$$= \begin{pmatrix} 17 & 0 & -102 \\ 102 & 68 & 204 \\ 0 & 0 & 68 \end{pmatrix} - \begin{pmatrix} 36 & 0 & -72 \\ 72 & 72 & 144 \\ 0 & 0 & 72 \end{pmatrix} + \begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix}$$

$$\therefore\ A^4 = \begin{pmatrix} 1 & 0 & -30 \\ 30 & 16 & 60 \\ 0 & 0 & 16 \end{pmatrix}$$

Exercises

1. Obtain the characteristic polynomial for the following matrices.

   (i) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

2. Find the characteristic equation of the following matrices.

   (i) $\begin{pmatrix} -b & -c \\ 1 & 0 \end{pmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

   (iii) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} -b & -c & -d \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

3. Verify Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and hence find A^{-1}.

4. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ prove that $A^3 - 2A^2 - 5A + 6I = 0$.

5. Verify Cayley-Hamilton theorem for A and hence find A^{-1}.

   (a) $A = \begin{bmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

6. If $A = \begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}$ find A^3 and A^{-3}.

7. Verify that the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix}$ satisfies its own characteristic equation and hence find A^{-1} and A^4.

7.8. Eigen Values And Eigen Vectors

Definition. Let A be an $n \times n$ matrix. A number λ is called an **eigen value** of A if there exists a **non-zero vector** $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ such that $AX = \lambda X$ and X is called an **eigen vector** corresponding to the eigen value λ.

Remark 1. If X is an eigen vector corresponding to the eigen value λ of A, then αX where α is any non-zero number, is also an eigen vector corresponding to λ.

Remark 2. Let X be an eigen vector corresponding to the eigen value λ of A. Then $AX = \lambda X$ so that $(A - \lambda I)X = \mathbf{0}$. Thus X is a non-trivial solution of the system of homogeneous linear equations $(A - \lambda I)X = \mathbf{0}$. Hence $|A - \lambda I| = 0$, which is the characteristic polynomial of A.

Let $|A - \lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n$.

The roots of this polynomial give the eigen values of A. Hence eigen values are also called **characteristic roots**.

Properties of Eigen Values

Property 1. Let X be an eigen vector corresponding to the eigen values λ_1 and λ_2. Then $\lambda_1 = \lambda_2$.

Proof. By definition $X \neq \mathbf{0}$, $AX = \lambda_1 X$ and $AX = \lambda_2 X$

$\therefore \quad \lambda_1 X = \lambda_2 X$

$\therefore \quad (\lambda_1 - \lambda_2)X = \mathbf{0}$

Since $X \neq \mathbf{0}$, $\quad \lambda_1 = \lambda_2$.

Property 2. Let A be a square matrix.

Then **(i)** the sum of the eigen values of A is equal to the sum of the diagonal elements (trace) of A.

(ii) Product of eigen values of A is $|A|$.

Proof. **(i)** Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

The eigen values of A are the roots of the characteristic equation

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0. \quad \ldots (1)$$

Let $|A - \lambda I| = a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n. \quad \ldots (2)$

From (1) and (2) we get

$$a_0 = (-1)^n; a_1 = (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn}); \cdots . \quad (3)$$

Also by putting $\lambda = 0$ in (2) we get $a_n = |A|$

Now let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of A.

$\therefore \quad \lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of (2).

$$\therefore \quad \lambda_1 + \lambda_2 + \cdots + \lambda_n = -\frac{a_1}{a_0}$$

$$= a_{11} + a_{22} + \cdots + a_{nn} \text{ (using (3))}$$

$\therefore \quad$ Sum of the eigen values $=$ trace of A.

(ii) Product of the eigen values $=$ product of the roots

$$= \lambda_1 \lambda_2 \cdots \lambda_n$$

$$= (-1)^n \frac{a_n}{a_0} = \frac{(-1)^n a_n}{(-1)^n}$$

$$= a_n$$

$$= |A|.$$

Property 3. The eigen values of A and its transpose A^T are the same.

Proof. It is enough if we prove that A and A^T have the same characteristic polynomial. Since for any square matrix M, $|M| = |M^T|$ we have,

$$|A - \lambda I| = |(A - \lambda I)^T| = |A^T - (\lambda I)^T|$$
$$= |A^T - \lambda I|.$$

Hence the result.

Property 4. If λ is an eigen value of a non singular matrix A then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Proof. Let X be an eigen vector corresponding to λ.

Then $AX = \lambda X$. Since A is non singular A^{-1} exists.

$$\therefore \quad A^{-1}(AX) = A^{-1}(\lambda X)$$
$$IX = \lambda A^{-1} X$$

$$\therefore \quad A^{-1} X = \left(\tfrac{1}{\lambda}\right) X.$$

$\therefore \quad \frac{1}{\lambda}$ is an eigen value of A^{-1}.

Corollary. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a non singular matrix A then

$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ are the eigen values of A^{-1}.

