodal points.

~~the~~ we ~~constant~~ construct linear (or) quadratic (or) cubic interpolating polynomials fitting the given data on each subinterval.

These polynomials define the piecewise linear or quadratic or cubic interpolation polynomial for the data $(x_i, f(x_i))$, $i = 0, 1, 2, \ldots k$ we can constant.

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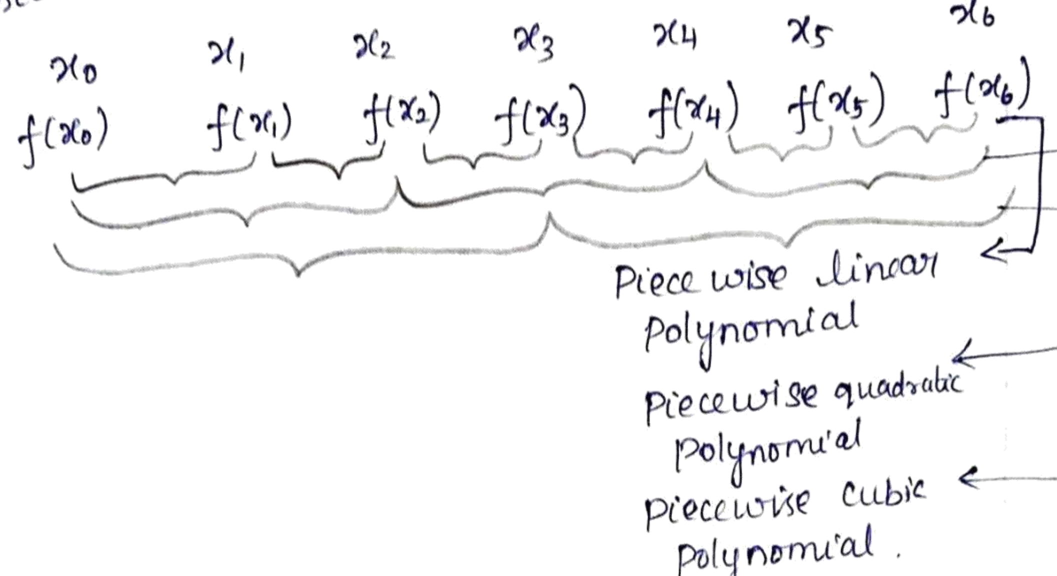


Piecewise Linear interpolation:

$(n+1)$ distinct nodal points $x_0, x_1, \ldots x_n$. The interpolation polynomial is linear in each subinterval (x_{i-1}, x_i) and it agrees with the function $f(x)$ at the $(n+1)$ nodal points.

The subintervals or line segments are called the elements in one space dimension and the nodal points are called knots.

use the linear Lagrange interpolating Polynomial

$$P(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

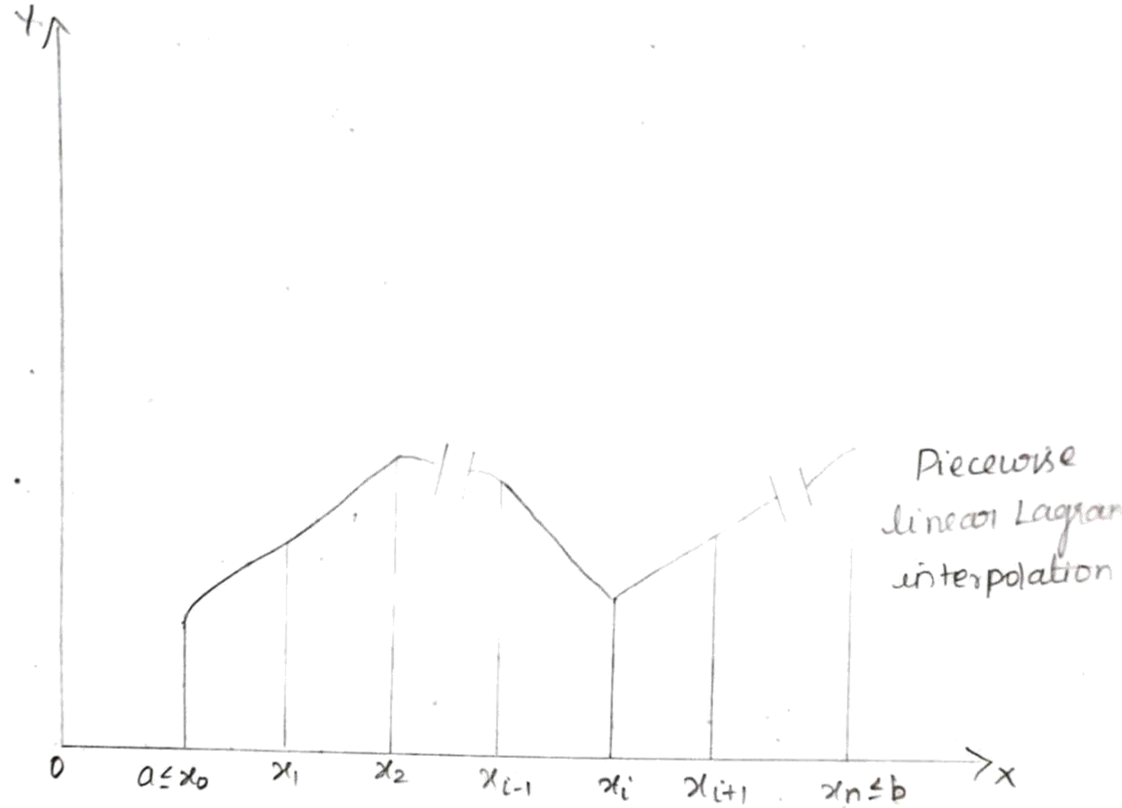
$$= l_0(x) f(x_0) + l_1(x) f(x_1)$$

For $x \in [x_{i-1}, x_i]$ the piecewise linear interpolation Polynomial.

$$P_{i,1}(x) = \frac{x - x_i}{x_{i-1} - x_i} f(x_{i-1}) + \frac{(x - x_{i-1})}{x_i - x_{i-1}} f(x_i) \quad , i=1,2,\dots n$$

(b) $x \in [x_i, x_{i+1}]$

$$P_{i+1,1}(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}} f(x_i) + \frac{x - x_i}{x_{i+1} - x_i} f(x_{i+1})$$



Define,

$$N_i(x) = \begin{cases} 0 & , \quad x \le x_{i-1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & , \quad x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & , \quad x_i \le x \le x_{i+1} \\ 0 & , \quad x \ge x_{i+1} \end{cases} \longrightarrow (2)$$

The non-zero terms in $N_i(x)$ are the co-efficient of $f(x_i)$ in $P_{i,1}(x) \times P_{i+1,1}(x)$

Then the interpolating polynobial.

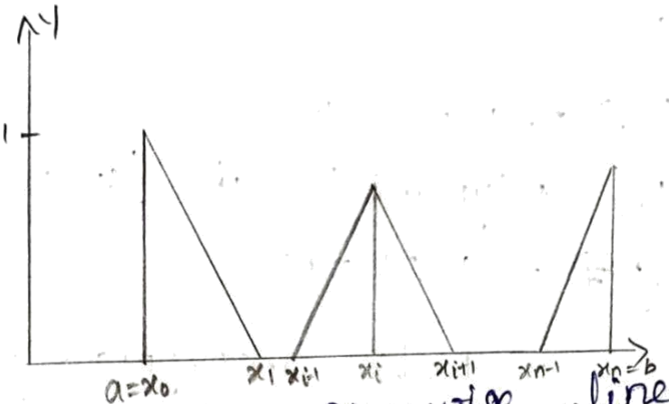
$$P(x) = \sum_{i=0}^{n} P_{i,1}(x) \longrightarrow (3)$$

$f(x)$ at x_i, $i = 0, 1, 2, \dots n$ and is linear in each sub interval $[x_{i-1}, x_i]$ can be written as,

$$P_1(x) = \sum_{i=0}^{n} N_i(x) f(x_i) \rightarrow (4)$$

The $N_i(x)$ is called a "shape function" and its shown as,

(7)



The error in the piecewise linear interpolating is given by,

$$f(x) - P_{i-1}(x) = \frac{1}{2!}(x - x_{i-1})(x - x_i) f''(\xi_0),$$

$$x_i \le \xi_i \le x, \rightarrow (5)$$

obtain the piecewise linear interpolating polynomial for the function $f(x)$ defined by the data:

5m A-18

x	1	2	4	8
$f(x)$	3	7	21	73

Hence estimate the value of $f(3) + f(7)$.

