Cauvery College for Women (Autonomous)

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Topics Covered	:	Ordinary differential equation-

discretization error,order of a method, Taylor series method , Rungi-kutta method- implicit &explicit Rungi-kutta method

UNIT-V Numerical Method

ORDINARY DIFFERENTIAL EQUATIONS

Single step methods:-

The methods for the solution of the initial value problem.

$$u' = f(t,u), \ u(t_0) = \eta_0 \cdot t \in [t_0, b] \rightarrow (1)$$

can be classified mainly into two types. They are

i) singlestep method.

ii) multistep method.

In singlestep method, the solution at any point is obtained using the solution at only the previous point.

Thus, a general singlestep method can be written as,

$$u_{j+1} = u_j + h\phi(t_{j+1}, t_j, u_{j+1}, u_j, h) \rightarrow (2)$$

where ϕ is a function of the arguments $t_j, t_{j+1}, u_j, u_{j+1}, h$ and also depends on f.

we often write it as $\phi(t,u,h)$. This function ϕ is called the increment function.

If u_{j+1} can be obtained simply by evaluating the right side of (2) then the method is called a explicit method. In this method is of the form,

$$u_{j+1} = u_j + h\phi(t_j, u_j, h) \rightarrow (3)$$

In the RHS of (3) depends on u_{j+1}, also then it is called an implicit method.

The general form in this case is as given in eqn (2).

Local Truncation error or Discretization error:-

The true (exact) value $u(t_j)$ satisfies the equation

$$u(t_{j+1}) = u(t_j) + h\phi(t_{j+1}, t_j, u(t_{j+1}), u(t_j), h) + T_{j+1}.$$

where T_{j+1} is called the local truncation error or discretization error of the method.

$\therefore$ the truncation error is given by.

$$T_{j+1} = u(t_{j+1}) - u(t_j) - h\phi(t_{j+1}, t_j, u(t_{j+1}), u(t_j), h) \rightarrow (4)$$

Order of a Method:-

The order of the method is the largest integer p for which,

$$\left| \frac{1}{h} T_{j+1} \right| = O(h^p)$$

We now derive singlestep methods which have different increment function.

Taylor's series method:-

The fundamental numerical method for the solution of,

$$u' = f(t,u),\ u(t_0) = \eta_0,\ t \in [t_0, b] \rightarrow (1)$$

is the Taylor series method.

We assume that the function $u(t)$ can be expanded in Taylor's series about any point t_j,

i.e., $u(t) = u(t_j) + (t-t_j)u'(t_j) + \frac{1}{2!}(t-t_j)^2 u''(t_j) + \dots$

$$+ \frac{1}{P!}(t-t_j)^P u^{(P)}(t_j) + \frac{1}{(P+1)!}(t-t_j)^{P+1} u^{(P+1)}(t_j + \theta h) \rightarrow (2)$$

This expansion holds for $t \in [t_0, b]$ and $0 < \theta < 1$.

sub $t = t_{j+1}$ in (2), we get.

$$u(t_{j+1}) = u(t_j) + h u'(t_j) + \frac{h^2}{2!} u''(t_j) + \dots + \frac{1}{P!} h^P u^P(t_j) + \frac{1}{(P+1)!} h^{P+1} u^{P+1}(t_j + \theta h)$$

$$= u(t_j) + h\phi(t_j, u(t_j), h) + \frac{1}{(P+1)!} h^{P+1} u^{(P+1)}(t_j + \theta h)$$

where, $h\phi(t_j, u(t_j), h) = h u'(t_j) + \frac{h^2}{2!} u''(t_j) + \dots + \frac{h^P}{P!} u^P(t_j)$

Denote by $h\phi(t_j, u_j, h)$ the value obtained from $h\phi(t_j, u(t_j), h)$ by using an approximate value u_j in place of the exact value $u(t_j)$

Neglecting the error term, we have the method

$$u_{j+1} = u_j + h\phi(t_j, u_j, h) \quad j = 0, 1 \dots N-1 \longrightarrow (3)$$

to approximate $u(t_{j+1})$

The error or truncation error of the method is given by,

$$T_{j+1} = \frac{1}{(P+1)!} h^{P+1} u^{(P+1)}(t_j + \theta h) \longrightarrow (4)$$

The method eqn (3) is called Taylor series method of order P.

sub $P=1$ in (3) we've

$$u_{j+1} = u_j + h u_j'$$

$$= u_j + h f(t_j, u_j)$$

which is euler method.

$\therefore$ Euler method can also called as Taylor series method of order 1.

To apply eqn (3), it necessary to know $u(t_j), u'(t_j) \dots u^{(P)}(t_j)$

If t_j and $u(t_j)$ are known then the derivatives can be calculated as follows.

First, the known values t_j and $u(t_j)$ are substituted into differential equation to give

$$u'(t_j) = f(t_j, u(t_j))$$

next, the differential equation

$$u' = f(t,u)$$

is differentiated to obtain expressions for the higher order derivatives of $u(t)$. This we've,

$$u' = f(t,u)$$
$$u'' = f_t + ff_u$$
$$u''' = f_{tt} + 2ff_{tu} + f^2 f_{uu} + f_u(f_t + ff_u)$$
$$\vdots$$

where $f_t, f_u, \ldots$ represent the partial derivatives of f with respect to t and u and so on.

The values $u''(t_j), u'''(t_j) \ldots$ can be computed by substituting $t = t_j$.

$\therefore$ If t_j and $u(t_j)$ are known exactly, then eqn (3) can be used to compute u_{j+1} with an error

$$\frac{h^{P+1}}{(P+1)!} u^{(P+1)}(t_j + \theta h)$$

The numbers of terms to be included in eqn (3) is fixed by the permissible error

If this error ε and the series is truncated at the term $u^{(P)}(t_j)$ then,

$$h^{P+1} / u^{(P+1)}(t_j + \theta h) \Big| < (P+1)!\, \varepsilon$$

(or)

$$h^{P+1} \left| f^{(P)}(t_j + \theta h) \right| < (P+1)!\, \varepsilon \rightarrow (5)$$

we assume that an estimate of

$|f^{(P)}(t_j+\theta h)|$ is Known.

For a given h and ε, eqn (5) will determine P and if P and ε are specified, then it will give an upper bound on h.

$\because t_j+\theta h$ is not known, $|f^{(P)}(t_j+\theta h)|$ in (5) is replaced by its maximum value in $[t_0, b]$.

A way of determining This value is as follows

write one more non-vanishing term in the series than is required and then differentiate this series P times.

The maximum value of this quality in $[t_0, b]$ gives a rough required bound.

1. Example 6.15:-

Given the initial problem $u' = t^2+u^2$, $u(0)=0$. Determine the first three non-zero terms in the Taylor series for $u(t)$ and hence obtain the value for $u(1)$.

Also determine t when the error $u(t)$ obtained from the fixed two non-zero terms is to be less than 10^{-6} after rounding.

soln:-

$$u(0)=0$$

$$u'(0)=0$$

Given: $u' = t^2+u^2$

$$u'' = 2t+2uu'$$

$$= 2t+2u(0) \Rightarrow 2(0)+2u(0)u'(0)$$

$$u''=0$$

$$u''' = 2+2(uu''+u'(u'))$$

$$= 2+2((uu'')+(u')^2)$$

$$= 2+2uu''+2(u')^2$$

$$= 2+2(0)+2(0)$$

$$u'' = 2$$

$$u^4 = 2[uu''' + u''u' + 2u'u'']$$
$$= 2[uu''' + 3u'u'']$$
$$= 2[0 + 2(0)]$$
$$u^4 = 0$$
$$u^5 = 2[uu^{(4)} + u'''u' + u'''u' + u''u'' + 2[u'u''' + u''u'']]$$
$$= 2[uu^{(4)} + u'''u' + u'u''' + u''u'' + 2u'u''' + 2(u'')^2]$$
$$= 2[uu^{(4)} + 4u'''u' + 3(u'')^2]$$
$$u^5 = 0$$
$$u^6 = 2[uu^{(5)} + u'u^{(4)} + 4[u'''u'' + u^{(4)}u'] + 3[2u''u''']$$
$$= 2[uu^{(5)} + u'u^{(4)} + 4u'''u'' + 4u^{(4)} + 6u''u''']$$
$$= 2[uu^{(5)} + 5u'u^{(4)} + 10u'''u'']$$
$$u^{(6)} = 0$$
$$u^{(7)} = 2[uu^{(6)} + u'u^{(5)} + 5(u'u^5 + u''u^4) + 10(u'''u''' + u^{(4)}u'')]$$
$$= 2[uu^{(6)} + u'u^{(5)} + 5u'u^{(5)} + 5u''u^{(4)} + 10(u''')^2 + 10(u^4u'')]$$
$$= 2[uu^{(6)} + 6u'u^{(5)} + 15u''u^{(4)} + 10(u''')^2]$$
$$= 2[0 + 0 + 0 + 10(2)^2]$$
$$= 2(40) = 80$$
$$u^{(7)} = 80$$
$$u^{(8)} = 2[uu^{(7)} + u'u^{(6)} + 6(u'u^{(6)} + u''u^{(5)}) + 15(u''u^{(5)} + u'''u^{(4)}) + 10(2u'''u^{(4)}))]$$
$$= 2[uu^{(7)} + u'u^{(6)} + 6u'u^6 + 6u''u^5 + 15u''u^5 + 15u'''u^4 + 20u'''u^4]$$
$$= 2[uu^7 + 7u'u^6 + 21u''u^5 + 15u'''u^4 + 20u'''u^4]$$
$$u^8 = 0$$

$$u^9 = 2[uu^8 + u'u^7 + 7[u'u^7 + u^6u''] + 21[u''u^6 + u'''u^5]$$
$$+ 15[u'''u^5 + u^4u^4] + 20[u'''u^5 + u^4u^4]]$$
$$= 2[uu^8 + u'u^7 + 7u'u^7 + 7u^6u'' + 21u''u^6 + 21u'''u^5 +$$
$$15u'''u^5 + 15u^4u^4 + 20u'''u^5 + 20(u^4)^2]$$
$$= 2[uu^8 + 8u'u^7 + 28u^6u'' + 56u'''u^5 + 35(u^4)^2]$$
$$u^9 = 0$$
$$u^{10} = 2[uu^9 + u'u^8 + 8u'u^8 + 8u''u^7 + 28u^6u''' +$$
$$28u^7u'' + 56u'''u^6 + 56u^4u^5 + 35(2u^4u^5)]$$
$$= 2[uu^9 + 84u'''u^6 + 126u^4u^5 + 9u'u^8 + 36u''u^7]$$
$$= 2[0]$$
$$u^{10} = 0$$
$$u^{11} = 2[uu^{10} + u'u^9 + 9u'u^9 + 9u''u^9 + 9u''u^8 + 36u''u^8 +$$
$$36u'''u^7 + 84u'''u^7 + 84u^4u^6 + 126u^4u^5 + 126u^5u^5]$$
$$= 2[uu^{10} + 10u'u^9 + 45u''u^8 + 120u''u^7 + 210u^4u^6 + 120(u^5)^2]$$
$$= 2[0+0+120(2\times 80)+0+0]$$
$$= 2\times 19200$$
$$u^{11} = 38400$$