Soln:

In the interval $[1, 2]$ we have

$$P_1(x) = \frac{x-2}{(-1)}(3) + (x-1)(7) = 4x - 1$$

in the interval $[2, 4]$

$$P_1(x) = \frac{x-4}{-2}(7) + \frac{x-2}{2}(21) = 7x - 7$$

In the interval $[4, 8]$.

$$P_1(x) = \frac{x-8}{-4}(21) + \frac{x-4}{4}(73) = 13x - 31$$

The piecewise linear interpolating polynomial are given by.

$$P_1(x) = \begin{cases} 4x-1, & 1 \le x \le 2 \\ 7x-7, & 2 \le x \le 4 \\ 13x-31, & 4 \le x \le 8 \end{cases}$$

The polynomial in the interval $[2,4]$ we obtain

$$f(3) = 21-7 = 14$$

Using the polynomial in $[4,8]$ we obtain

$$f(7) = 91-31 = 60$$

Piece wise quadratic interpolation:

Let the no of distinct nodal point be $2n+1$ with $a = x_0 < x_1 < x_2 \dots < x_n = b$

we conclude the graps of 3 consecutive nodal points as $[x_0, x_2]$ $[x_2, x_4] \dots [x_{i-1}, x_{i+1}]$ $\dots [x_{2n-2}, x_n]$

for $x \in [x_{i-1}, x_{i+1}]$

the quadratic interpolating polynomial

$$P_2(x) = \frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} f(x_{i-1}) + \frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_{i-1}-x_{i+1})} f(x_i) + \frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)} f(x_{i+1}) \rightarrow (1)$$

The error in the piecewise quadratic interpolation is given by,

$$f(x) - P_{i,2}(x) = \frac{1}{3!}(x-x_{i-1})(x-x_i)(x-x_{i+1}) f'''(\xi_i) \quad x_{i-1} < \xi_i < x_{i+1}$$

obtain the piecewise quadratic interpolating polynomial for the function $f(x)$ defined by the data.

10m A-18

x	-3	-2	-1	1	3	6	7
$f(x)$	369	222	171	165	207	990	1779.

9 Hence find an approximate value of $f(-2.5)$ & $f(6.5)$

Soln:

nodal points are $\{-3,-2,-1\}$ $\{-1,1,3\}$ $\{3,6,7\}$

on each subintervals we write the quadratic interpolating polynomial $[-3,-1]$.

$$P_{1,2} = \frac{(x+2)(x+1)}{(-3+2)(-3+1)}(369) + \frac{(x+3)(x+1)}{(-2+3)(-2+1)}(222) + \frac{(x+3)(x+2)}{(-1+3)(-1+2)}(17)$$

$$= \frac{369}{2}(x^2+3x+2) - 222(x^2+4x+3) + \frac{171}{2}(x^2+5x+6)$$

$$= 48x^2+93x+216 \quad \text{on } [-1,3]$$

$$P_{22}(x) = \frac{(x-1)(x-3)}{(-1-1)(-1-3)}(17) + \frac{(x+1)(x-3)}{(1+1)(1-3)}(165) + \frac{(x+1)(x-1)}{(3+1)(3-1)}(207)$$

$$= \frac{171}{8}(x^2-4x+3) - \frac{165}{4}(x^2-2x-3) + \frac{207}{8}(x^2-1)$$

$$= 6x^2-3x+162 \quad \text{on } [3,7]$$

$$P_{3,2}(x) = \frac{(x-6)(x-7)}{(3-6)(3-7)}(207) + \frac{(x-3)(x-7)}{(6-3)(6-7)}(990) + \frac{(x-3)(x-6)}{(7-3)(7-6)}(1779)$$

$= \frac{207}{12}(x^2-13x+42) - \frac{990}{3}(x^2-10x+21) + \frac{1779}{4}(x^2-9x+18)$

$= 132x^2 - 927x + 1800$

The point -2.5 lies in the interval $[-3,-1]$

Hence use $P_{2,1}(x)$ we obtain

(10) $f(-2.5) = P_{2,1}(-2.5)$

$= 48(-2.5)^2 + 93(-2.5) + 216$

$= 283.5$

The point 6.5 lies in $(3,7)$

$f(6.5) = P_{2,3}(6.5)$

$= 132(6.5)^2 - 927(6.5) + 1800$

$= 1351.5$

Piecewise cubic interpolation:-

Let the no of distinct nodal points be $3n+1$ with $a = x_0 < x_1 < x_2 \dots < x_{3n} = b$

Consider graps of 4 nodal points as $[x_0, \dots x_3]$ $[x_3, \dots x_6] \dots [x_{3n-3} \dots x_{3n}]$

Each of the subintervals we write the cubic interpolating polynomial.

For $x \in [x_i, x_{i+3}]$

The cubic interpolating polynomial

$$P_{i,3}(x) = l_{i,0} f(x_i) + l_{i,1} f(x_{i+1}) + l_{i,2} f(x_{i+2}) + l_{i,3} f(x_i+3)$$

where,

$$l_{i,0} = \frac{(x-x_{i+1})(x-x_{i+2})(x-x_{i+3})}{(x_i-x_{i+1})(x_i-x_{i+2})(x_i-x_{i+3})}$$

$$l_{i,1} = \frac{(x-x_i)(x-x_{i+2})(x-x_{i+3})}{(x_{i+1}-x_i)(x_{i+1}-x_{i+2})(x_{i+1}-x_{i+3})}$$

we can use newton's divided difference interpolation.

$$l_{i,2} = \frac{(x-x_i)(x-x_{i+1})(x-x_{i+3})}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})(x_{i+2}-x_{i+3})}$$

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$$l_{i,3} = \frac{(x-x_i)(x-x_{i+1})(x-x_{i+2})}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})}$$

The error in the piecewise cubic interpolation

$$f(x) - P_{i,3}(x) = \frac{1}{4!}(x-x_i)(x-x_{i+1})(x-x_{i+2})(x-x_{i+3}) f^{iv}(\xi_i) \quad x_i < \xi_i < x_{i+3}$$

In eg :- 4.22
obtain the approximate values of $f(-2.5)$ and $f(6.5)$ using the piecewise cube interpolation.

Soln:
Problem
nodal points as $\{-3,-2,-1,1\}$ & $\{1,3,6,7\}$

Newton's divided difference interpolation

x	$f(x)$	1^{st} d.d	2^{nd} d.d	3^{rd} d.d
-3	369	$\frac{222-369}{-2+3} = -147$	48	-8
-2	222	$\frac{171-222}{-1+2} = -51$	16	
-1	171	$\frac{165-171}{1+1} = -3$		
1	165			

on $[-3,1]$

$$P_{1,3} = 369 + (x+3)(-147) + (x+3)(x+2)(48) + (x+3)(x+2)(x+1)(-8)$$

$$= -8x^3 + 5x + 168$$

$$f(-2.5) = P_{1,3}(-2.5) = -8(-2.5)^3 + 5(-2.5) + 168$$

$$= 280.5$$

on $[1, 7]$.

x	$f(x)$	1st d.d	2nd d.d	3rd d.d
1	165			
		21		
			48	
				14
3	207			
		261		
			132	12
6	990			
		789		
7	1779			

$$P_{2,3} = 165 + (x-1)(21) + (x-1)(x-3)(48) + (x-1)(x-3)(x-6)(14)$$

$$= 14x^3 - 92x^2 + 207x + 36$$

$$f(6.5) = 14(6.5)^3 - 92(6.5)^2 + 207(6.5) + 36$$

$$= 1339.25$$

Piecewise cubic interpolation using Hermite Type data:

Let the following Hermite type of data be given on each interval $[x_{i-1}, x_i]$ $i = 1, 2, \ldots n$.

$$\left.\begin{aligned} P_{i,3}(x_{i-1}) &= f_{i-1} \quad P_{i,3}(x_i) = f_i \\ P'_{i,3}(x_{i-1}) &= f'_{i-1} \quad P'_{i,3}(x_i) = f'_i \end{aligned}\right\} \rightarrow (1)$$

We construct a cubic polynomial $P_{i,3}(x)$ on each of the sub-intervals. The polynomial obtained is called piecewise cubic Hermite interpolating polynomial.

Use $$P(x) = \sum_{i=0}^{n} \left[1 - 2(x - x_i) l_i'(x_i)\right] l_i^2(x) f(x_i) + \sum_{i=0}^{n} (x - x_i) l_i^2(x) f'(x_i)$$

We can write this polynomial in the form

$P_{i,3}(x) = A_{i-1}(x) f_{i-1} + A_i(x) f_i + B_{i-1}(x) f'_{i-1} + B_i(x) f'_i \rightarrow (2)$

where $x_{i-1} \le x \le x_i$ and

$$A_{i-1}(x) = \frac{(x-x_i)^2}{(x_{i-1}-x_i)^2}\left[1+\frac{2(x_{i-1}-x)}{x_{i-1}-x}\right]$$

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$$A_i(x) = \frac{(x-x_{i-1})^2}{(x_{i-1}-x_i)^2}\left[1+\frac{2(x-x_i)}{x_{i-1}-x_i}\right]$$

$$B_{i-1}(x) = \frac{(x-x_{i-1})(x-x_i)^2}{(x_{i-1}-x_i)^2}$$

$$B_i(x) = \frac{(x-x_i)(x-x_{i-1})^2}{(x_{i-1}-x_i)^2} \rightarrow (3)$$

on the interval $[x_i, x_{i+1}]$

$$P_{i+1,3} = A_i^*(x) f_i + A^*_{i+1}(x) f_{i+1} + B_i^*(x) f_i' + B^*_{i+1}(x) f'_{i+1}$$

where A_i^*, B_i^*, A^*_{i+1} & B^*_{i+1} are obtained by setting $i=i+1$ respectively in A_{i-1}, B_{i-1}, A_i & B_i in eqn (3)

define,

$$N_i(x) = \begin{cases} 0, & x \le x_{i-1} \\ \dfrac{(x-x_{i-1})^2}{(x_i-x_{i-1})^2}\left[1+\dfrac{2(x-x_0)}{x_{i-1}-x_i}\right], & x_{i-1} \le x \le x_i \\ \dfrac{(x-x_{i+1})^2}{(x_{i+1}-x_i)^2}\left[1+\dfrac{2(x-x_i)}{x_{i+1}-x_i}\right], & x_i \le x \le x_{i+1} \\ 0, & x \ge x_{i+1} \end{cases} \quad (4)$$

&

$$H_i(x) = \begin{cases} 0, & x_{i-1} \le x \le x_i \\ \dfrac{(x-x_{i-1})(x-x_i)}{(x_i-x_{i-1})^2}, & x_{i-1} \le x \le x_i \\ \dfrac{(x-x_{i+1})^2(x-x_i)}{(x_{i+1}-x_i)^2}, & x_i \le x \le x_{i+1} \\ 0, & x \ge x_{i+1} \end{cases}$$

The non-zero terms in $N_i(x)$ & $H_i(x)$ are the co-efficients of $f(x_i)$ & $f'(x_i)$ in $P_{i,3}(x)$ & $P_{i+1,3}(x)$ respectively.

Then the interpolating polynomial

(iv) $P_3(x) = \sum_{i=1}^{n} P_{i,3}(x) \rightarrow (5)$

with $f(x)$ & $f'(x)$ at x_i, $i = 0, 1, 2, \dots n$ and is cubic in each subinterval $[x_{i-1}, x_i]$ can be written as

$$P_3(x) = \sum_{i=0}^{n} N_i(x) f(x_i) + \sum_{i=0}^{n} H_i(x) f'(x_i) \rightarrow (6)$$

$P_{i,3}(x_i) = P_{i+1,3}(x_i) = f_i$, $i = 1, 2, \dots n-1$

& $P'_{i,3}(x_i) = P'_{i+3}(x_i) = f_i'$, $i = 1, 2, \dots n-1$

$P_3(x)$ is continuously differentiable on (a, b).

The error in the piecewise cubic Hermite interpolation is given by

$$f(x) - P_{i,3}(x) = \frac{1}{4!}(x - x_{i-1})^2 (x - x_i)^2 f^4(\xi_i),$$

$$x_{i-1} < \xi_i < x_i \rightarrow (7)$$

∴ Using the following value of $f(x)$ & $f'(x)$

10m (4.)

x	$f(x)$	$f'(x)$
-1	1	-5
0	1	1
1	3	7

Estimate the value of $f(-0.5)$ & $f(0.5)$ using Piecewise cubic Hermite interpolation.

Soln:

$$x_{i-1} = -1, \quad x_i = 0, \quad x_{i+1} = 1$$

$x = -0.5 \in [x_{i-1}, x_i]$ piecewise cubic Hermite interpolation becomes,

$$P_3(x) = [1+2(x+1)]x^2(1) + [1-2(x-0)](x+1)^2(0) + (x+1)x^2(-5) + x(x+1)^2(1)$$

(15)

$$= -(4x^3+3x^2-x-1)$$

we get $f(-0.5) = 0.25$

$\|^{ly}$ $x = 0.5 \in [x_i, x_{i+1}]$ Piecewise cubic Hermite interpolation

$$P_3(x) = [1+2(x-0)](x-1)^2(1) + [1-2(x-1)]x^2(3) + x(x-1)^2(1) + (x-1)x^2(7)$$

$$= 4x^3 - 3x^2 + x + 1$$

$$f(0.5) = P_3(0.5) = 1.25$$

$S_3(x)$ is the piecewise cubic Hermite interpolating approximate of $f(x) = \sin x \cos x$ in the $0, 1, 1.5, 2, 3$. Estimate the error $\max_{0 \le x \le 3} |f(x) - S_3(x)|$ as

The error E_i in approximating $f(x)$ by the Hermite interpolating polynomial based on the x_{i-1}, x_i is given by,

$$E_i = \frac{1}{4!}(x - x_{i-1})^2 (x - x_i)^2 f^{iv}(\xi), \quad x_{i-1} < \xi < x_i$$

we have

$$f(x) = \frac{1}{2}\sin 2x$$

$$f'(x) = \cos 2x$$

$$f''(x) = -2\sin 2x, \quad f'''(x) = -4\cos 2x,$$

$$f^{IV}(x) = -8\sin 2x$$

Hence $M_i = \max_{x_{i-1} \le x \le x_i} |f^{iv}(x)|$

$$= \max_{x_{i-1} \le x \le x_i} |8 \sin 2x|$$

(b) $M_1 = \max_{0 \le x \le 1} |8 \sin 2x|$

$M_2 = \max_{1 \le x \le 1.5} |8 \sin 2x| = 8|\sin 2|$

$= 7.2744$

$M_3 = \max_{1.5 \le x \le 2} |8 \sin 2x| = 8|\sin 4| = 6.0544$

$M_4 = \max_{2 \le x \le 3} |8 \sin 2x| = 8$

Max value of $(x-x_{i-1})(x-x_i)$ is obtained at $x = \frac{(x_i + x_{i-1})}{2}$ &

$$\max_{x_{i-1} \le x \le x_i} |(x-x_{i-1})(x-x_i)| = \frac{1}{4}(x_i - x_{i-1})^2$$

Hence we have,

$$|E_i| \le \frac{1}{24} \max_{x_{i-1} \le x \le x_i} |(x-x_{i-1})^2 (x-x_i)^2| . \max_{x_{i-1} \le x \le x_i} |8 \sin 2x|$$

$$\le \frac{1}{384} (x_i - x_{i-1})^4 M_i$$

& $|E_1| \le \frac{1}{384} (1-0)^4 M_1 = 0.0208$

$|E_2| \le \frac{1}{384} (1.5-1)^4 M_2 = 0.0012$

$|E_3| \le \frac{1}{384} (2-1.5)^4 M_3 = 0.00099$

$|E_4| \le \frac{1}{384} (3-2)^4 M_4 = 0.0208$

$$\max_{0 \le x \le 3} |f(x) - S_3(x)| = \max(|E_1|, |E_2|, |E_3|, |E_4|)$$
$$= 0.0208$$

Spline Interpolation :-

A smooth curve through a given set of points such that the slope of curvature are also continuous along the curve,
(ie) $f(x)$, $f'(x)$, $f''(x)$ are continuous on the curve such a device is called a spline and
17 plotting of the curve is called spline fitting.

The given interval $[a,b]$ is subinterval into n subintervals $[x_0, x_1]\ [x_1, x_2] \dots [x_{n-1}, x_n]$. where $a = x_0 < x_1 < x_2 < \dots < x_n = b$ the nodes $x_1, x_2, \dots x_{n-1}$ are called internal nodes.

Definition: spline function:

A spline function of degree n with knots (nodes) x_i, $i = 1, 0, 2 \dots n$ is a function $F(x)$ satisfy the properties.

(i) $F(x_i) = f(x_i)$ $\quad i = 0, 1, 2 \dots n$

(ii) On each subinterval $[x_{i-1}, x_i]$, $1 \le i \le n$, $F(x)$ is a polynomial of deg n.

(iii) $F(x)$ and it's 1st $(n-1)$ derivatives are continuous on (a,b) linear spline interpolation is a linear piecewise interpolation.