Thus the Taylor series $u(t)$ becomes

$$u(t) = u(t_j) + (t-t_j)u'(t_j) + \dots + \frac{1}{(P+1)!}(t-t_j)^{P+1}u^{P+1}(t_j+\theta h)$$

$$u(t) = 0+0+0+\frac{1}{3!}t^3(2) + 0+0+0+\frac{1}{7!}t^7(80) + 0+0+0+$$
$$\frac{1}{11!}t^{11}(38400)$$

$$= \frac{1}{6}t^3(2) + \frac{1}{5040}t^7(80) + \frac{1}{39916800}t^{11}(38400)$$
$$= \frac{1}{3}t^3 + \frac{1}{63}t^7 + \frac{2}{2079}t^{11}$$
$$\therefore u(t) = \frac{1}{3}t^3 + \frac{1}{63}t^7 + \frac{2}{2079}t^{11}$$

the approximate value of $u(1)$ is given by,

$$u(1) = \frac{1}{3} + \frac{1}{63} + \frac{2}{2079}$$

$$= \frac{693+33+2}{2079} = \frac{728}{2079}$$

$$= 0.350168$$

If only the first two terms are used then the value of t is obtained from,

$$\left| \frac{2}{2079} + t^{(11)} \right| < 0.5 \times 10^{-7}$$

2. Find the three taylor series solution for the third order initial value problem

$$w''' + w w'' = 0 \quad w(0) = 0,\ w'(0) = 0,\ w''(0) = 1.$$

Find the bound on the error for $c \in [0, 0.2]$

Soln:-

$$w(0) = 0,\ w'(0) = 0,\ w''(0) = 1$$

$$w''' + w w''' = 0$$

$$w''' = -w w''$$

$$w'''(0) = -0.1 = 0$$

$$w^{(4)} = -(w w''' + w' w'')$$

$$w^{(4)}(0) = -(0.0 + 0.1)$$

$$= 0$$

$$w^5 = -(w w^4 + w''' w' + w' w''' + w'' w'')$$

$$= -(w w^4 + 2 w''' w' + (w'')^2]$$

$$w^5(0) = -[0.0 + 2(0)(0) + 1]$$

$$w^5(0) = -1$$

$$w^6 = -[w w^{(5)} + w^4 w' + 2(w''' w'' + w^4 w') + 2 w'' w''']$$

$$= -[w w^{(5)} + w^4 w' + 2 w''' w'' + 2 w' w^4 + 2 w'' w''']$$

$$w^6 = -[w w^5 + 3 w^4 w' + 4 w'' w''']$$

$$w^6(0) = -[0(-1)+3(0)(0)+4(0)(1)]$$

$$= 0+0+0$$

$$w^6(0)=0$$

$$w^7 = -[w'w^5+ww^6+3(w^4w''+w'w^5)+4(w''w^4+w'''w''')]$$

$$= -[w'w^5+ww^6+3w^4w''+3w'w^5+4w''w^4+4(w''')^2]$$

$$w^7 = -[4w'w^5+ww^6+7w''w^4+4(w''')^2]$$

$$w^7(0) = -[4(0)(-1)+(0)(0)+7(1)(0)+4(0)^2]$$

$$w^7(0)=0$$

$$w^8 = -[4(w'w^6+w''w^5)+w'w^6+ww^7+7(w''w^5+w'''w^4)+ 8w'''w^4]$$

$$= -[4w'w^6+4w''w^5+w'w^6+ww^7+7w''w^5+7w''w^4+8w'''w^4]$$

$$= -[5w'w^6+11w''w^5+ww^7+15w'''w^5]$$

$$w^8(0) = -[5(0)(0)+11(1)(-1)+(0)(0)+15(0)(0)]$$

$$w^8(0)=11$$

$$w^9 = -[5(w''w^6+w'w^7)+11(w''w^6+w'''w^5)+w'w^7+ ww^8+15(w^4w^4+w'''w^5)]$$

$$= -[5w''w^6+5w'w^7+11w''w^6+11w'''w^5+w'w^7+ww^8+ 15(w^4)^2+15w'''w^5]$$

$$w^9(0) = -[16(1)(0)+6(0)(0)+26(0)(-1)+0(8)+15(0)^2]$$

$$w^9(0)=0$$

$$w^{10} = -[16(w'''w^6+w''w^7)+6(w''w^7+w'w^8)+26(w^4w^5+ w'''w^6)+w'w^8+ww^9+30w^4w^5]$$

$$w^{10} = -[16w'''w^6+16w''w^7+6w''w^7+6w'w^8+26w^4w^5+ 26w'''w^6+w'w^8+ww^9+30w^4w^7]$$

$$w^{10} = -[42w'''w^6+22w''w^7+7w'w^8+ww^9+56w^4w^5]$$

$$\omega^{10}(0) = -[42(0)(0)+22(1)(0)+7(0)(11)+0(0)+56(0)(-1)]$$

$$\omega^{10}(0) = 0$$

$$\omega'' = -[16(\omega^4\omega^6+\omega'''\omega^7)+16(\omega''\omega^7+\omega''\omega^8)+6(\omega'''\omega^7+\omega''\omega^8)+26(\omega^4\omega^6+\omega'''\omega^7)$$
$$+6(\omega''\omega^8+\omega'\omega^9)+26(\omega^5\omega^5+\omega^4\omega^5)+30\omega^5\omega^5+30\omega^4\omega^6]$$
$$+\omega''\omega^8+\omega'\omega^9+\omega'\omega^9+\omega'\omega^9+\omega\omega^{10}$$

$$= -[16\omega^4\omega^6+16\omega'''\omega^7+16\omega''\omega^8+6\omega'''\omega^7+6\omega''\omega^8+$$
$$6\omega''\omega^8+6\omega'\omega^9+26(\omega^5)^2+26\omega^4\omega^5+26\omega^4\omega^6+$$
$$26\omega'''\omega^7+\omega''\omega^8+\omega''\omega^8+\omega'\omega^9+\omega'\omega^9+\omega\omega^{10}+30(\omega^5)^2$$
$$+30\omega^4\omega^6]$$

$$\omega'' = -[72\omega^4\omega^6+64u'''u^7+29\omega''\omega^8+8\omega'\omega^9+26(\omega^5)^2+$$
$$26\omega^4\omega^5+\omega\omega^{10}+30(\omega^5)^2]$$

$$\omega''(0) = -[72(0)(0)+64(0)(0)+29(1)(11)+8(0)(0)+26(-1)^2$$
$$+26(0)(-1)+0(0)+30(-1)^2]$$
$$= -[319+26+39]$$
$$= -375.$$

Thus the Taylor's series $\omega(t)$ becomes

$$\omega(t) = u(t_j)+(t-t_j)u'(t_j)+\ldots+\frac{1}{(P+1)!}(t-t_j)^{P+1}u^{(P+1)}(t_j+\theta h)$$

$$\omega(t) = 0+0+\frac{1}{2!}t^2(1)+0+0+\frac{1}{5!}t^5(-1)+0+0+\frac{1}{8!}t^8(11)+$$
$$0+0+\frac{1}{11!}t^{11}(-375)$$

$$\omega(t) = \frac{t^2}{2!}-\frac{t^5}{5!}+\frac{11}{8!}(t^8)-\frac{375}{11!}t^{11}$$

$$\omega(t) = \frac{t^2}{2!}-\frac{t^5}{5!}+\frac{11}{8!}t^8+E_8$$

where, $|E_8| \leq \max|\omega^9(t)|\,\frac{t^9}{9!}$

writing the next term we've

$$w(t) = \frac{t^2}{2!} - \frac{t^5}{5!} + \frac{11}{8!}t^8 - \frac{375}{11!}t^{11} \ldots$$

$$w'(t) = \frac{2t}{2} - \frac{5t^4}{5!} + \frac{11\times 8t^7}{8!} - \frac{375(11)t^{10}}{11!}$$

$$= t - \frac{t^4}{4!} + \frac{11t^7}{7!} - \frac{375t^{10}}{10!}$$

$$w''(t) = 1 - \frac{4t^3}{4!} + \frac{11(7)t^6}{7!} - \frac{375(10)t^9}{10!}$$

$$= -\frac{t^3}{3!} + \frac{11t^6}{6!} - \frac{375t^9}{9!}$$

$$w'''(t) = \frac{-3t^2}{3!} + 11\times\frac{6t^5}{6!} - \frac{375\times 9t^8}{8!}$$

$$w^4(t) = \frac{-2t}{2} + 11\times\frac{5t^4}{5!} - \frac{375\times 8t^7}{8!}$$

$$w^5(t) = -1 + 11\times\frac{4t^3}{4!} - \frac{375\times 7t^6}{7!}$$

$$w^6(t) = 11\times\frac{3t^2}{3!} - \frac{375\times 6t^5}{6!}$$

$$w^7(t) = 11\times\frac{2t}{2} - \frac{375\times 5t^4}{5!}$$

$$w^8(t) = 11\times t - \frac{375\times 4t^3}{4!}$$

$$w^8(t) = 11\times t - \frac{375\times 3t^2}{3!}$$

$$w^9(t) = \frac{-375\times 3t^2}{3!}$$

$$w^9(t) = -\frac{375}{2}\times t^2$$

Put $t = 0.2$

$$\max_{0\le t\le 0.2} |w^9(t)| = 7.5$$

$$|E_8| \le \frac{7.5(0.2)^9}{9!} \le (1.06)10^{-11}$$

Runge- Kutta methods:-

From the application point of view, the Taylor series method has a major disadvantage. the method requires evaluation of partial derivatives of higher orders manually this is not possible in any practical application.

$\therefore$ we need to develop methods which do not require evaluation and computation of higher order derivatives. the most important class of methods in this direction are the Runge-Kutta methods. However, all these methods should compare with the Taylor series method when they are expanded about a point $t = t_j$. we shall first explain the principle involved in the Runge-Kutta methods.

Integrating the differential equation $u' = f(t, u)$ on the interval $[t_j, t_{j+1}]$, we get

$$\int_{t_j}^{t_{j+1}} \frac{du}{dt} \, dt = \int_{t_j}^{t_{j+1}} f(t, u) dt \quad \rightarrow (1)$$

By the mean value theorem of integral calculus, we obtain

$$u(t_{j+1}) = u(t_j) + h f(t_j + \theta h, u(t_j + \theta h)), \; 0 < \theta < 1 \quad \rightarrow (2)$$

Any value $\theta \in [0,1]$ produces a numerical method. consider the following cases

case $\theta = 0$ when $\theta = 0$ we obtain the approximation

$$u_{j+1} = u_j + h f(t_j, u_j) \rightarrow (3)$$

~~we have~~ the ~~numerical method as~~ which is the Euler method.

case $\theta = 1$ ~~t_{j+1}~~ when $\theta = 1$, we obtain

$$u(t_{j+1}) = u(t_j) + h f(t_j + h, u(t_{j+1}))$$

we have the numerical method as

$$u_{j+1} = u_j + h f_{j+1} \quad \rightarrow (4)$$

which is the backward Euler method.

If we approximate $u_{j+1} = u_j + h f_j$ in (4), i.e by the Euler method, in the argument of f, we get

$$u_{j+1} = u_j + hf(t_{j+1}, u_j + hf_j) \rightarrow (5)$$

If we set $K_1 = hf_j$

$$K_2 = hf(t_{j+1}, u_j + K_1)$$

we get the method as $u_{j+1} = u_j + K_2 \rightarrow (6)$

case $\theta = 1/2$ when $\theta = 1/2$, we obtain

$$u(t_{j+1}) = u(t_j) + hf\left(t_j + \frac{h}{2}, u\left(t_j + \frac{h}{2}\right)\right) \rightarrow (7)$$

However. $t_j + \left(\frac{h}{2}\right)$ is not a nodal point

If we approximate $u\left(t_j + \frac{h}{2}\right)$ in (7) by Euler method with spacing $h/2$, we get

$$u\left(t_j + \frac{h}{2}\right) = u_j + \frac{h}{2} f_j$$

Then, we have the approximation

$$u_{j+1} = u_j + hf\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} f_j\right) \rightarrow (8).$$

If we set $K_1 = hf_j$

$$K_2 = hf\left(t_j + \frac{h}{2}, u_j + \frac{1}{2} K_1\right)$$

Then (8) can be written as

$$u_{j+1} = u_j + K_2 \rightarrow (9)$$

Alternatively, if we use the approximation

$$u'\left(t_j + \frac{h}{2}\right) = \frac{1}{2}\left[u'(t_j) + u'(t_j + h)\right]$$

and the Euler method, we obtain

$$u'\left(t_j + \frac{h}{2}\right) = \frac{1}{2}\left[f(t_j, u_j) + f(t_{j+1}, u_j + hf_j)\right]$$

Thus (7) may be approximated by

$$u_{j+1} = u_j + \frac{h}{2}\left[f(t_j, u_j) + f(t_{j+1}, u_j + hf_j)\right] \rightarrow (10)$$

If we set $K_1 = hf_j$

$$K_2 = hf(t_{j+1}, u_j + K_1)$$

Then (10) can be written as

$$u_{j+1} = u_j + \frac{h}{2}[K_1 + K_2] \rightarrow (11)$$

this method is also called Euler-cauchy method.

consider now, the eqn (1). The integrand on the RHS $f(t,u)$ is the slope of the solution curve which varies continuously in the interval $[t_j, t_{j+1}]$. In deriving the Euler method given by (3), we may interpre that the slope of the solution curve, which varies continuously on $[t_j, t_{j+1}]$, is approximated by the slope at the initial point, that is by $f(t_j, u_j) \approx f(t_j, u(t_j))$. similarly, the methods given in (4) & (7) may be interpreted as the cases when the slope of the solution curve on $[t_j, t_{j+1}]$ is approximated by a single slope at the terminal point and mid point respectively.