Quadratic Spline interpolation:

A quadratic spline interpolation satisfies the following properties

(i) $F(x_i) = f(x_i)$ $\quad , i = 0, 1, 2 \dots n$

(ii) On each subinterval $[x_{i-1}, x_i]$, $1 \le i \le n$, $F(x)$ is a 2nd degree poly except in the 1st or the last interval.

(iii) $F(x)$ & $F'(x)$ are continuous on (a,b)

Denote $F''(x_i) = M_i$

Each sub interval $[x_{i-1}, x_i]$ we approximate form by a 2^{nd} degree polynomial as

$$F(x) = P_i(x) = a_i x^2 + b_i x + c_i \quad , i = 1, 2, \dots n$$

There are 3 unknowns to be determined which are a_1, b_1, c_1 ; a_2, b_2, c_2 ; .... a_n, b_n, c_n

Since $F(x)$ is Continuous at the internal nodes $x_1, x_2, \dots x_{n-1}$ is Continuous at the internal nodes.

$x_1, x_2, \dots x_{n-1}$ we obtain the equations

on $[x_{i-1}, x_i]$; $P_i(x_i) = f_i = a_i x_i^2 + b_i x_i + c_i \rightarrow (1)$

on $[x_i, x_{i+1}]$; $P_{i+1}(x_i) = f_i = a_{i+1} x_i^2 + b_{i+1} x_i + c_{i+1} \rightarrow (2)$

$i = 1, 2, \dots n-1$

we have $2n-2$ equations.

Since $F'(x)$ is continuous at the internal nodes we obtain the eqn. Continuity at x_i.

$$P_i'(x_i) = P_{i+1}'(x_i)$$

$$2a_i x_i + b_i = 2a_{i+1} x_i + b_{i+1} \longrightarrow (3)$$

$i = 1, 2 \dots n-1$

From this ~~not~~ set we have $n-1$ equations.

At the end points x_0, x_n interpolating conditions given the eqns.

$$f_0 = a_1 x_0^2 + b_1 x_0 + c_1 \longrightarrow (4)$$

$$\& \; f_n = a_n x_n^2 + b_n x_n + c_n \longrightarrow (5)$$

Total $(2n-2) + (n-1) + 2 = 3n-1$ equations to determine the $3n$ unknowns

(a) prescribe $M_0 = f''(x_0) = p''(x_0) = P$

$$\Rightarrow f''(x_0) = 2a_1 = P \quad (\text{or}) \quad \boxed{a_1 = P/2}$$

The value $P = 0$ is choosen

we get $a_1 = 0$

In the first subinterval $[x_0, x_1]$ we are using linear approximation.

i.e) the 1st 2 points are joined by a straight line

b) prescribe $M_n = f''(x_n) = P_n''(x_n) = q$

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$$f''(x_n) = 2a_n = q \Rightarrow a_n = q/2$$

again $q = 0$ is choosen

$$\Rightarrow a_n = 0.$$

Hence, in the last subinterval $[x_{n-1}, x_n]$ we are using linear approximation, that is the last two points are joined by a straight line.

Now, the system of $(3n)\times(3n)$ linear algebraic equations are solved for a_i, b_i, c_i, $i = 1, 2, 3 \ldots n$

However, by arranging the equations in a proper order, it is possible to solve 3×3 equations for each set of unknowns a_i, b_i, c_i, $i = 1, 2, \ldots n$.

We illustrate this procedure through an example.

Suppose that we have 3 subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$ then from ① to ⑤ we have the equations.

$$a_1 x_0^2 + b_1 x_0 + c_1 = f_0, \quad a_2 x_1^2 + b_2 x_1 + c_2 = f_1 \rightarrow (6\ a, b)$$

$$a_2 x_2^2 + b_2 x_2 + c_2 = f_2, \quad a_3 x_3^2 + b_3 x_3 + c_3 = f_3 \rightarrow (7\ a, b)$$

$$2a_1 x_0 + b_1 = 2a_2 x_1 + b_2, \quad 2a_2 x_2 + b_2 = 2a_3 x_2 + b_3 \rightarrow (8\ a, b)$$

$$a_1 x_0^2 + b_1 x_0 + c_1 = f_0 \quad a_3 x_3^2 + b_3 x_3 + c_3 = f_3 \rightarrow (9\ a, b)$$

Let us choose $M_0 = f''(x_0) = 0$ as the extra condition. This gives $a_1 = 0$ using the equations (6 a), (6 b), (7 a), (7, b), (8 a), (9 a), (9.b)

We write them in the following order.

$$\left.\begin{aligned} b_1 x_0 + c_1 &= f_0 \\ b_2 x_1 + c_2 &= f_1 \end{aligned}\right\} \rightarrow (10)$$

$$\left.\begin{aligned} a_2 x_1^2 + b_2 x_1 + c_2 &= f_1 \\ a_2 x_2^2 + b_2 x_2 + c_2 &= f_2 \\ 2a_2 x_1 + b_2 &= 2a_1 x_1 + b_1 \end{aligned}\right\} \rightarrow (11)$$

and

$$\left.\begin{aligned} a_3 x_2^2 + b_3 x_2 + c_3 &= f_2 \\ 2a_3 x_2 + b_3 &= 2a_2 x_2 + b_2 \\ a_3 x_3^2 + b_3 x_3 + c_3 &= f_3 \end{aligned}\right\} \rightarrow (12)$$

The system of equations (10) are solved for b_1, c_1 Using these solution the system of the ~~soln~~ equation (11) are solved. The system of equation are ~~such~~ Solved in the forward direction

If $M_3 = f''(x_3) = 0$ is prescribed, then we rearrange the equations. So that solution is obtained in the backward direction, that is, we solve for b_3, c_3 first, then for a_2, b_2, c_2 etc.

Quadratic splines have two disadvantages they are,

(i) a straight line connects the first two or the last two points.

(ii) The spline for the last interval may swing high in the above case for these reasons, quadratic splines are not often used.

1\. Given the data

10m

x:	0	1	2	3
$f(x)$:	1	2	33	244

fit quadratic splines with $M(0) = f''(0) = 0$; Hence, find an estimate of $f(2.5)$

Soln:

We write the spline approximation as

(21) $P_1(x) = a_1x^2 + b_1x + c_1$

$P_2(x) = a_2x^2 + b_2x + c_2$, $1 \le x \le 2$

$P_3(x) = a_3x^2 + b_3x + c_3$, $2 \le x \le 3$

Since $M(0) = f''(0) = 0$. we get $a_1 = 0$

Substitute $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$

$f_0 = 1$, $f_1 = 2$, $f_2 = 33$, $f_3 = 244$ in equations (10), (11) & (12). we obtain

$$b_1(0) + c_1 = 1$$

$$b_1(1) + c_1 = 2$$

$$\boxed{c_1 = 1}$$

$$b_1 + 1 = 2$$

$$\boxed{b_1 = 1}$$

$$a_2 + b_2 + c_2 = 2$$

$$4a_2 + 2b_2 + c_2 = 33$$

$$2a_2 + b_2 = 2a_1 + b_1$$

Put $b_1 = 1$, $a_1 = 0$

$$a_2 + b_2 + c_2 = 2$$

$$4a_2 + 2b_2 + c_2 = 33$$

$$2a_2 + b_2 = 1$$

$$a_2 + b_2 + c_2 = 2$$

$$4a_2 + 2b_2 + c_2 = 33$$

$$(-) \quad (-) \quad (-) \quad (-)$$

$$-3a_2 - b_2 = -31$$

$$3a_2 + b_2 = 31$$

$$3a_2 + \cancel{b_2} = 31$$
$$\begin{array}{r} 2a_2 + \cancel{b_2} = 1 \\ (-) \quad (-) \quad (-) \\ \hline a_2 = 30 \end{array}$$

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$$3(30) + b_2 = 31$$
$$b_2 = 31 - 90$$
$$b_2 = -59$$
$$30 - 59 + C_2 = 2$$
$$\boxed{C_2 = 31}$$

∴ The solution to the system is

$a_2 = 30,\quad b_2 = -59,\quad C_2 = 31$

$$4a_3 + 2b_3 + C_3 = 33$$
$$9a_3 + 3b_3 + C_3 = 244$$
$$4a_3 + b_3 = 4a_2 + b_2$$

Put $a_2 = 30, \quad b_2 = -59$

$$4a_3 + b_3 = 4(30) - 59$$
$$4a_3 + b_3 = 61$$

$$\begin{array}{r} 9a^3 + 3b^3 + C_3 = 244 \\ 4a^3 + 2b^3 + C_3 = 33 \\ (-) \quad (-) \quad (-) \quad (-) \\ \hline 5a_3 + b_3 = 211 \\ 4a_3 + b_3 = 61 \\ (-) \quad (-) \quad (-) \\ \hline \boxed{a_3 = 150} \end{array}$$

$$A(150) + b_3 = 61$$
$$b_3 = -539$$
$$A(150) + 2(-539) + C_3 = 33$$
$$\boxed{C_3 = 511}$$

The quadratic splines in the corresponding intervals can be written as

$P_1(x) = x+1$ $0 \le x \le 1$

$P_2(x) = 30x^2 - 59x + 31$, $1 \le x \le 2$

$P_3(x) = 150x^2 - 539x + 511$, $2 \le x \le 3$

An estimate at 2.5 is

(23) $f(2.5) = P_3(2.5)$

$= 150(2.5)^2 - 539(2.5) + 511$

$= 101$

(i)

x :	0	1	2	3
$f(x)$:	1	3	11	311

Assume $f''(0) = M(0) = 0$

Interpolate at $x = 1.5$ and 2.5

(ii)

x :	-1	0	1	2
$f(x)$:	-4	1	0	5

Assume $f''(2) = M(2) = 0$

Interpolate at $x = -0.5$

Cubic spline interpolation:

A cubic spline satisfies the following properties

(i) $F(x_i) = f_i$, $i = 0, 1, \ldots n$

(ii) On each subinterval $[x_{i-1}, x_i]$, $1 \le i \le n$
$F(x)$ is a third degree polynomial.

(iii) $F(x)$, $F'(x)$ & $F''(x)$ are continuous on (a,b)

We denote $F'(x_i) = m_i$ and $F''(x_i) = M_i$ Cubic splines do not have the disadvantages of the quadratic splines. On each subinterval $[x_{i-1}, x_i]$ we approximate $f(x)$ by a cubic

Polynomial as

$$F(x) = P_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i, \quad i = 1, 2, \dots n$$

We have $4n$ unknowns a_i, b_i, c_i, d_i, $i = 1, 2 \dots n$ to be determined.

Using the Continuity of $F(x)$, $F'(x)$ and $F''(x)$ we have the following equations.

(a) Continuity of $F(x)$.

on $[x_{i-1}, x_i]$ $\quad P_i(x_i) = f_i = a_i x_i^3 + b_i x_i^2 + c_i x_i + d_i$

On $[x_i, x_{i+1}]$,

$$P_{i+1}(x_i) = f_i = a_{i+1} x_i^3 + b_{i+1} x_i^2 + c_{i+1} x_i + d_{i+1}$$
$$i = 1, 2, \dots n-1$$

(b) Continuity of $f'(x)$:-

$$3a_i x_i^2 + 2b_i x_i + c_i = 3a_{i+1} x_i^2 + 2b_{i+1} x_i + c_{i+1},$$
$$i = 1, 2 \dots n-1 \rightarrow (2)$$

(c) Continuity of $F''(x)$.

$$6a_i x_i + 2b_i = 6a_{i+1} x_i + 2b_{i+1}, \quad i = 1, 2 \dots n-1 \rightarrow (3)$$

At the end points x_0 and x_n, we have the interpolatory Conditions.

$$f_0 = a_1 x_0^3 + b_1 x_0^2 + c_1 x_0 + d_1 \rightarrow (4)$$

$$f_n = a_n x_n^3 + b_n x_n^2 + c_n x_n + d_n \rightarrow (5)$$

We have $2(n-1)$ equations from (1), $2(n-1)$ eqns from (2) & (3) and 2 equations from (4) & (5) is a total of $4n-2$ equations. We need two more equation to obtain the polynomial uniquely. In most cases we prescribe $F''(x)$ at the two end points that is,

$$F''(x_0) = M_0 = p \text{ and } F''(x_n) = M_n = q$$

The end Conditions, $M_0 = 0$, $M_n = 0$, lead to a natural spline. It is called a natural spline.

since the drafting spline always behaves in this fashion. However we can use the conditions as D to or / and q70.

If the above two conditions are imposed, then we have 4n equations in 4n unknowns. These equations can be written in matrix form and solution can be obtained.

How will you construct a cubic spline method

Construction of cubic Spline Method :-

5m (7) Let $f(x)$ be a piecewise cubic polynomial, $F''(x)$ is a linear function of x in the interval $x_{i-1} \le x \le x_i$

$$F''(x) = \frac{(x_i - x)}{(x_i - x_{i-1})} F''(x_{i-1}) + \frac{(x - x_{i-1})}{(x_i - x_{i-1})} F''(x_i) \rightarrow (1)$$

Integrating (1) two times with respect to x, we get

$$F(x) = \frac{(x_i - x)^3}{6h_i} M_{i-1} + \frac{(x - x_{i-1})^3}{6h_i} M_i + c_1 x + c_2 \rightarrow (2)$$

where $M_i = F''(x_i)$ & c_1, c_2 are arbitrary constants to be determined by using the conditions we have $F(x_{i-1}) = f(x_{i-1})$ and $F(x_i) = f(x_i)$

$$f_{i-1} = \frac{1}{6h_i} (x_i - x_{i-1})^3 M_{i-1} + c_1 x_{i-1} + c_2$$

(or)

$$f_{i-1} = \frac{1}{6} h_i^2 M_{i-1} + c_1 x_{i-1} + c_2 \quad \text{and}$$

$$f_i = \frac{1}{6h_i} (x_i - x_{i-1})^3 M_i + c_1 x_i + c_2$$

(or)

$$f_i = \frac{1}{6} h_i^2 M_i + c_1 x_i + c_2$$

Subtracting these two equations we obtain

$$c_1 (x_i - x_{i-1}) = \left[(f_i - f_{i-1}) - \frac{1}{6} (M_i - M_{i-1}) h_i^2 \right]$$

(or) $$c_1 = \frac{1}{h_i} (f_i - f_{i-1}) - \frac{1}{6} (M_i - M_{i-1}) h_i \rightarrow (3)$$

Solving for c_2 we obtain

$$c_2 = \frac{1}{h_i}(x_i f_{i-1} - x_{i-1} f_i) - \frac{1}{6}(x_i M_{i-1} - x_{i-1} M_i) h_i$$

Substituting ③ & ④ in ② we obtain

$$F(x) = \frac{1}{6h_i}(x_i - x)^3 M_{i-1} + \frac{1}{6h_i}(x - x_{i-1})^3 M_i + \frac{x}{h_i}(f_i - f_{i-1}) - \frac{x}{6}(M_i - M_{i-1}) h_i + \frac{1}{h_i}(x_0 f_{i-1} - x_{i-1} f_i) - \frac{1}{6}(x_i M_{i-1} - x_{i-1} M_i) h_i = \cancel{\frac{1}{6h_i}\left[(x_i - x)\{(x_i-x)^2\right.}$$

$$= \frac{1}{6h_i}\left[(x_i - x)\{(x_i - x)^2 - h_i^2\}\right] M_{i-1} + \frac{1}{6h_i}\left[(x - x_{i-1})\{(x - x_{i-1})^2 - h_i^2\}\right] M_i + \frac{1}{h_i}(x_i - x) f_{i-1} + \frac{1}{h_i}(x - x_{i-1}) f_i \rightarrow ⑤$$

where $x_{i-1} \le x \le x_i$

Differentiating ⑤ we get

$$F'(x) = \frac{-(x_i - x)^2}{2h_i} M_{i-1} + \frac{(x - x_{i-1})^2}{2h_i} M_i - \frac{(M_i - M_{i-1})}{6} h_i + \frac{f_i - f_{i-1}}{h_i}, \quad x_{i-1} \le x \le x_i \rightarrow ⑥$$

Setting $i = i+1$, we get

$$F'(x) = \frac{-(x_{i+1} - x)^2}{2h_{i+1}} M_i + \frac{(x - x_i)}{2h_{i+1}} M_{i+1} - \frac{1}{6}(M_{i+1} - M_i) h_{i+1} + \frac{f_{i+1} - f_i}{h_{i+1}}, \quad x_i \le x \le x_{i+1} \rightarrow ⑦$$

Now, we require that the derivative $F'(x)$ be continuous at $x = x_i + \epsilon$ as $\epsilon \to 0$.