Runge-Kutta methods use a weighted average of slopes on the given interval $[t_j, t_{j+1}]$, instead of a single slope. thus, the general Runge-Kutta methods may be defined as

$u_{j+1} = u_j + h$ [weighted average of slopes on the given interval],

consider v slopes on $[t_j, t_{j+1}]$. Define

$$\left.\begin{aligned} K_1 &= hf(t_j + c_1 h,\ u_j + a_{11}K_1 + a_{12}K_2 + \dots + a_{1v}K_v) \\ K_2 &= hf(t_j + c_2 h,\ u_j + a_{21}K_1 + a_{22}K_2 + \dots + a_{2v}K_v) \\ &\vdots \\ K_v &= hf(t_j + c_v h,\ u_j + a_{v1}K_1 + a_{v2}K_2 + \dots + a_{vv}K_v) \end{aligned}\right\} \rightarrow (12)$$

the Runge-Kutta method is now defined by

$$u_{j+1} = u_j + w_1 K_1 + w_2 K_2 + \dots + w_v K_v \quad \rightarrow (13)$$

this is also called a v-stage Runge-Kutta method. It is a fully implicit method which uses v evaluations of f. the matrix of coefficients a_{ij} is the full $v \times v$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1v} \\ a_{21} & a_{22} & \dots & a_{2v} \\ \vdots & & & \\ a_{v1} & a_{v2} & \dots & a_{vv} \end{bmatrix} \rightarrow (14)$$

It is very difficult to derive the fully implicit methods. If in A, we set the elements in the upper triangular part as zeros, then (12),(13) define semi-implicit methods, where

$$\left.\begin{aligned} K_1 &= hf(t_j + c_1 h,\ u_j + a_{11} K_1) \\ K_2 &= hf(t_j + c_2 h,\ u_j + a_{21} K_1 + a_{22} K_2) \\ &\vdots \\ K_v &= hf(t_j + c_v h,\ u_j + a_{v1} K_1 + a_{v2} K_2 + \dots + a_{vv} K_v) \end{aligned}\right\} \rightarrow (15)$$

If in A, we set the elements on the diagonal and the upper triangular part as zeros, then (12),(13) define explicit methods, where

$$\left.\begin{aligned} K_1 &= hf(t_j, u_j) \\ K_2 &= hf(t_j + c_2 h,\ u_j + a_{21} K_1) \\ K_3 &= hf(t_j + c_3 h,\ u_j + a_{31} K_1 + a_{32} K_2) \\ &\vdots \\ K_v &= hf(t_j + c_v h,\ u_j + a_{v1} K_1 + a_{v2} K_2 + \dots + a_{v,v-1} K_{v-1}) \end{aligned}\right\} \rightarrow (16)$$

As mentioned earlier, the methods defined by (12)(13) or (15),(13) or (16),(13) should compare with the Taylor series method. Hence, to determine the parameters c_i's, a_i's and w_i's in the Runge-Kutta methods, we expand u_{j+1} and f's in powers of h $\ni$ it agrees with the Taylor series expansion of the solution of the differential eqn upto a certain number of terms.

We shall first consider the derivation of explicit Runge-Kutta methods.

Explicit Runge-Kutta Methods

Second order methods:-

Consider the following Runge-Kutta method with two slopes

$$\left.\begin{aligned} K_1 &= hf(t_j, u_j) \\ K_2 &= hf(t_j + c_2 h,\ u_j + a_{21} K_1) \\ u_{j+1} &= u_j + w_1 K_1 + w_2 K_2 \end{aligned}\right\} \rightarrow (1)$$

where the parameters c_2, a_{21}, w_1 & w_2 are chosen to make u_{j+1} closer to $u(t_{j+1})$. There are 4 parameters to be determined.

Now, Taylor series expansion about t_j gives

$$u(t_{j+1}) = u(t_j)+hu'(t_j)+\frac{h^2}{2!}u''(t_j)+\frac{h^3}{3!}u'''(t_j)+\cdots$$

$$= u(t_j)+hf(t_j,u(t_j))+\frac{h^2}{2!}(f_t+ff_u)_{t_j}+\frac{h^3}{3!}[f_{tt}+2ff_{tu}+$$

$$f^2f_{uu}+f_u(f_t+ff_u)]_{t_j}+\cdots \rightarrow (2)$$

we also have $\quad K_1=hf_j$

$$K_2=hf(t_j+c_2h,u_j+a_{21}hf_j)$$

$$=h[f_j+h(c_2f_t+a_{21}ff_u)_{t_j}+\frac{h^2}{2}(c^2f_{tt}+2c_2a_{21}ff_{tu}+a_{21}^2f^2f_{uu})_{t_j}+\cdots]$$

sub the value of K_1 & K_2 in (1) we get

$$u_{j+1}=u_j+(w_1+w_2)hf_j+h^2(w_2c_2f_t+w_2a_{21}ff_u)_{t_j}+$$

$$\frac{h^3}{2}w_2(c_2^2f_{tt}+2c_2a_{21}ff_{tu}+a_{21}^2f^2f_{uu})_{t_j}\cdots \rightarrow (3)$$

comparing the coefficients of h + h^2 in (2) & (3) we obtain

$$w_1+w_2=1$$

$$c_2w_2=\frac{1}{2}$$

$$a_{21}w_2=\frac{1}{2}$$

The solution of this system is $a_{21}=c_2,\ w_2=\frac{1}{2c_2},\ w_1=1-\frac{1}{2c_2}$ $\rightarrow (4)$

where $c_2\neq 0$ is arbitrary. It is not possible to compare the coefficients of h^3, as there are 5 terms in (2) + only 3 terms in (3). $\therefore$ the Runge-Kutta method using 2 evaluation of f is

$$u_{j+1}=u_j+\left(1-\frac{1}{2c_2}\right)K_1+\frac{1}{2c_2}K_2 \rightarrow (5)$$

$$K_1=hf(t_j,u_j)$$

$$K_2=hf(t_j+c_2h,u_j+c_2K_1)$$

sub (4) in (3) we get

$$u_{j+1}=u_j+hf_j+\frac{h^2}{2}(f_t+ff_u)_{t_j}+\frac{h^3c_2}{4}(f_{tt}+2ff_{tu}+f^2f_{uu})_{t_j}+\cdots \rightarrow (6)$$

the local truncation error is gn by $T_{j+1} = u(t_{j+1}) - u_{j+1}$

$$= h^3\left[\left(\frac{1}{6} - \frac{c_2}{4}\right)(f_{tt} + 2ff_{tu} + f^2 f_{uu})_{t_j} + \frac{1}{6}\{f_u(f_t + ff_u)\}_{t_j} + \cdots\right] \quad \to (7)$$

which shows that the method (1) is of 2^{nd} order. It may be noted that every Runge-Kutta method should reduce to a quadrature formula when $f(t,u)$ is independent of u, with w_i's as weights & c_i's as abscissas.

the free parameter c_2 is usually taken b/w 0 & 1. sometimes c_2 is chosen $\ni$ one of the w_i's in the method (1) is zero. For eg, the choice $c_2 = 1/2$ makes $w_1 = 0$.

If $c_2 = 1/2$ we get

$$u_{j+1} = u_j + K_2$$

where

$$K_1 = hf(t_j, u_j)$$

$$K_2 = hf\left(t_j + \frac{h}{2}, u_j + \frac{1}{2}K_1\right)$$

which is the Euler method with spacing $h/2$. It is also called the modified Euler-cauchy method. It reduces to the mid point quadrature rule when $f(t,u)$ is independent of u.

For $c_2 = 1$, we get $\quad u_{j+1} = u_j + \frac{1}{2}(K_1 + K_2)$

where

$$K_1 = hf(t_j, u_j)$$

$$K_2 = hf(t_j + h, u_j + K_1)$$

which reduces to the trapezoidal rule when $f(t,u)$ is independent of u. this method is also called as the Euler cauchy or Heun method.

Minimization of local Truncation Error:-

An alternative way of choosing the arbitrary parameter is to minimize the sum of the absolute values of the coefficients in the term T_{j+1}. such a choice gives an optimal method in the sense of minimum truncation error. Here we use the Lotkin's bounds which are defined by

$$\left|\frac{\partial^{i+j} f}{\partial t^i \partial u^j}\right| < \frac{L^{i+j}}{M^{j-1}}, \quad i,j = 0,1,2\ldots$$

we find that $|f| < M$, $|f_u| < L$, $|f_1| < LM$

$$|f_{tt}| < L^2M,\ |f_{tu}| < L^2,\ |f_{uu}| < \frac{L^2}{M}$$

Thus, $|T_{j+1}|$ becomes $\quad |T_{j+1}| < ML^2h^3\left[4\left|\frac{1}{6}-\frac{c_2}{4}\right|+\frac{1}{3}\right]$

The minimum value of T_{j+1} occurs for $c_2 = 2/3$ in which case $|T_{j+1}| < ML^2h^3/3$. we have the optimal method as

$$u_{j+1} = u_j + \frac{1}{4}(K_1 + 3K_2) \quad \rightarrow (1)$$

$$K_1 = hf(t_j, u_j)$$

$$K_2 = hf\left(t_j + \frac{2}{3}h,\ u_j + \frac{2}{3}K_1\right)$$

If the arbitrary parameters are determined by putting the leading coefficients in $|T_{j+1}|$ to zero, then such a formula is called nearly optimal. It may be noted that the explicit Runge-Kutta methods using 2 evaluations of f have one arbitrary parameter & have produced 2nd order methods.

Third order methods:-

we now use 3 evaluations of f & define the method as

$$u_{j+1} = u_j + W_1K_1 + W_2K_2 + W_3K_3 \quad \rightarrow (1)$$

$$K_1 = hf(t_j, u_j)$$

$$K_2 = hf(t_j + c_2h,\ u_j + a_{21}K_1)$$

$$K_3 = hf(t_j + c_3h,\ u_j + a_{31}K_1 + a_{32}K_2) \quad \rightarrow (2)$$

Expand K_2, K_3 in Taylor series about t_j & sub in (1). collecting the terms & comparing upto $O(h^3)$, we obtain 6 eqns in 8 parameters $W_1, W_2, W_3, c_2, c_3, a_{21}, a_{31}$ & a_{32}.