Letting $F'(x_i - \epsilon) = F'(x_i + \epsilon)$ as $\epsilon \to 0$

we get

$$\frac{h_i}{6} M_{i-1} + \frac{h_i}{3} M_i + \frac{1}{h_i}(f_i - f_{i-1}) =$$ next page

$$= -\frac{h_{i+1}}{3} M_i - \frac{h_{i+1}}{6} M_{i+1} + \frac{1}{h_{i+1}} (f_{i+1} - f_i)$$

which may be written as,

$$\frac{h_i}{6} M_{i-1} + \frac{h_i - h_{i+1}}{3} M_i + \frac{h_{i+1}}{6} M_{i+1} = \frac{1}{h_{i+1}} (f_{i+1} - f_i) - \frac{1}{h_i} (f_i - f_{i-1}) \rightarrow (8)$$

This gives a system of $(n-1)$ linear equation is $n+1$ unknowns $M_0, M_1, M_2 \dots M_n$ the two additional Conditional may be taken in one of the following forms.

(i) $M_0 = M_n = 0$ (natural spline) $\rightarrow (9)$

(ii) $M_0 = M_n$, $M_1 = M_{n+1}$, $f_0 = f_n$

$f_1 = f_{n+1}$, $h_1 = h_{n+1}$ $\rightarrow (10)$

(a spline satisfying these condition is called a periodic spline) 2m (xi) A-18

(iii) For a non-periodic spline, we use the conditions $F'(a) = f'(a) = f_0'$ and $F'(b) = f'(b) = f_n'$

using (6) we get,

$$2M_0 + M_1 = \frac{6}{h_1}\left(\frac{f_1 - f_0}{h_1} - f_0'\right)$$

$$M_{n-1} + 2M_n = \frac{6}{h_n}\left(f_n' - \frac{f_n - f_{n-1}}{h_n}\right) \rightarrow (11)$$

For equispaced knots $h_i = h$ and for all i, equations (5) & (8) reduce to

$$F(x) = \frac{1}{6h_i}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left(f_{i-1} - \frac{h^2}{6} M_{i-1}\right) + \frac{1}{h}(x - x_{i-1})\left(f_i - \frac{h^2}{6} M_i\right) \rightarrow (12)$$

and

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(f_{i+1} - 2f_i + f_{i-1}) \rightarrow (13)$$

This method gives the values of $M_i = f''(x_i)$ $i = 1, 2 \dots N-1$. The Solutions obtained for M_i, $i = 1, 2 \dots N-1$ are Substituted in ⑤ or ② to obtain the cubic spline interpolation.

It may be noted that in this method also, we need to solve only an $(n-1) \times (n-1)$ tridiagonal system of equations for finding M_i.

Splines usually provide a better approximation of the behaviour of functions that have abrupt local changes. Further, Splines perform better than higher order polynomial approximation.

Problem:

1. Obtain the cubic Spline approximation for the function defined by the data.

10m

h = difference = 1

$i = 1, 2$

x :	0	1	2	3
$f(x)$:	1	2	33	244

with $M(0) = 0$, $M(3) = 0$. Hence find an estimate of (2.5).

Soln:

Since the points are equispaced with $n = 1$, we obtain from (13)

$$M_{i-1} + 4M_i + M_{i+1} = 6(f_{i+1} - 2f_i + f_{i-1}), \quad i = 1, 2, \dots$$

There fore,

$$M_0 + 4M_1 + M_2 = 6(f_2 - 2f_1 + f_0)$$

$$M_1 + 4M_2 + M_3 = 6(f_3 - 2f_2 + f_1)$$

Using $M_0 = 0$, $M_3 = 0$ and the given function values we get

$$4M_1 + M_2 = 6(33 - 4 + 1) = 180$$

$M_1 + 4M_2 = 6(244 - 66 + 2) = 1080$

$16M_1 + 4M_2 = 720$

$M_1 + 4M_2 = 1080$

$(-) \quad (-) \quad (-)$

$15M_1 = -360$

$\boxed{M_1 = -24}$

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$4M_2 = 1080 + 24$

$= 1104$

$M_2 = 276$

From the eqn (12) the cubic splines the corresponding intervals on $[0,1]$

$F(x) = \frac{1}{6}\left[(1-x)^3 M_0 + (x-0)^3 M_1\right] + (1-x)\left(f_0 - \frac{1}{6}M_0\right) + (x-0)\left(f_1 - \frac{1}{6}M_1\right)$

$= \frac{1}{6}\left[(1-x)^3(0) + x^3(-24)\right] + (1-x)\left(1 - \frac{1}{6}(0)\right) + x\left(2 - \frac{1}{6}(-24)\right)$

$= -4x^3 + 1 - x + 6x$

$= -4x^3 + 5x + 1$

$F(x) = \frac{1}{6}\left[(2-x)^3 M_1 + (x-1)^3 M_2\right] + (2-x)\left(f_1 - \frac{1}{6}M_1\right) + (x-1)\left(f_2 - \frac{1}{6}M_2\right)$

$= \frac{1}{6}\left[(2-x)^3(-24) + (x-1)^3(276)\right] + (2-x)\left(2 - \frac{1}{6}(-24)\right) + (x-1)\left[33 - \frac{1}{6}(276)\right]$

$= !$

on $[2,3]$

$F(x) = \frac{1}{6}\left[(3-x)^3 M_2 + (x-2)^3 M_3\right] + (3-x)\left(f_2 - \frac{1}{6}M_2\right) + (x-2)\left(f_3 - \frac{1}{6}M_3\right)$

$$= \frac{1}{6}\left[(3-x)^3(276) + (x-2)^3(0)\right] + (3-x)$$

$$\left[33 - \frac{1}{6}(276)\right] + (x-2)\left[244 - \frac{1}{6}(0)\right]$$

(31)

$$= -46x^3 + 414x^2 - 985x + 715$$

$F(2.5) = 121.25$

Bivariate Interpolation:

The problem of polynomial interpolation for functions of several independent variables is quite important. so we shall only consider the functions of two variables the extension the two higher dimension is straight forward.

Lagrange bivariate Interpolation:

Let $f(x, y)$ be defined at $(m+1)(n+1)$ distinct points (x_i, y_j) $i = 0, 1, \ldots m$, $j = 0, 1, \ldots n$ and denote $f(x_i, y_j)$ by $f(i, j)$ (or) $f_{i,j}$.

To obtain a polynomial $P(x, y)$ of degree atmost m in x and n in y, such that

$$P(x_i, y_j) = f_{i,j} \quad \begin{matrix} i = 0, 1, \ldots m \\ j = 0, 1, \ldots n \end{matrix} \quad (1)$$

Using the lagrange fundamental polynomials of a single variable we defined

$$X_{m,i}(x) = \frac{w(x)}{(x - x_i)\, w'(x_i)}, \quad i = 0, 1, 2 \ldots m$$

$$Y_{n,j}(y) = \frac{w^*(y)}{(y - y_j)\, w^{*\prime}(y_j)}, \quad j = 0, 1, \ldots n$$

where $w(x) = (x - x_0)(x - x_1) \ldots (x - x_m)$

$$w^*(y) = (y - y_0)(y - y_1) \ldots (y - y_n)$$

Obviously $X_{m,i}(x)$ and $Y_{n,j}(y)$ are polynomials of degree m in x and n in y respectively.

These polynomials satisfying the following Properties,

$$X_{m,i}(x_k) = \delta_{ik}$$

$$Y_{n,j}(y_k) = \delta_{jk}.$$

The polynomials which satisfy eqn ① can be written as

$$P_{m,n}(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} X_{m,i}(x)\, Y_{n,j}(y) \cdot f_{i,j} \rightarrow ②$$

This polynomial is called the Lagrange Bivariate interpolation polynomial.

It may also be interpreted as a double application of the Lagrange interpolating polynomial in a single variable.