$\therefore$ the method contain 2 arbitrary parameters. The eqns are

$$a_{21} = c_2,\quad a_{31} + a_{32} = c_3,\quad W_1 + W_2 + W_3 = 1,$$

$$c_2W_2 + c_3W_3 = \frac{1}{2},\quad c_2^2W_2 + c_3^2W_3 = \frac{1}{3},\quad c_2a_{32}W_3 = \frac{1}{6} \quad \rightarrow (3)$$

Multiplying the (4th) & (5th) by c_2a_{32} & using 6th eqn of (3) we get

$c_2^2 a_{32} w_2 + c_3\left(\frac{1}{6}\right) = \frac{1}{2} c_2 a_{32}$, $c_2^3 a_{32} w_2 + c_3^2\left(\frac{1}{6}\right) = \frac{1}{3} c_2 a_{32}$

eliminating w_2 from these 2 eqns, we find that no soln exists unless

$$\frac{3c_2a_{32}-c_3}{6c_2^2a_{32}} = \frac{2c_2a_{32}-c_3^2}{6c_2^3a_{32}} \quad \text{or} \quad a_{32} = \frac{c_3(c_3-c_2)}{c_2(2-3c_2)} \rightarrow (4)$$

usually c_2, c_3 are arbitrarily chosen & a_{32} is determined from (4). if $c_2 = c_3$, then we immediately obtain from 4^{th} & 5^{th} eqn of (4), that $c_2 = 2/3$. the values of the remaining parameters are obtained from (3)

when $c_2 = c_3$, we get $c_2 = 2/3$ & $a_{21} = 2/3$. we get $a_{31} = 0, a_{32} = 2/3$, $w_1 = 2/8$, $w_2 = 3/8$ & $w_3 = 3/8$. The Runge kutta method is obtained as

$$u_{j+1} = u_j + \frac{1}{8}(2K_1 + 3K_2 + 3K_3) \longrightarrow (5)$$

$$K_1 = hf(t_j, u_j)$$

$$K_2 = hf\left(t_j + \frac{2h}{3}, u_j + \frac{2}{3}K_1\right)$$

$$K_3 = hf\left(t_j + \frac{2h}{3}, u_j + \frac{2}{3}K_2\right)$$

we note that the methods use 3 evaluations of f, has 2 arbitrary parameters & produce 3^{rd} order methods.

Fourth order methods:-

we now use 4 evaluations of f & define the method as

$$u_{j+1} = u_j + w_1K_1 + w_2K_2 + w_3K_3 + w_4K_4 \longrightarrow (1)$$

where $K_1 = hf(t_j, u_j)$

$$K_2 = hf(t_j + c_2h, u_j + a_{21}K_1)$$

$$K_3 = hf(t_j + c_3h, u_j + a_{31}K_1 + a_{32}K_2)$$

$$K_4 = hf(t_j + c_4h, u_j + a_{41}K_1 + a_{42}K_2 + a_{43}K_3)$$

the parameters $c_2, c_3, c_4, a_{21}, \dots a_{43}$ & $w_1, \dots w_4$ are chosen to make u_{j+1} closer to $u(t_{j+1})$. Expanding K_2, K_3, K_4 in Taylor series about t_j, sub in (1) & matching the coefficients of h, h^2, h^3 & h^4, we obtain the following system of equations

$$c_2 = a_{21}$$

$$c_3 = a_{31} + a_{32}$$

$$c_4 = a_{41} + a_{42} + a_{43}$$

$$w_1 + w_2 + w_3 + w_4 = 1$$

$$w_2 c_2 + w_3 c_3 + w_4 c_4 = \frac{1}{2}$$

$$w_2 c_2^2 + w_3 c_3^2 + w_4 c_4^2 = \frac{1}{3}$$

$$w_3 c_2 a_{32} + w_4 (c_2 a_{42} + c_3 a_{43}) = \frac{1}{6}$$

$$w_2 c_2^3 + w_3 c_3^3 + w_4 c_4^3 = \frac{1}{4}$$

$$w_3 c_2^2 a_{32} + w_4 (c_2^2 a_{42} + c_3^2 a_{43}) = \frac{1}{12}$$

$$w_3 c_2 c_3 a_{32} + w_4 (c_2 a_{42} + c_3 a_{43}) c_4 = \frac{1}{8}$$

$$w_4 c_2 a_{32} a_{43} = \frac{1}{24} \qquad \longrightarrow (2)$$

we have 11 eqns in 13 unknowns. ∴ there are 2 arbitrary parameter terms upto $o(h^4)$ are compared, the truncation error is of $o(h^5)$ & the order of the method is 4. The simplest soln of the eqns (2) is gn by

$c_2 = c_3 = 1/2$, $c_4 = 1$, $w_2 = w_3 = 1/3$

$w_1 = w_4 = 1/6$, $a_{21} = 1/2$, $a_{31} = 0$, $a_{32} = 1/2$, $a_{41} = 0$, $a_{42} = 0$, $a_{43} = 1$

Thus, the 4^{th} order method (1) becomes

$$u_{j+1} = u_j + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \qquad \rightarrow (3)$$

$$K_1 = hf(t_j, u_j)$$

$$K_2 = hf\left(t_j + \frac{1}{2}h,\ u_j + \frac{1}{2}K_1\right)$$

$$K_3 = hf\left(t_j + \frac{1}{2}h,\ u_j + \frac{1}{2}K_2\right)$$

$$K_4 = hf(t_j + h,\ u_j + K_3)$$

we now list the 2^{nd}, 3^{rd} & 4^{th} order Runge-Kutta methods

second order methods:-

$$\begin{array}{c|cc} c_2 & a_{21} & \\ \hline & w_1 & w_2 \\ \hline \end{array}$$

$$\begin{array}{c|cc} 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

Euler cauchy or Heun

$$\begin{array}{c|cc} 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$

Modified Euler-cauchy

$$\begin{array}{c|cc} 2/3 & 2/3 & \\ \hline & 1/4 & 3/4 \end{array}$$

optimal

Third order methods :-

$$\begin{array}{c|ccc} c_2 & a_{21} & & \\ c_3 & a_{31} & a_{32} & \\ \hline & w_1 & w_2 & w_3 \end{array}$$

$$\begin{array}{c|ccc} 2/3 & 2/3 & & \\ 2/3 & 0 & 2/3 & \\ \hline & 2/8 & 3/8 & 3/8 \end{array}$$

Nystrom

$$\begin{array}{c|ccc} 1/2 & 1/2 & & \\ 3/4 & 0 & 3/4 & \\ \hline & 2/9 & 3/9 & 4/9 \end{array}$$

Nearly optimal

$$\begin{array}{c|ccc} 1/2 & 1/2 & & \\ 1 & -1 & 2 & \\ \hline & 1/6 & 4/6 & 1/6 \end{array}$$

classical

$$\begin{array}{c|ccc} 1/3 & 1/3 & & \\ 2/3 & 0 & 2/3 & \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

Heun

Fourth order methods:-

$$\begin{array}{c|cccc} c_2 & a_{21} & & & \\ c_3 & a_{31} & a_{32} & & \\ c_4 & a_{41} & a_{42} & a_{43} & \\ \hline & w_1 & w_2 & w_3 & w_4 \end{array}$$

$$\begin{array}{c|cccc} 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 2/6 & 2/6 & 1/6 \end{array}$$

classical

$$\begin{array}{c|cccc} 1/3 & 1/3 & & & \\ 2/3 & -1/3 & 1 & & \\ 1 & 1 & -1 & 1 & \\ \hline & 1/8 & 3/8 & 3/8 & 1/8 \end{array}$$

Kutta

It can be observed that all explicit Runge-Kutta methods satisfy the equations. $c_i = \sum_{j=1}^{i-1} a_{ij}$ & $\sum_{i=1}^{v} w_i = 1 \rightarrow (1)$

$\because$ c_i's can be obtained from a_{ij}'s the no. of independent parameters is equal to (no. of a_{ij}'s) + (no. of w_i's) $= [1+2+\dots+(v-1)]+v$

$$= \frac{1}{2}v(v+1)$$

High order Methods:-

we note that the 2^{nd} order method requires 2 function evaluations for each step of integration.

$\|^{ly}$, we find that the 3^{rd} & 4^{th} order methods require 3 & 4 function evaluations respectively for each step of integration.

The minimum no of function evaluations v for a given order p is gn in the following table

p	2	3	4	5	6	....
v	2	3	4	6	8	....

There is a jump in v from 4 to 6 when p goes from 4 to 5. Hence, methods with $p<v$ are not generally used.

Eg: 6.17. Given the initial value problem $u'=-2tu^2; u(0)=1$ estimate $u(0.4)$ using (i) modified Euler-Cauchy method and ii) Heun method with $h=0.2$. compare the results with the exact solution $u(t)=1/(1+t^2)$

i) The modified Euler-cauchy method is gn by $u_{j+1}=u_j+K_2$.

$$u_{j+1}=u_j+K_2$$

$$K_1=hf(t_j,u_j)=0.2\,[-2t_ju_j^2]=-0.4t_ju_j^2$$

$$\& \; K_2=hf\left(t_j+\tfrac{h}{2},u_j+\tfrac{1}{2}K_1\right)=-0.4(t_j+0.1)\left(u_j+\tfrac{1}{2}K_1\right)^2$$

For $j=0$ we've $t_0=0$, $u_0=1$ $K_1=0$ $K_2=-0.4(0.1)(1)^2=-0.04$

$$u(0.2)\approx u_1=u_0+K_2=1-0.04=0.96$$

For $j=1$, we've $t_1=0.2$, $u_1=0.96$, $K_1=-0.4(0.2)(0.96)^2=-0.073728$

$$K_2=-0.4(0.2+0.1)(0.96-0.036864)^2=-0.102262$$

$$u(0.4)\approx u_2=u_1+K_2=0.96-0.102262=0.857738.$$

ii) The Heun method is gn by $u_{j+1}=u_j+\frac{1}{2}(K_1+K_2)$

where $K_1=hf(t_j,u_j)=-0.4t_ju_j^2$

$$K_2=hf(t_j+h,u_j+K_1)=-0.4(t_j+0.2)(u_j+K_1)^2$$

For $j=0$, we've $t_0=0$, $u_0=1$, $K_1=0$, $K_2=-0.4(0.2)(1)^2=-0.08$

$$u(0.2)\approx u_1=u_0+\tfrac{1}{2}(K_1+K_2)=1+\tfrac{1}{2}(-0.08)=0.96.$$

For j=1 we've $t_1=0.2$, $u_1=0.96$, $K_1=-0.4(0.2)(0.96)^2=-0.073728$

$K_2=-0.4(0.2+0.2)(0.96-0.073728)^2=-0.125676$

$u(0.4)\approx u_2=u_1+\frac{1}{2}(K_1+K_2)=0.96+\frac{1}{2}(-0.073728-0.125676)=0.860298$

the exact soln is $u(0.2)=0.961538$, $u(0.4)=0.862069$

the absolute errors in the numerical solns are

Modified Euler cauchy method : $\epsilon(0.2)=0.001538$, $\epsilon(0.4)=0.004331$

Heun' method: $\epsilon(0.2)=0.001538$, $\epsilon(0.4)=0.001771$.

Eg: 6.18: solve the initial value problem $u'=-2tu^2$, $u(0)=1$ with $h=0.2$ on the interval $[0, 0.4]$. use the 4^{th} order classical Runge-Kutta method. compare with the exact solution.

For j=0, we get

$t_0=0,\ u_0=1$

$K_1=hf(t_0,u_0)=-2(0.2)(0)(1)^2=0$

$K_2=hf(t_0+\frac{h}{2},u_0+\frac{1}{2}K_1)=-2(0.2)\left(\frac{0.2}{2}\right)(1)^2=-0.04$

$K_3=hf(t_0+\frac{h}{2},u_0+\frac{1}{2}K_2)=-2(0.2)\left(\frac{0.2}{2}\right)(0.98)^2=-0.038416$

$K_4=hf(t_0+h,u_0+K_3)=-2(0.2)(0.2)(0.961584)^2=-0.0739715$

$u(0.2)\approx u_1=1+\frac{1}{6}[0-0.08-0.076832-0.0739715]=0.9615328$

For j=1, we get $t_1=0.2$, $u_1=0.9615328$

$K_1=hf(t_1,u_1)=-2(0.2)(0.2)(0.9615328)^2=-0.0739636$

$K_2=hf(t_1+\frac{h}{2},u_1+\frac{K_1}{2})=-2(0.2)(0.3)(0.924551)^2=-0.1025753$

$K_3=hf(t_1+\frac{h}{2},u_1+\frac{K_2}{2})=-2(0.2)(0.3)(0.9102451)^2=-0.0994255$

$K_4=hf(t_1+h,u_1+K_3)=-2(0.2)(0.4)(0.8621073)^2=-0.1189166$

$u(0.4)\approx u_2=0.9615328+\frac{1}{6}[-0.0739636-0.2051506-0.1988510-0.1189166]$

$=0.8620525$

the absolute errors in the numerical solutions are

$\epsilon(0.2)=|0.961539-0.961533|=0.000006$

$\epsilon(0.4)=|0.862069-0.862053|=0.000016$.