Newton's Bivariable interpolation for equi-spaced points:

With equi-space points, with spacing h in x and k in y we defined

$$\Delta_x f(x,y) = f(x+h,y) - f(x,y)$$
$$= (E_x - 1) f(x,y)$$

$$\Delta_y f(x,y) = f(x,y+k) - f(x,y)$$
$$= (E_y - 1) f(x,y)$$

$$\Delta_{xx} f(x,y) = \Delta_x f(x+h,y) - \Delta_x f(x,y)$$
$$= (E_x - 1)^2 f(x,y)$$

$$\Delta_{yy} f(x,y) = \Delta_y f(x,y+k) - \Delta_y f(x,y)$$
$$= (E_y - 1)^2 f(x,y)$$

$$\Delta_{xy} f(x,y) = \Delta_x [f(x,y+k) - f(x,y)]$$
$$= \Delta_x \Delta_y f(x,y)$$

$= (E_x - 1)(E_y - 1) f(x,y)$

$= (E_y - 1)(E_x - 1) f(x,y)$

$= \Delta_y \Delta_x f(x,y)$

$= \Delta_{yx} f(x,y)$

Now $f(x_0+mh, y_0+nk) = E_x^m E_y^n f(x_0, y_0)$

$$= (1+\Delta x)^m (1+\Delta y)^n f(x_0, y_0)$$

$$= \left[1 + \binom{m}{1}\Delta x + \binom{m}{2}\Delta xx + \cdots\right]\left[1 + \binom{n}{1}\Delta y + \binom{n}{2}\Delta yy + \cdots\right] f(x_0, y_0).$$

$$= 1 + \binom{m}{1}\Delta x + \binom{n}{1}\Delta y + \binom{m}{2}\Delta xx + \binom{m}{1}\binom{n}{1}\Delta xy + \binom{n}{2}\Delta yy + \cdots \Big] f(x_0, y_0) \quad \text{①}$$

Let $x = x_0 + mh$

$y = y_0 + nk$

$m = \frac{x - x_0}{h}$

$n = \frac{y - y_0}{k}$

Then the eqn ① we have the interpolating Polynomial,

$$P(x,y) = f(x_0,y_0) + \left[\frac{1}{h}(x-x_0)\Delta x + \frac{1}{k}(y-y_0)\Delta y\right] f(x_0,y_0) + \cdots + \frac{1}{2!}\Big[\frac{1}{h^2}(x-x_0)(x-x_1)\Delta xx + \frac{2}{hk}(x-x_0)(y-y_0)\Delta xy + \frac{1}{k^2}(y-y_0)(y-y_1)\Delta yy\Big] f(x_0,y_0) + \cdots \rightarrow \text{②}$$

This is called the Newton's Bivariate interpolating polynomial for equispaced points.

1. The following data for a function $f(x,y)$ is given

$y \backslash x$	0	1
0	1	1.414214
1	1.732051	2

Find $f(0.25, 0.75)$ using linear interpolation

Soln:

The linear interpolating polynomial is given by

$$p(x,y) = f(x_0,y_0) + \frac{1}{h}(x-x_0)\,\Delta x\, f(x_0,y_0) + \frac{1}{k}(y-y_0)\,\Delta y\, f(x_0,y_0)$$

$$\Delta x\, f(x_0,y_0) = f(x_0+h, y_0) - f(x_0,y_0)$$
$$= 1.414214 - 1$$
$$= 0.414214$$

$$\Delta y\, f(x_0,y_0) = f(x_0, y_0+k) - f(x_0,y_0)$$
$$= 1.732051 - 1$$
$$= 0.732051$$

Now $h = k = 1$

$$p(0.25, 0.75) = 1 + 0.25(0.414214) + 0.75(0.732051)$$
$$= 1.652592$$

2. The following data for a function $f(x,y)$ is given

$y \backslash x$	0	1	3
0	1	2	10
1	2	4	14
3	10	14	28

Construct the bivariate interpolating polynomial and hence find $f(0.5, 0.5)$.

Soln:

$$p(x,y) = \sum_{i=0}^{2} \sum_{j=0}^{2} X_{2i} \, Y_{2j} \, f_{ij}$$

$$P_{2,2}(x,y) = X_{2,0}[Y_{2,0} f_{0,0} + Y_{2,1} f_{0,1} + Y_{2,2} f_{0,2}] +$$
$$X_{2,1}[Y_{2,0} f_{1,0} + Y_{2,1} f_{1,1} + Y_{2,2} f_{1,2}] +$$
$$X_{2,2}[Y_{2,0} f_{2,0} + Y_{2,1} f_{2,1} + Y_{2,2} f_{2,2}] \rightarrow ①$$

$$X_{2,0} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{x^2-4x+3}{3}$$

$$X_{2,1} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = \frac{x^2-3x}{-2}$$

$$X_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(3-0)(3-1)}$$

$$= \frac{x^2-x}{6}$$

$$Y_{2,0}(y) = \frac{(y-1)(y-3)}{(-1)(-3)} = \frac{1}{3}(y^2-4y+3)$$

$$Y_{2,1}(y) = \frac{y(y-3)}{(1)(-2)} = -\frac{1}{2}(y^2-3y)$$

$$Y_{2,2}(y) = \frac{y(y-1)}{3(2)} = \frac{y^2-y}{6}$$

$$P_{2,2}(x,y) = \frac{1}{3}(x^2-4x+3)\Big[\frac{1}{3}(y^2-4y+3)(1) -$$
$$\frac{1}{2}(y^2-3y)(2) + \frac{1}{6}(y^2-y)(10)\Big]$$
$$-\frac{1}{2}(x^2-3x)\Big[\frac{1}{3}(y^2-4y+3)(2) - \frac{1}{2}$$
$$(y^2-3y)(4) + \frac{1}{6}(y^2-y)(14)\Big] +$$
$$\frac{1}{6}(x^2-x)\Big[\frac{1}{3}(y^2-4y+3)(10) - \frac{1}{2}(y^2-3y)$$
$$(14) + \frac{1}{6}(y^2-y)(28)\Big]$$

$$= \frac{1}{3}(x^2-4x+3)\left[\frac{2y^2-8y+6-6y^2+18y+10y^2-10y}{6}\right].$$

$$-\frac{1}{2}(x^2-3x)\left[\frac{4y^2-16y+12-12y^2+36y+14y^2-14y}{6}\right]$$

$$+\frac{1}{6}(x^2-x)\left[\frac{20y^2-80y+60-42y^2+126y+28y^2-28y}{6}\right]$$

$$= \frac{1}{3}(x^2-4x+3)(y^2+1)-\frac{1}{2}(x^2-3x)(y^2+y+2)+\frac{1}{6}(x^2-x)(y^2+8y+10)$$

$$= \frac{1}{6}\left[(2y^2+2)(x^2-4x+3)-(3x^2-9x)(y^2+y+2)+(x^2-x)(y^2+3y+10)\right]$$

$$= \frac{1}{6}\left[2x^2y^2-8xy^2+6y^2+2x^2-8x+6-3x^2y^2-3x^2y-6x^2+9xy^2+9xy+18x+x^2y^2+3x^2y+10x^2-xy^2-3xy-10x\right].$$

$$= \frac{1}{6}\left[6x^2+6y^2+6xy+6\right]$$

$$= x^2+y^2+xy+1$$

Hence $f(0.5, 0.5) = 1.75$

3. Find newton's bivariate interpolating polynomial from the following data

y|x	0	1	2
0	1	3	7
1	3	6	11
2	7	11	17

Soln:

$$P(x,y) = f(x_0,y_0) + \frac{1}{1!}\left(\frac{1}{h}(x-x_0)\Delta x + \frac{1}{k}(y-y_0)\Delta y\right) f(x_0,y_0)$$

$$+\frac{1}{2!}\left[\frac{(x-x_0)(x-x_1)}{h^2}\Delta xx + \frac{(x-x_0)(y-y_0)}{hk}\Delta xy + \frac{(y-y_0)(y-y_1)}{k^2}\Delta yy\right] f(x_0,y_0) + \cdots \rightarrow (1)$$

Here $x_0 = 0 \quad x_1 = 1 \quad x_2 = 2 \quad h = 1$

$y_0 = 0 \quad y_1 = 1 \quad y_2 = 2 \quad k = 1$

$$\Delta x\, f(x_0, y_0) = f(x_0+h, y_0) - f(x_0, y_0)$$
$$= f(1,0) - f(0,0)$$
$$= 3 - 1 = 2$$

$$\Delta y\, f(x_0, y_0) = f(x_0, y_0+k) - f(x_0, y_0)$$
$$= f(0,1) - f(0,0)$$
$$= 3 - 1 = 2$$

$$\Delta xy\, f(x_0, y_0) = f(x_0+h, y_0+k) - f(x_0+h, y_0) - f(x_0, y_0+k) + f(x_0, y_0)$$
$$= f(1,1) - f(1,0) - f(0,1) + f(0,0)$$
$$= 6 - 3 - 3 + 1 = 1$$

$$\Delta xx\, f(x_0, y_0) = f(x_0, y_0+2k) - 2f(x_0+h, y_0) + f(x_0, y_0)$$
$$= f(0,2) - 2f(1,0) + f(0,0)$$
$$= 7 - 2\times 3 + 1 = 2$$

$$\Delta yy\, f(x_0, y_0) = f(x_0, y_0+2k) - 2f(x_0, y_0+k) + f(x_0, y_0)$$
$$= f(0,2) - 2f(0,1) + f(0,0)$$
$$= 7 - 2\times 3 + 1 = 2$$

$$P(x,y) = 1 + [(x-0) + (y-0)] + \frac{1}{2}[(x-0)(x-1)(2) + (x-0)(y-0) + (y-0)(y-1)(2)]$$

$$= 1 + [x+y] + \frac{1}{2}[2x(x-1)] + \frac{1}{2}(xy) + \frac{\cancel{2}y}{\cancel{2}}(y-1)$$

$$= 1 + x + y + x^2 + y^2 + \frac{xy}{2}$$

Least squares Approximation:

Least squares approximations are the most commonly used approximations for approximating a function $f(x)$ which may be given in tabular form (or) known explicity over a given interval. In this case, we use the Euclidean norm,

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

$$\|8\| = \left(\int_a^b W(x) f^2(x) dx\right)^{1/2}$$

The best approximation in the least square is a define as a constant c_i, $i=0,1,\ldots n$ are determine so that aggregate of $W(x)$, over a given Domain D is as small as possible where $W(x) > 0$ is a weight function.