Estimation of Local Truncation error:-

In the numerical soln of differential eqns, it is desirable to have an estimate of the local truncation error of the soln at each step. This enables us to adjust the step size. In our discussion, the rounding error will be ignored. A simple procedure for estimating the truncation error is called extrapolation. Here, we find the difference b/w 2 solns u_{j+1}^* & u_{j+1} where u_{j+1}^* & u_{j+1} are calculated using the step sizes $h/2$ & h respectively. we've $|u(t_{j+1}) - u_{j+1}^*| < |u(t_{j+1}) - u_{j+1}|$

& $\epsilon_{j+1} = u(t_{j+1}) - u_{j+1} \propto u_{j+1}^* - u_{j+1}$.

For Runge Kutta method, using 1-step & 2 half steps (extrapolation) procedure can be very expensive as the cost of computation increases with no of fun evaluations. we list the total no. of function evaluations required per step for a pth order method, using 1-step & 2-half step to calculate u_{j+1} & u_{j+1}^* in the following table.

p	2	3	4	5
v^*	5	8	11	14

where v^* is the total evaluations of f per step.

This method is generally used to calculate the total (not local) error for any method. An economical procedure is the Runge-Kutta Fehlberg method. we use Runge-Kutta method of higher order to compute u_{j+1}^* & a lower order Runge-Kutta method to compute u_{j+1} using same step size. An important achievement of this method is that the lower order method has the K_i's common with the higher order method. The Runge-Kutta-Fehlberg 4th order pair of formulas is

$$u_{j+1} = u_j + \left[\frac{25}{216}K_1 + \frac{1408}{2565}K_3 + \frac{2197}{4104}K_4 - \frac{1}{5}K_5\right],$$

with $T_{j+1} = O(h^5)$ & $\quad \rightarrow (1)$

$$u^*_{j+1} = u_j + \left[\frac{16}{135}K_1 + \frac{6656}{12825}K_3 + \frac{28561}{56430}K_4 - \frac{9}{50}K_5 + \frac{2}{55}K_6\right] \rightarrow (2)$$

with $T^*_{j+1} = O(h^6)$

where $K_1 = hf(t_j, u_j)$

$$K_2 = hf\left(t_j + \frac{1}{4}h,\ u_j + \frac{1}{4}K_1\right)$$

$$K_3 = hf\left(t_j + \frac{3}{8}h,\ u_j + \frac{3}{32}K_1 + \frac{9}{32}K_2\right)$$

$$K_4 = hf\left(t_j + \frac{12}{13}h,\ u_j + \frac{1932}{2197}K_1 - \frac{7200}{2197}K_2 + \frac{7296}{2197}K_3\right)$$

$$K_5 = hf\left(t_j + h,\ u_j + \frac{439}{216}K_1 - 8K_2 + \frac{3680}{513}K_3 - \frac{845}{4104}K_4\right)$$

$$K_6 = hf\left(t_j + \frac{1}{2}h,\ u_j - \frac{8}{27}K_1 + 2K_2 - \frac{3544}{2565}K_3 + \frac{1859}{4104}K_4 - \frac{11}{40}K_5\right)$$

the formula for u_{j+1} is of 4^{th} order & requires 5 fun evaluations & with an additional fun evaluation we obtain u^*_{j+1}. thus $u^*_{j+1} - u_{j+1}$ requires only 6 as compared to 11 fun evaluations in case of 1-step & 2-1/2 steps procedure. thus, an estimate of the 4^{th} order accuracy may be determined during the course of computation with only 6 fun evaluations this method can also be used to construct a variable mesh method, the mesh varying with the accuracy of the solns required.

System of equations:-

Consider the system of n eqns

$$\frac{du}{dt} = f(t, u_1, u_2 \dots u_n)$$

$$u(t_0) = \eta \qquad \rightarrow (1)$$

where $u = [u_1, u_2 \dots u_n]^T$, $f = [f_1, f_2 \dots f_n]^T$, $\eta = [\eta_1, \eta_2 \dots \eta_n]^T$

All the methods derived earlier can be used to solve the system of eqns (1) by writing the methods in vector form. let us now apply some of these methods to solve (1)

<u>Taylor series method</u>:-

we write (2) in vector form as

$$u_{j+1} = u_j + hu'_j + \frac{h^2}{2!}u''_j + \dots + \frac{h^p}{p!}u_j^{(p)},\ j = 0, 1, 2 \dots N-1 \rightarrow (1)$$

where

$$u_j^{(K)} = \begin{bmatrix} u_{1,j}^{(K)} \\ u_{2,j}^{(K)} \\ \vdots \\ u_{n,j}^{(K)} \end{bmatrix} = \begin{bmatrix} \frac{d^{K-1}}{dt^{K-1}} f_1(t_j, u_{1,j}, u_{2,j} \cdots u_{n,j}) \\ \vdots \\ \frac{d^{K-1}}{dt^{K-1}} f_n(t_j, u_{1,j}, u_{2,j} \cdots u_{n,j}) \end{bmatrix}$$

In particular, the Euler method can be written as

$$u_{j+1} = u_j + h u_j' \quad , j = 0, 1, \ldots N-1. \quad \longrightarrow (2)$$

Runge-Kutta method of second order:-

consider the Euler cauchy or the Heun method. we write it in vector form as $u_{j+1} = u_j + \frac{1}{2}(K_1 + K_2) \quad \rightarrow (1)$

where $K_{i1} = hf_i(t_j, u_{1,j}, u_{2,j} \cdots u_{n,j})$.

$K_{i2} = hf_i(t_j + h, u_{1,j} + K_{11}, u_{2,j} + K_{21} \cdots u_{n,j} + K_{n1})$ $i = 1, 2 \ldots n$

In an explicit form (1) becomes

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{n,j+1} \end{bmatrix} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{n,j} \end{bmatrix} + \frac{1}{2}\left[\begin{bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{n,1} \end{bmatrix} + \begin{bmatrix} K_{12} \\ K_{22} \\ \vdots \\ K_{n,2} \end{bmatrix}\right] j = 0, 1 \ldots N-1 \longrightarrow (2)$$

Runge-Kutta (classical) Method of Fourth order:-

we write the vector format of (3) as

$$u_{j+1} = u_j + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \quad \longrightarrow (1)$$

where

$$K_1 = \begin{bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{n1} \end{bmatrix}, K_2 = \begin{bmatrix} K_{12} \\ K_{22} \\ \vdots \\ K_{n2} \end{bmatrix}, K_3 = \begin{bmatrix} K_{13} \\ K_{23} \\ \vdots \\ K_{n3} \end{bmatrix}, K_4 = \begin{bmatrix} K_{14} \\ K_{24} \\ \vdots \\ K_{n4} \end{bmatrix}$$

& $K_{i1} = hf_i(t_j, u_{1,j}, u_{2,j} \ldots u_{n,j})$.

$K_{i2} = hf_i(t_j + \frac{h}{2}, u_{1,j} + \frac{1}{2}K_{11}, u_{2,j} + \frac{1}{2}K_{21}, \ldots u_{n,j} + \frac{1}{2}K_{n1})$

$K_{i3} = hf_i(t_j + \frac{h}{2}, u_{1,j} + \frac{1}{2}K_{12}, u_{2,j} + \frac{1}{2}K_{22}, \cdots u_{n,j} + \frac{1}{2}K_{n2})$

$K_{i4} = hf_i(t_j + h, u_{1,j} + K_{13}, u_{2,j} + K_{23} \cdots u_{n,j} + K_{n3})$

$i = 1(1)n$

In an explicit form, (6.117)$^{(1)}$ becomes

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{n,j+1} \end{bmatrix} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{n,j} \end{bmatrix} + \frac{1}{6}\left(\begin{bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{n1} \end{bmatrix} + 2\begin{bmatrix} K_{12} \\ K_{22} \\ \vdots \\ K_{n2} \end{bmatrix} + 2\begin{bmatrix} K_{13} \\ K_{23} \\ \vdots \\ K_{n3} \end{bmatrix} + \begin{bmatrix} K_{14} \\ K_{24} \\ \vdots \\ K_{n4} \end{bmatrix} \right) \rightarrow (2)$$

$j = 1, 2 \dots N-1$

Example 6.19:- compute an approximation to $u(1), u'(1)$ and $u''(1)$ with the Taylor series method of 2^{nd} order and step length $h=1$, for the initial value problem

$$u''' + 2u'' + u' - u = \cos t, \quad 0 \le t \le 1$$
$$u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 2$$

after reducing it to a system of first order equations.

set $u = v_1$, $v_1' = v_2$, $v_2' = v_3$ note that $v_2 = v_1' = u'$, $v_3 = v_2' = u''$ and $v_3' = v_2'' = u'''$.

The system of eqns is

$$v_1' = v_2 \qquad v_1(0) = 0$$
$$v_2' = v_3 \qquad v_2(0) = 1$$
$$v_3' = \cos t - 2v_3 - v_2 + v_1, \quad v_3(0) = 2$$

or $$v' = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}' = \begin{bmatrix} v_2 \\ v_3 \\ \cos t - 2v_3 - v_2 + v_1 \end{bmatrix}, \quad v(0) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

The Taylor series method of 2^{nd} order is

$$v(t_0 + h) = v_0 + h v_0' + \frac{h^2}{2} v_0'' = v_0 + v_0' + \frac{1}{2} v_0''$$

$\because h = 1$, we've

$$v_0' = \begin{bmatrix} v_2(0) \\ v_3(0) \\ 1 - 2v_3(0) - v_2(0) + v_1(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

$$v'' = \begin{bmatrix} v_2' \\ v_3' \\ -\sin t - 2v_3' - v_2' + v_1' \end{bmatrix} \text{ and } v_0'' = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$$

Hence, $v(1) = v_0 + v_0' + \frac{1}{2} v_0'' = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3/2 \end{bmatrix}$

$\therefore$ $u(1) = 2$, $u'(1) = 1$, $u''(1) = 3/2$.

Eg: 6.20 solve the system of equations

$$u' = -3u + 2v, \quad u(0) = 0$$
$$v' = 3u - 4v, \quad v(0) = 0.5$$

with $h = 0.2$ on the interval $[0, 0.4]$ use the i) Euler-cauchy method and ii) classical Runge-Kutta 4^{th} order method.

i) the Euler-cauchy method is gn by

$$u_{j+1} = u_j + \frac{1}{2}(K_1 + K_2), \quad j = 0, 1$$
$$K_1 = h f(t_j, u_j)$$
$$K_2 = h f(t_j + h, u_j + K_1)$$

For $j = 0$, we've $t_0 = 0$, $u_0 = 0$, $v_0 = 0.5$

$K_{11} = h f_1(t_0, u_0, v_0) = 0.2(-3u_0 + 2v_0)$
$= 0.2[-3\times 0 + 2\times 0.5] = 0.2$

$K_{21} = h f_2(t_0, u_0, v_0) = 0.2(3u_0 - 4v_0)$
$= 0.2[3\times 0 - 4\times 0.5] = -0.4$

$K_{12} = h f_1(t_0 + h, u_0 + K_{11}, v_0 + K_{21})$
$= 0.2[-3(u_0 + K_{11}) + 2(v_0 + K_{21})]$
$= 0.2[-3(0 + 0.2) + 2(0.5 - 0.4)] = -0.08$

$K_{22} = h f_2(t_0 + h, u_0 + K_{11}, v_0 + K_{21})$
$= 0.2[3(u_0 + K_{11}) - 4(v_0 + K_{21})] = 0.2[3(0 + 0.2) - 4(0.5 - 0.4)]$
$= 0.04$

$u(0.2) \approx u_1 = u_0 + \frac{1}{2}(K_{11} + K_{12}) = 0 + \frac{1}{2}(0.2 - 0.08) = 0.06$

$v(0.2) \approx v_1 = v_0 + \frac{1}{2}(K_{21} + K_{22}) = 0.5 + \frac{1}{2}(-0.4 + 0.04) = 0.32$

For $j = 1$, we've $t_1 = 0.2$, $u_1 = 0.06$, $v_1 = 0.32$

$K_{11} = hf_1(t_1, u_1, v_1) = 0.2(-3u_1 + 2v_1)$

$= 0.2(-3\times 0.06 + 2\times 0.32) = 0.092$

$K_{21} = hf_2(t_1, u_1, v_1) = 0.2(3u_1 - 4v_1)$

$= 0.2(3\times 0.06 - 4\times 0.32) = -0.22$

$K_{12} = hf_1(t_1+h, u_1+K_{11}, v_1+K_{21})$

$= 0.2[-3(u_1+K_{11}) + 2(v_1+K_{21})]$

$= 0.2[-3(0.06+0.092) + 2(0.32-0.22)] = 0.0512$

$K_{22} = hf_2(t_1+h, u_1+K_{11}, v_1+K_{21}) = 0.2[3(u_1+K_{11}) - 4(v_1+K_{21})]$

$= 0.2[3(0.06+0.092) - 4(0.32-0.22)] = 0.0112.$

$u(0.4) \approx u_2 = u_1 + \frac{1}{2}(K_{11}+K_{12}) = 0.06 + \frac{1}{2}(0.092 - 0.0512) = 0.0804$

$v(0.4) \approx v_2 = v_1 + \frac{1}{2}(K_{21}+K_{22}) = 0.32 + \frac{1}{2}(-0.22 + 0.0112) = 0.2156$

ii) For the classical Runge-Kutta 4^{th} order method, we obtain the following results

For $j=0$, we've $\quad t_0 = 0, u_0 = 0, v_0 = 0.5$

$K_{11} = hf_1(t_0, u_0, v_0) = h(-3u_0 + 2v_0) = 0.2(-3\times 0 + 2\times 0.5) = 0.2$

$K_{21} = hf_2(t_0, u_0, v_0) = h(3u_0 - 4v_0) = 0.2(3\times 0 - 4\times 0.5) = -0.4$

$K_{12} = hf_1(t_0 + \frac{h}{2}, u_0 + \frac{1}{2}K_{11}, v_0 + \frac{1}{2}K_{21})$

$= h[-3(u_0 + \frac{1}{2}K_{11}) + 2(v_0 + \frac{1}{2}K_{21})] = 0.2[-3\times 0.1 + 2\times 0.3] = 0.06$

$K_{22} = hf_2(t_0 + \frac{h}{2}, u_0 + \frac{1}{2}K_{11}, v_0 + \frac{1}{2}K_{21})$

$= h[3(u_0 + \frac{1}{2}K_{11}) - 4(v_0 + \frac{1}{2}K_{21})] = 0.2[3\times 0.1 - 4\times 0.3] = -0.18.$

$K_{13} = hf_1(t_0 + \frac{h}{2}, u_0 + \frac{1}{2}K_{12}, v_0 + \frac{1}{2}K_{22})$

$= h[-3(u_0 + \frac{1}{2}K_{12}) + 2(v_0 + \frac{1}{2}K_{22})] = 0.2[-3\times 0.03 + 2\times 0.41]$

$= 0.146$

$K_{23} = hf_2(t_0 + \frac{h}{2}, u_0 + \frac{1}{2}K_{12}, v_0 + \frac{1}{2}K_{22})$

$= h[3(u_0 + \frac{1}{2}K_{12}) - 4(v_0 + \frac{1}{2}K_{22})] = 0.2[3\times 0.03 - 4\times 0.41] = -0.31$

$K_{14} = hf_1(t_0+h, u_0+K_{13}, v_0+K_{23}) = h[-3(u_0+K_{13}) + 2(v_0+K_{23})]$

$= 0.2[-3\times 0.146 + 2\times 0.19] = -0.0116$

$K_{24} = hf_2(t_0+h, u_0+K_{13}, v_0+K_{23}) = h[3(u_0+K_{13}) - 4(v_0+K_{23})]$

$= 0.2[3\times 0.146 - 4\times 0.19] = -0.0644.$

$$u(0.2) \approx u_1 = u_0 + \frac{1}{6}(K_{11} + 2K_{12} + 2K_{13} + K_{14})$$

$$= 0 + \frac{1}{6}(0.2 + 0.12 + 0.292 - 0.0116) = 0.1001.$$

$$v(0.2) \approx v_1 = v_0 + \frac{1}{6}(K_{21} + 2K_{22} + 2K_{23} + K_{24})$$

$$= 0.5 + \frac{1}{6}(-0.4 - 0.36 - 0.62 - 0.0644) = 0.2593$$

For $j=1$, we've

$t_1 = 0.2, \; u_1 = 0.1001, \; v_1 = 0.2593$

$$K_{11} = hf(t_1, u_1, v_1) = h(-3u_1 + 2v_1) = 0.2(-3 \times 0.1001 + 2 \times 0.2593)$$

$$= 0.0437$$

$$K_{21} = hf_2(t_1, u_1, v_1) = h(3u_1 - 4v_1) = 0.2[3 \times 0.1001 - 4 \times 0.2593]$$

$$= -0.1474$$

$$K_{12} = hf_1\left(t_1 + \frac{h}{2}, u_1 + \frac{1}{2}K_{11}, v_1 + \frac{1}{2}K_{21}\right)$$

$$= h\left[-3\left(u_1 + \frac{1}{2}K_{11}\right) + 2\left(v_1 + \frac{1}{2}K_{21}\right)\right] = 0.2[-3 \times 0.1220 + 2 \times 0.1856]$$

$$= 0.0010$$

$$K_{22} = hf_2\left(t_1 + \frac{h}{2}, u_1 + \frac{1}{2}K_{11}, v_1 + \frac{1}{2}K_{21}\right)$$

$$= h\left[3\left(u_1 + \frac{1}{2}K_{11}\right) - 4\left(v_1 + \frac{1}{2}K_{21}\right)\right]$$

$$= 0.2[3 \times 0.1220 - 4 \times 0.1856] = -0.0753$$

$$K_{13} = hf_1\left(t_1 + \frac{h}{2}, u_1 + \frac{1}{2}K_{12}, v_1 + \frac{1}{2}K_{22}\right) = h\left[-3\left(u_1 + \frac{1}{2}K_{12}\right) + 2\left(v_1 + \frac{1}{2}K_{22}\right)\right]$$

$$= 0.2[-3 \times 0.1006 + 2 \times 0.2217] = 0.0283$$

$$K_{23} = hf_2\left[t_1 + \frac{h}{2}, u_1 + \frac{1}{2}K_{12}, v_1 + \frac{1}{2}K_{22}\right) = h\left[3\left(u_1 + \frac{1}{2}K_{12}\right) - 4\left(v_1 + \frac{1}{2}K_{22}\right)\right]$$

$$= 0.2[3 \times 0.1006 - 4 \times 0.2217] = -0.1170$$

$$K_{14} = hf_1(t_1 + h, u_1 + K_{13}, v_1 + K_{23}) = h[-3(u_1 + K_{13}) + 2(v_1 + K_{23})]$$

$$= 0.2[-3 \times 0.1284 + 2 \times 0.1423] = -0.0201$$

$$K_{24} = hf_2[t_1 + h, u_1 + K_{13}, v_1 + K_{23}] = h[3(u_1 + K_{13}) - 4(v_1 + K_{23})]$$

$$= 0.2[3 \times 0.1284 - 4 \times 0.1423] = -0.0368.$$

$$u(0.4) \approx u_2 = u_1 + \frac{1}{6}(K_{11} + 2K_{12} + 2K_{13} + K_{14})$$

$$= 0.1001 + \frac{1}{6}[0.0437 + 0.0020 + 0.0566 - 0.0201] = 0.1138$$

$$v(0.4) \approx v_2 = v_1 + \frac{1}{6}(K_{21} + 2K_{22} + 2K_{23} + K_{24})$$

$= 0.2593 + \frac{1}{6}(-0.1474 - 0.1506 - 0.2340 - 0.0368) = 0.1645$

the exact solution is $u(t) = \frac{1}{5}(e^{-t} - e^{-6t})$, $v(t) = \frac{1}{10}(2e^{-t} + 3e^{-6t})$

and $u(0.2) = 0.1035$, $v(0.2) = 0.2541$.

$u(0.4) = 0.1159$, $v(0.4) = 0.1613$.

Implicit Runge-Kutta methods :-

The implicit Runge-Kutta method using v slopes is defined as

$$K_i = hf\left(t_j + c_i h, u_j + \sum_{m=1}^{v} a_{im} K_m\right)$$

$$u_{j+1} = u_j + \sum_{m=1}^{v} W_m K_m \quad \rightarrow (1)$$

where $c_i = \sum_{j=1}^{v} a_{ij}$, $i = 1, 2 \dots v$

and a_{ij}, $1 \le i, j \le v$, $w_1, w_2 \dots w_v$ are arbitrary parameters. The slopes K_m are defined implicitly. The no. of unknown parameters are $v(v+1)$. we now give the derivation for the case $v = 1$

we've

$$u_{j+1} = u_j + w_1 K_1 \quad \rightarrow (2)$$

$$K_1 = hf(t_j + c_1 h, u_j + a_{11} K_1)$$

the Taylor series gives

$$u(t_{j+1}) = u(t_j) + hu'(t_j) + \frac{h^2}{2} u''(t_j) + \dots$$

$$= u(t_j) + hf(t_j, u(t_j)) + \frac{h^2}{2}(f_t + ff_u)_{t_j} + \dots$$

and $K_1 = h[f(t_j, u_j) + (c_1 h f_t + a_{11} K_1 f_u + \dots)_{t_j}]$

$$= (hf + c_1 h^2 f_t + h a_{11} f_u K_1)_{t_j} + O(h^3)$$

$$= hf_j + h^2(c_1 f_t + a_{11} ff_u)_{t_j} + O(h^3) \quad \rightarrow (3)$$

sub (3) into (2) we get

$$u(t_{j+1}) = u(t_j) + w_1[hf_j + h^2(c_1 f_t + a_{11} ff_u)_{t_j} + O(h^3)]$$

comparing the coefficients of h and h^2, we get

$$w_1 = 1, \quad w_1 c_1 = 1/2, \quad w_1 a_{11} = 1/2.$$

we obtain $w_1 = 1$, $c_1 = a_{11} = 1/2$.

The 2nd order implicit Runge-Kutta method becomes

$$u_{j+1} = u_j + K_1 \quad \rightarrow (4)$$

$$K_1 = hf\left(t_j + \frac{1}{2}h, u_j + \frac{1}{2}K_1\right)$$

For obtaining the value of K_1, we need to solve a nonlinear algebraic equation in one variable K_1.

For $v=2$, the implicit Runge-Kutta method (1) becomes

$$u_{j+1} = u_j + W_1 K_1 + W_2 K_2 \longrightarrow (5)$$
$$K_1 = hf(t_j + c_1 h, u_j + a_{11} K_1 + a_{12} K_2)$$
$$K_2 = hf(t_j + c_2 h, u_j + a_{21} K_1 + a_{22} K_2)$$

Taylor series expansion of K_1, K_2 given in this recursive form is very difficult. $\therefore K_1, K_2$ can be expanded in powers of h, we can write

$$\left.\begin{aligned} K_1 &= hA_1 + h^2 B_1 + h^3 C_1 + \cdots \\ K_2 &= hA_2 + h^2 B_2 + h^3 C_2 + \cdots \end{aligned}\right\} \longrightarrow (6)$$

sub (6) in (5) expand in Taylor series & compare the coefficients of h, h^2, h^3 and h^4 solving the resulting eqns, we obtain the parameter values as

$$W_1 = 1/2, \quad W_2 = 1/2, \quad c_1 = (3-\sqrt{3})/6, \quad c_2 = (3+\sqrt{3})/6$$
$$a_{11} = 1/4, \quad a_{12} = (3-2\sqrt{3})/12, \quad a_{21} = (3+2\sqrt{3})/12, \quad a_{22} = 1/4$$

$\because$ the truncation error is $O(h^5)$, the order of the method is 4. the method is gr by

$$u_{j+1} = u_j + \frac{1}{2}(K_1 + K_2) \longrightarrow (7)$$
$$K_1 = hf\left(t_j + \frac{3-\sqrt{3}}{6} h, u_j + \frac{1}{4} K_1 + \frac{3-2\sqrt{3}}{12} K_2\right)$$
$$K_2 = hf\left(t_j + \frac{3+\sqrt{3}}{6} h, u_j + \frac{3+2\sqrt{3}}{12} K_1 + \frac{1}{4} K_2\right)$$

For obtaining the values of K_1, K_2. we need to solve a system of 2 nonlinear algebraic eqns in 2 unknowns K_1, K_2.