For the functions whose values are given at $(n+1)$ points, $x_0, x_1, \ldots x_n$ we have,

$$I(c_0, c_1, \ldots c_n) = \sum_{K=0}^{N} W(x_K)\left[f(x_K) - \sum_{i=0}^{n} c_i Q_i(x_K)\right]^2 \rightarrow ①$$

The functions which are continuous on $[a,b]$ and given explicitly

$$I(c_0, c_1, \ldots c_n) = \int_a^b W(x_K)\left[f(x_K) - \sum_{i=0}^{n} c_i Q_i(x_K)\right]^2 dx \rightarrow ②$$

$= \text{minimum}$

The coordinate function $Q_i(x) = x^i \rightarrow ③$ $i=0,1,\ldots n$

It gives a system of $(n+1)$ linear equations in $(n+1)$ unknowns $c_0, c_1, \ldots c_n$. These eqns are called normal equations.

The normal equations for ① & ② becomes,

$$\sum_{K=0}^{N} W(x_K)\left[f(x_K) - \sum_{i=0}^{n} c_i Q_i(x_K)\right] Q_j(x_K) = 0$$

$j = 0$ to n $\rightarrow ④$

$$\int_a^b W(x)\left[f(x) - \sum_{i=0}^{n} C_i Q_i(x)\right] Q_j(x)\,dx = 0, \quad j = 0 \text{ to } n. \rightarrow (5)$$

1) obtain a linear polynomial approximation to the function $f(x) = x^3$ on $[0,1]$ using the least square approximation with $w(x) = 1$.

Soln:

Consider a linear polynomial

$$p(x) = a_0 x + a_1$$

where a_0 & a_1 are arbitrary parameters.

Use eqn (2) we get,

$$I(a_0, a_1) = \int_0^1 [x^3 - (a_0 x + a_1)]^2\,dx$$

$$= \text{minimum}.$$

$$I = \frac{1}{7} + \frac{a_0^2}{3} + a_1^2 - \frac{2a_0}{5} - \frac{2a_1}{4} + a_0 a_1$$

The normal equations are

$$\frac{\partial I}{\partial a_0} = 0 \quad \& \quad \frac{\partial I}{\partial a_1} = 0$$

$$\Rightarrow \frac{2a_0}{3} - \frac{2}{5} + a_1 = 0$$

$$2a_1 - \frac{1}{2} + a_0 = 0$$

$$a_1 = -\frac{1}{5}, \quad a_0 = \frac{9}{10}$$

$$p(x) = \left(\frac{9}{10}\right)x - \frac{1}{5}$$

2) obtain the least square polynomial approximation of degree 1 and 2 for $f(x) = \sqrt{x}$ on $[0,1]$.

5m Soln:

(A) A18 i) For $n=1$ we have

10m (A) $P(x) = C_0 + C_1 x$ and $f(x) = x^{1/2}$

$$I[C_0, C_1] = \int_0^1 \left(x^{1/2} - (C_0 + C_1 x)\right)^2 dx$$

4. Find the least square approximation of 2nd degree for the data.

x :	-2	-1	0	1	2
$f(x)$:	15	1	1	3	19

Soln:

$$P_2(x) = c_0 + c_1 x + c_2 x^2$$

x	$f(x)$	x^2	x^3	x^4	$xf(x)$	$x^2 f(x)$
-2	15	4	-8	16	-30	60
-1	1	1	-1	1	-1	1
0	1	0	0	0	0	0
1	3	1	1	1	3	3
2	19	4	8	16	38	76
0	39	10	0	34	10	140

The normal eqn of 2nd degree

$$(N+1)c_0 + c_1 \Sigma x_i + c_2 \Sigma x_i^2 = \Sigma f(x_i)$$

$$5c_0 + 0c_1 + 10c_2 = 39$$

$$5c_0 + 10c_2 = 39 \longrightarrow (1)$$

$$c_0 \Sigma x_i + c_1 \Sigma x_i^2 + c_2 \Sigma x_i^3 = \Sigma x_i f(x_i)$$

$$c_0(0) + 10c_1 + c_2(0) = 10$$

$$10c_1 = 10$$

$$\boxed{c_1 = 1} \longrightarrow (2)$$

$$c_0 \Sigma x_i^2 + c_1 \Sigma x_i^3 + c_2 \Sigma x_i^4 = \Sigma x_i^2 f(x_i)$$

$$10c_0 + 34c_2 = 140 \longrightarrow (3)$$

$(1) \times (2) \Rightarrow$

$$10c_0 + 20c_2 = 78$$

$$\begin{array}{rrr} 10c_0 & +34c_2 & = 140 \\ (-) & (-) & (-) \\ \hline & -14c_2 & = -62 \end{array}$$

$c_2 = \frac{31}{7}$ in (1) we get,

$$5c_0 + \frac{310}{7} = 39$$

$$c_0 = \frac{-37}{35}$$

$$\therefore p(x) = -\frac{37}{35} + x + \frac{31}{7}x^2$$

$= \frac{1}{35}(155x^2 + 35x - 37)$

5. obtain the least square straight line fit to the following data.

x:	0.2	0.4	0.6	0.8	1
$f(x)$:	0.447	0.632	0.775	0.894	1

Soln:

x	$f(x)$	x^2	$xf(x)$
0.2	0.447	0.04	0.0894
0.4	0.632	0.16	0.2528
0.6	0.775	0.36	0.465
0.8	0.894	0.64	0.752
1	1	1	1
3	3.748	2.2	2.5224

$$p(x) = c_0 + c_1 x$$

The normal eqn are,

$$c_0(N+1) + c_1 \Sigma x_i = \Sigma f(x_i)$$

$$5c_0 + 3c_1 = 3.748 \rightarrow ①$$

$$c_0 \Sigma x_i + c_1 \Sigma x_i = \Sigma x_i f(x_i)$$

$$3c_0 + 2.2c_1 = 2.5224$$

$$c_0 = 0.3392$$

$$c_1 = 0.684$$

$$\therefore P_1(x) = 0.3392 + 0.684x$$

Least square error

$$= \sum_{i=0}^{4} \left[f(x_i) - (0.3392 + 0.684x_i)\right]^2$$

$$= 0.00245$$

6. Use the method of least square to fit the Curve $f(x) = C_0 x + (C_1/\sqrt{x})$ for the following data.

x:	0.2	0.3	0.5	1	2
$f(x)$:	16	14	11	6	3

Find the least square error.

Soln:

$$I(c_0, c_1) = \Sigma\left[f(x_i) - c_0 x_i - \frac{c_1}{\sqrt{x_i}}\right]^2$$

$$= \text{Minimum}$$

We obtain the normal eqn

$$C_0 \Sigma x_i^2 + C_1 \Sigma x_i^2 = \Sigma x_i f(x_i)$$

$$C_0 \Sigma x_i^{1/2} + C_1 \Sigma(1/x_i) = \Sigma\left[f(x_i)/x_i^{1/2}\right]$$

We have

$$\Sigma x_i^{1/2} = 4.1163, \quad \Sigma(1/x_i) = 11.8333$$

$$\Sigma x_i^2 = 5.38, \quad \Sigma x_i f(x_i) = 24.9$$

$$\Sigma\left[f(x_i)/(x_i)^{1/2}\right] = 85.0151$$

The normal eqn are given by,

$$5.38\, C_0 + 4.1163\, C_1 = 24.9$$

$$4.1163\, C_0 + 11.8333\, C_1 = 85.0151$$

$\therefore C_0 = -1.1836, \quad C_1 = 7.5961$

The least square fit is given by,

$$f(x) = (7.5961/x^{1/2}) - 1.1836x$$

and least square error

$$= \Sigma\left[f(x_i) - \left\{\frac{7.5961}{x_i^{1/2}} - 1.1836 x_i\right\}\right]^2$$

$$= 1.6887.$$