Example 6.21. solve the initial value problem $u' = -2tu^2, u(0) = 1$ with $h = 0.2$ on the interval $[0, 0.4]$. use the 2nd order implicit Runge Kutta method.

The 2nd order implicit Runge-Kutta method is gr by

$$u_{j+1} = u_j + K_1, \quad j = 0, 1 \longrightarrow (1)$$
$$K_1 = hf\left(t_j + \frac{h}{2}, u_j + \frac{1}{2} K_1\right)$$

which gives $K_1 = -h(2t_j + h)\left(u_j + \frac{1}{2} K_1\right)^2$

this is an implicit eqn in K_1 & can be solved by using an iterative method. we generally use the Newton-Raphson method we write

$$F(K_1)=K_1+h(2t_j+h)(u_j+\tfrac{1}{2}K_1)^2 = K_1+0.2(2t_j+0.2)(u_j+\tfrac{1}{2}K_1)^2$$

we've $F'(K_1)=1+h(2t_j+h)(u_j+\frac{1}{2}K_1) = 1+0.2(2t_j+0.2)(u_j+\frac{1}{2}K_1)$

the Newton-Raphson method gives

$$K_1^{(s+1)} = K_1^{(s)} - \frac{F(K_1^{(s)})}{F'(K_1^{(s)})}, \quad s=0,1,\ldots \qquad \rightarrow (2)$$

we assume $K_1^{(0)} = hf(t_j,u_j),\ j=0,1.$

Now, we obtain from (1)&(2) $j=0$: $t_0=0,\ u_0=1,\ K_1^{(0)} = -h(2t_0u_0^2) = 0$

$F(K_1^{(0)})=0.04,\ F'(K_1^{(0)})=1.04,\ K_1^{(1)}=-0.03846150$

$F(K_1^{(1)})=0.00001483,\ F'(K_1^{(1)})=1.03923077,\ K_1^{(2)}=-0.03847567$

$F(K_1^{(2)})=0.30\times10^{-8}$

$\therefore K_1=K_1^{(2)}=-0.03847567$ and

$u(0.2)\approx u_1=u_0+K_1=0.96152433$

$j=1$: $t_1=0.2,\ u_1=0.96152433,\ K_1^{(0)}=-h(2t_1u_1^2)=-0.07396231$

$F(K_1^{(0)})=0.02861128,\ F'(K_1^{(0)})=1.11094517,\ K_1^{(1)}= \cancel{-0.07396231}\ -0.09971631$

$F(K_1^{(1)})=0.00001989,\ F'(K_1^{(1)})=1.10939993,\ K_1^{(2)}=-0.09973423$

$F(K_1^{(2)})=0.35\times10^{-7},\ F'(K_1^{(2)})=1.10939885,\ K_1^{(3)}=-0.099773420$

$\therefore K_1=K_1^{(3)}=-0.09973420,$

and $u(0.4)\approx u_2=u_1+K_1=0.86179013.$

Second order Equations;

The second order and higher order equations can be solved by considering an equivalent system of first order equations. However, we can also derive single step methods to solve second order or higher order equations directly. Such methods are useful when we consider oscillatory systems, which are usually governed by 2nd order equations consider the 2nd order initial value problem

$$u''=f(t,u,u'),\ t_0\le t\le b$$

$u(t_0) = u_0,\ u'(t_0) = u_0' \longrightarrow (1)$

we present a few methods to solve (1) directly.

$$u_2' = u_1' + \frac{1}{96h}(23K_1 + 125K_2 - 81K_3 + 125K_4)$$

$$= 0.2040329 + \frac{1}{19.2}[23(0.0212202) + 125(0.0224290) - 81(0.0234853) + 125(0.0241015)]$$

$$= 0.2040329 + \frac{1}{19.2}(4.4020678) = 0.4333073$$

$\therefore$ we have $u(0.2) \approx u_1 = 1.0202010$, $u'(0.2) \approx u_1' = 0.2040329$

$u(0.4) \approx u_2 = 1.0832855$, $u'(0.4) \approx u_2' = 0.4333073$

the exact soln is $u(t) = e^{t^2/2}$. the exact values are gn by

$u(0.2) = 1.02020134$ $\quad$ $u'(0.2) = 0.20404027$

$u(0.4) = 1.0832871$ $\quad$ $u'(0.4) = 0.43331484$

6.5 stability Analysis of single step methods:-

The analytical solution $u(t_j)$ of the differential eqn, the difference soln u_j of the difference eqn & the numerical soln $\bar{u}_j$ can be related by a relation of the form

$$|u(t_j) - \bar{u}_j| \le |u(t_j) - u_j| + |u_j - \bar{u}_j| \longrightarrow (1)$$

In practice, we would like the difference b/w the analytical & numerical soln to be small. From (1), we find that this difference depends on the values $|u(t_j) - u_j|$ & $|u_j - \bar{u}_j|$. The value $|u(t_j) - u_j|$ is the truncation error which arises because the differential eqn is replaced by the difference eqn. A method, is said to be consistent if it is at least of order 1. For a consistent method, the truncation error tends to zero as h approaches zero. the numerical error $|u_j - \bar{u}_j|$ arises because in actual computation, we cannot compute the difference solution exactly as we are faced with the round off errors In fact, in some cases the numerical. soln may differ considerably from the difference soln. If the effect of the total error including the round-off error remains bounded as $j \to \infty$ with fixed step size, then the difference method is said to be stable, otherwise unstable. we now study the stability of the single step methods when applied to the test eqn (6.10)

$$u' = \lambda u, \quad u(t_0) = u_0$$

where λ may be a real or complex number. In the previous section, it was shown that the analytical soln of the test eqn satisfies the eqn

$$u(t_{j+1}) = e^{\lambda h} u(t_j) \longrightarrow (2)$$

If we apply any single step method to solve the test eqn $u' = \lambda u$, then we get a 1st order difference eqn of the form

$$u_{j+1} = E(\lambda h)\, u_j, \; j = 0, 1, 2 \ldots \longrightarrow (3)$$

let $\epsilon_j = u_j - u(t_j)$. Then we obtain

$$\epsilon_{j+1} = [E(\lambda h) - e^{\lambda h}]\, u(t_j) + E(\lambda h)\, \epsilon_j \longrightarrow (4)$$

The 1st term on the RHS is the local truncation error & the 2nd term on the RHS is the error propagated from the step t_j to t_{j+1}. The error on the next step t_{j+2} satisfies the eqn

$$\epsilon_{j+2} = [E^2(\lambda h) - e^{2\lambda h}]\, u(t_j) + E^2(\lambda h)\, \epsilon_j \longrightarrow (5)$$

where again the 1st term on RHS is the local truncation error & the 2nd term is the propagated error. The errors in computations do not grow, if the propagated error tends to zero or is at least bounded.

we call a singlestep method, when applied to test eqn $u' = \lambda u$

absolutely stable if $|E(\lambda h)| \le 1, \lambda < 0$

relatively stable if $|E(\lambda h)| \le e^{\lambda h}, \lambda > 0$

periodically stable if $|E(\lambda h)| = 1$, λ pure imaginary $\longrightarrow (6)$

asymptotically stable (A-stable) if $u_j \to 0$ as $j \to \infty$. This implies that the stability interval is $h\lambda \in (-\infty, 0)$, ie) the entire left half $h\lambda$ plane.

when $\lambda<0$, the exact soln decreases as t increases & the necessary condition is absolute stability, $\therefore$ the numerical soln must also decrease with t. when $\lambda>0$, the exact soln increases with t & we do not need the condition $|E(\lambda h)|\le 1$. so that the relative stability is the necessary condition to be satisfied.

when λ is pure imaginary & $|E(\lambda h)|=1$, the absolute stability is called periodic stability (P-stability)

Euler Method:-

Applying the Euler method to the test eqn $u'=\lambda u$, we obtain

$$u_{j+1} = u_j + hf_j = (1+\lambda h)u_j = E(\lambda h)u_j$$

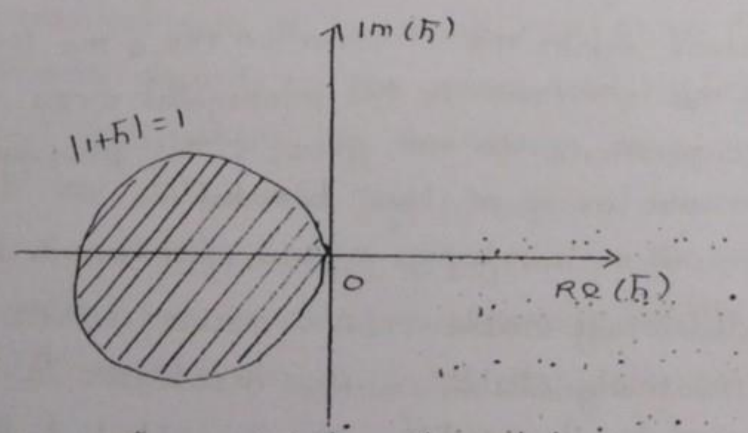
where $E(\lambda h) = 1+\lambda h$

when λ is real & $\lambda<0$, we get the condition $-2<h\lambda<0$,

when λ is complex with $R_e(\lambda)<0$, let $\bar{h}=\lambda h = x+iy$. then

$$|1+\lambda h| = |1+x+iy| = \sqrt{(1+x)^2+y^2}$$

Hence, $|1+\lambda h|<1$ gives $(x+1)^2+y^2<1$, which is the region inside the circle with centre at $(-1,0)$ & radius 1.



stability region for Euler's method, $\bar{h}=h\lambda$

Backward Euler method:-

Applying the backward Euler method $u_{j+1} = u_j + hf_{j+1}$, to the test eqn $u' = \lambda u$, we get

$$u_{j+1} = u_j + \lambda h u_{j+1}$$

or $$u_{j+1} = \frac{1}{1-\lambda h} u_j = E(\lambda h) u_j$$

where $$E(\lambda h) = 1/(1-\lambda h)$$

where λ is real & $\lambda < 0$, we find that the condition

$$\frac{1}{|1-\lambda h|} < 1$$

is always satisfied. $\therefore$ the method is absolutely stable for $-\infty < \lambda h < 0$. when λ is complex with $Re(\lambda) < 0$, let $\lambda h = x + iy$. Then

$$\frac{1}{|1-\lambda h|} < 1 \text{ or } |1-\lambda h| > 1 \text{ gives } (1-x)^2 + y^2 > 1 \rightarrow (1)$$

which the region outside the circle with centre at $(1,0)$ & radius 1. $\because$ $Re(\lambda h) = x < 0$, this condition is always satisfied. Hence, the region of stability is the entire left half of λh-plane.

Runge-Kutta second order method:-

Applying the Runge-Kutta 2^{nd} order methods

$$u_{j+1} = u_j + \left(1 - \frac{1}{2c_2}\right) K_1 + \frac{1}{2c_2} K_2$$

$$K_1 = hf(t_j, u_j)$$

$$K_2 = hf(t_j + c_2 h, u_j + c_2 K_1)$$

to test eqn $u' = \lambda u$, we get

$$K_1 = h\lambda u_j, \quad K_2 = h\lambda(u_j + c_2 K_1) = h\lambda[1 + c_2 h\lambda] u_j$$

$$u_{j+1} = u_j + \left(1 - \frac{1}{2c_2}\right) h\lambda u_j + \frac{1}{2c_2} h\lambda (1 + c_2 h\lambda) u_j$$

$$= \left[1 + h\lambda\left(1 - \frac{1}{2c_2}\right) + \frac{h\lambda}{2c_2}(1 + c_2 h\lambda)\right] u_j$$

$$= \left[1 + h\lambda + \frac{h^2\lambda^2}{2}\right] u_j = E(\lambda h) u_j \rightarrow (2)$$

where $E(\lambda h) = 1 + h\lambda + (h^2\lambda^2/2)$.

Hence, the propagation factor $E(\lambda h)$ is independent of the parameter c_2.

$\therefore$ the stability intervals or regions of all the second order methods is same. Now, for λ real & $\lambda<0$, the condition

$$|E(\lambda h)| = \left|1+h\lambda+\frac{h^2\lambda^2}{2}\right| < 1 \quad \rightarrow (2)$$

is satisfied when $h\lambda \in (-2,0)$ which is the required stability interval.

Runge-Kutta Fourth order Method:-

consider now, the Runge-Kutta 4^{th} order methods defined by along with the conditions. Applying this method to test eqn $u'=\lambda u$, we get

$K_1 = h\lambda u_j = \bar{h} u_j$, where $\bar{h} = h\lambda$,

$K_2 = h\lambda (u_j + a_{21} K_1) = \bar{h}[1+c_2\bar{h}]u_j$

$K_3 = h\lambda (u_j + a_{31}K_1 + a_{32}K_2)$

$= \bar{h}[u_j + a_{31}\bar{h}u_j + a_{32}\bar{h}(1+c_2\bar{h})u_j] = \bar{h}[1+a_{31}\bar{h}+a_{32}\bar{h}+a_{32}c_2\bar{h}^2]u_j$

$= \bar{h}[1+c_3\bar{h}+a_{32}c_2\bar{h}^2]u_j$

$K_4 = h\lambda(u_j + a_{41}K_1 + a_{42}K_2 + a_{43}K_3)$

$= \bar{h}[u_j + a_{41}\bar{h}u_j + a_{42}\bar{h}(1+c_2\bar{h})u_j + a_{43}\bar{h}(1+c_3\bar{h}+a_{32}c_2\bar{h}^2)u_j]$

$= \bar{h}[1+(a_{41}+a_{42}+a_{43})\bar{h} + (a_{42}c_2+a_{43}c_3)\bar{h}^2 + a_{43}a_{32}c_2\bar{h}^3]u_j$

$= \bar{h}[1+c_4\bar{h}+(a_{42}c_2+a_{43}c_3)\bar{h}^2+a_{43}a_{32}c_2\bar{h}^3]u_j$

and $\quad u_{j+1} = u_j + W_1K_1 + W_2K_2 + W_3K_3 + W_4K_4$

$= u_j + W_1\bar{h}u_j + W_2\bar{h}(1+c_2\bar{h})u_j + W_3\bar{h}[1+c_3\bar{h}+a_{32}c_2\bar{h}^2]u_j$

$+ W_4\bar{h}[1+c_4\bar{h}+(a_{42}c_2+a_{43}c_3)\bar{h}^2+a_{43}a_{32}c_2\bar{h}^3]u_j$

$= [1+\bar{h}(W_1+W_2+W_3+W_4)+\bar{h}^2(W_2c_2+W_3c_3+W_4c_4)+$

$\bar{h}^3\{W_3a_{32}c_2+W_4(a_{42}c_2+a_{43}c_3)\}+W_4a_{43}a_{32}c_2\bar{h}^4]u_j$

$$= \left[1+\bar{h}+\frac{\bar{h}^2}{2}+\frac{\bar{h}^3}{6}+\frac{\bar{h}^4}{24}\right]u_j = E(\lambda h)u_j$$

$\therefore$ the propagation factor of the 4^{th} order methods is independent of the arbitrary parameters. Hence, the stability intervals or regions of all the 4^{th} order methods are same.

For absolute stability, we require

$$|E(\lambda h)| = \left| 1+\bar{h} + \frac{\bar{h}^2}{2} + \frac{\bar{h}^3}{6} + \frac{\bar{h}^4}{24} \right| < 1 \quad \rightarrow (1)$$

when λ is real & $\lambda < 0$, we obtain the stability interval as $\lambda h \in (-2.78, 0)$. when λh is pure imaginary, set $\lambda = iy$.

$$\left| 1 + i(yh) - \frac{(yh)^2}{2} - i\frac{(yh)^3}{6} + \frac{(yh)^4}{24} \right| < 1$$

or $\left(1 - \frac{t^2}{2} + \frac{t^4}{24}\right)^2 + \left(t - \frac{t^3}{6}\right)^2 < 1$, where $t = yh$

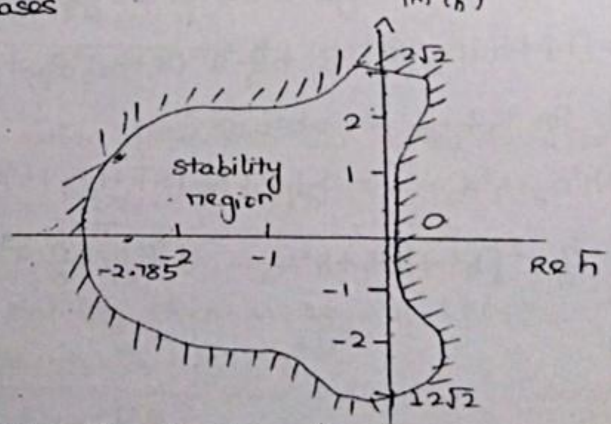
or $1 - \frac{t^6}{72} + \frac{t^8}{576} < 1$.

This eqn is satisfied for $|t| < 2\sqrt{2}$. Hence, the stability interval in this case is $0 < |\lambda h| < 2\sqrt{2}$ when λ is complex with $Re(\lambda) < 0$, it is difficult to derive the stability region analytically. we set $\lambda = x+iy$ in (1) & plot the boundary region (real & Imaginary)

It is easy to verify that Euler method, Backward Euler method & all Runge-Kutta methods are relatively stable

$$\because |E(\lambda h)| \le e^{\lambda h}, \lambda > 0$$

in all these cases



stability region for fourth order Runge Kutta method $\bar{h} = h\lambda$

consider now, the stability of the implicit method

$$u_{j+1} = u_j + K_1$$
$$K_1 = hf\left(t_j + \frac{1}{2}h,\ u_j + \frac{1}{2}K_1\right)$$

Applying it on the test eqn $u' = \lambda u$ we get

$$K_1 = h\lambda\left[u_j + \frac{1}{2}K_1\right] \quad \text{or} \quad K_1 = \frac{h\lambda}{1-(h\lambda/2)}\, u_j$$

$$u_{j+1} = \left[1 + \frac{h\lambda}{1-(h\lambda/2)}\right]u_j = \left[\frac{1+(h\lambda/2)}{1-(h\lambda/2)}\right]u_j$$

Hence $\quad |E(\lambda h)| = \left|\dfrac{1+(h\lambda/2)}{1-(h\lambda/2)}\right|$

$\because$ for real $\lambda < 0$, $|E(\lambda h)| < 1$ always, the method is stable for all $\lambda h \in (-\infty, 0)$. $\therefore$ The method is also A-stable.

Eg 6·24. Find the implicit Runge Kutta method of the form

$$y_{j+1} = y_j + W_1 K_1 + W_2 K_2$$
$$K_1 = hf(y_j) \qquad K_2 = hf(y_j + a(K_1 + K_2))$$

for a soln of initial value problem $y' = f(y)$, $y(t_0) = y_0$. Obtain the interval of absolute stability when the method is applied on $y' = \lambda y$, $\lambda < 0$

we have $\quad K_1 = hf_j$

$\because K_2$ can be expanded in powers of h, we write it as

$$K_2 = hA_1 + h^2A_2 + h^3A_3 + \cdots$$

where A_i's are independent of h.

Expanding K_2 in Taylor's series we get

$$K_2 = \left[hf + ha(K_1+K_2)f_y + \frac{h}{2}a^2(K_1+K_2)^2 f_{yy} + \frac{h}{6}a^3(K_1+K_2)^3 f_{yyy} + \cdots\right]_{t_j}$$

sub for K_1 & K_2 we obtain

$$hA_1 + h^2A_2 + h^3A_3 + \cdots = \Big\{hf + ha[hf + hA_1 + h^2A_2 + \cdots]f_y + \frac{h}{2}a^2[hf + hA_1 + h^2A_2 + \cdots]^2 f_{yy} + \frac{h}{6}a^3[hf + hA_1 + \cdots]^3 f_{yyy} + \cdots\Big\}_{t_j}$$

equating the coefficients of various powers of h, we've

$$A_1 = f_j$$

$$A_2 = (2aff_y)_{t_j}$$

$$A_3 = (aA_2 f_y + 2a^2 f_{yy} f^2)_{t_j} = (2a^2 ff_y^2 + 2a^2 f_{yy} f^2)_{t_j}$$

Hence, $y_{j+1} = y_j + hy_j' + \frac{h^2}{2} y_j'' + \frac{h^3}{6} y_j''' + \dots$

$$= y_j + \left[hf + \frac{h^2}{2} ff_y + \frac{h^3}{6}(ff_y^2 + f^2 f_{yy}) + \dots\right]_{t_j}$$

$$= y_j + w_1 hf_j + w_2 \left[hf + 2ah^2 ff_y + h^3(2a^2 ff_y^2 + 2a^2 f^2 f_{yy}) + \dots\right]_{t_j}$$

comparing terms corresponding to various powers of h, we obtain

$$w_1 + w_2 = 1$$

$$2aw_2 = \frac{1}{2}$$

$$2a^2 w_2 = \frac{1}{6}$$

whose soln is gn by $a = 1/3$, $w_2 = 3/4$, $w_1 = 1/4$

The implicit Runge Kutta method becomes

$$K_1 = hf(y_j) \qquad K_2 = hf\left(y_j + \frac{1}{3}(K_1 + K_2)\right)$$

$$y_{j+1} = y_j + \frac{1}{4}(K_1 + 3K_2)$$

the order of the method is 3.

Apply the method to test eqn $y' = \lambda y$, $\lambda < 0$ we've

$$K_1 = \bar{h} y_j \qquad K_2 = \bar{h}\left(y_j + \frac{1}{3}(\bar{h} y_j + K_2)\right)$$

$$K_2 = \frac{1 + (1/3)\bar{h}}{1 - (1/3)\bar{h}} (\bar{h} y_j)$$

$$y_{j+1} = y_j + \frac{1}{4}\bar{h} y_j + \frac{3}{4} \frac{1 + (1/3)\bar{h}}{1 - (1/3)\bar{h}} (\bar{h} y_j)$$

$$= \frac{1 + (2/3)\bar{h} + (1/6)\bar{h}^2}{1 - (1/3)\bar{h}} y_j \quad \text{where } \bar{h} = h\lambda$$

This is a 1^{st} order difference eqn & the characteristic eqn is gn by

$$\xi = \frac{1 + (2/3)\bar{h} + (1/6)\bar{h}^2}{1 - (1/3)\bar{h}}$$

For absolute stability $(\lambda < 0)$, we require $|\xi| \leq 1$

$$\therefore -1 \leq \frac{1 + (2/3)\bar{h} + (1/6)\bar{h}^2}{1 - (\bar{h}/3)} \leq 1$$

or $\quad -1+\frac{\bar{h}}{3} \le 1+\frac{2}{3}\bar{h}+\frac{1}{6}\bar{h}^2 \le 1-\frac{\bar{h}}{3}$

The right inequality gives

$$\bar{h}+\frac{1}{6}\bar{h}^2 = \frac{\bar{h}}{6}(6+\bar{h}) \le 0$$

$\because \bar{h}=\lambda h<0$, we require $6+\bar{h}\ge 0$ or $\bar{h}\ge -6$

The left inequality $\quad 2+\bar{h}+\frac{1}{6}\bar{h}^2 \ge 0$

is satisfied for $\bar{h}\ge -6$. Hence the stability interval is $(-6,0)$.