Property 5. If λ is an eigen value of A then $k\lambda$ is an eigen value of kA where k is a scalar.

Proof. Let X be an eigen vector corresponding to λ.

Then $AX = \lambda X$. ...(1)

Now, $(kA)X = k(AX)$

$= k(\lambda X)$ (by (1))

$= (k\lambda)X$.

$\therefore$ $k\lambda$ is an eigen value of kA.

Property 6. If λ is an eigen value of A then λ^k is an eigen value of A^k where k is any positive integer.

Proof. Let X be an eigen vector corresponding to λ.

Then $AX = \lambda X$. ...(1)

Now, $A^2X = (AA)X = A(AX)$

$= A(\lambda X)$ (by (1))

$= \lambda(AX)$

$= \lambda(\lambda X)$ (by (1))

$= \lambda^2 X$.

$\therefore$ λ^2 is an eigen value of A^2.

Proceeding like this we can prove that λ^k is an eigen value of A^k for any positive integer.

Corollary. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values of A then $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ are eigen values of A^k for any positive integer k.

Property 7. Eigen vectors corresponding to distinct eigen values of a matrix are linearly independent.

Proof. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be distinct eigen values of a matrix and let X_i be the *eigen vector* corresponding to λ_i.

Hence $AX_i = \lambda_i X_i$ $\quad (i = 1, 2, \ldots, k)$ ... (1)

Now, suppose $X_1, X_2, \ldots, X_k$ are linearly dependent. Then there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, not all zero, such that $\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_k X_k = 0$. Among all such relations, we choose one of shortest length, say j.

By rearranging the vectors $X_1, X_2, \ldots, X_k$ we may assume that

$\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_j X_j = 0$. ... (

$\therefore$ $A(\alpha_1 X_1) + A(\alpha_2 X_2) + \cdots + A(\alpha_j X_j) = 0$

$\therefore$ $\alpha_1(AX_1) + \alpha_2(AX_2) + \cdots + \alpha_j(AX_j) = 0$

$\therefore$ $\alpha_1\lambda_1 X_1 + \alpha_2\lambda_2 X_2 + \cdots + \alpha_j\lambda_j X_j = 0$ ... (

Multiplying (2) by λ_1 and subtracting from (3), we g

$$\alpha_2(\lambda_1 - \lambda_2)X_2 + \alpha_3(\lambda_1 - \lambda_3)X_3 + \cdots + \alpha_j(\lambda_1 - \lambda_j)X_j = 0 \quad \ldots (}$$

and since $\lambda_1, \lambda_2, \ldots, \lambda_j$ are distinct and $\alpha_2, \ldots,$ (are non-zero we have

$\alpha_i(\lambda_1 - \lambda_i) \neq 0; \quad i = 2, 3, \ldots, j.$

Thus (4) gives a relation whose length is $j - 1$, givi a contradiction.

Hence $X_1, X_2, \ldots, X_k$ are linearly independent

Property 8. The characteristic roots of a Hermiti matrix are all real.

Proof. Let A be a Hermitian matrix.

Hence

$A = \overline{A}^T$ (by theorem 7.13) ... ...

Let λ be a characteristic root of A and let X b characteristic vector corresponding to λ.

$\therefore$ $AX = \lambda X$ ... ...

Now,

$AX = \lambda X \Rightarrow \overline{X}^T AX = \lambda \overline{X}^T X$

$\Rightarrow (\overline{X}^T AX)^T = \lambda \overline{X}^T X$ (since $X^T A$ a 1×1 mat

$\Rightarrow X^T A^T (\overline{X}^T)^T = \lambda \overline{X}^T X$

$\Rightarrow X^T A^T \overline{X} = \lambda \overline{X}^T X$

$\Rightarrow \overline{X^T A^T \overline{X}} = \overline{\lambda X^T X}$

$\Rightarrow \overline{X}^T \overline{A}^T X = \overline{\lambda} X^T \overline{X}$

$\Rightarrow \overline{X}^T AX = \overline{\lambda} X^T \overline{X}$ (using 1)

$\Rightarrow \overline{X}^T \lambda X = \overline{\lambda} X^T \overline{X}$ (using 2)

$\Rightarrow \lambda(\overline{X}^T X) = \overline{\lambda}(X^T \overline{X})$ ... ...

Now,

$$\overline{X}^T X = X^T \overline{X} = \overline{x_1}x_1 + \overline{x_2}x_2 + \ldots\ldots + \overline{x_n}x_n$$
$$= |x_1|^2 + |x_2|^2 + \ldots\ldots + |x_n|^2$$
$$\neq 0$$

From (3) we get $\lambda = \overline{\lambda}$

Hence λ is real.

Corollary. The characteristic roots of a real symmetric matrix are real.

Proof. We know that any real symmetric matrix is Hermitian. Hence the result follows from the above property.

Property 9. The characteristic roots of a skew Hermitian matrix are either purely imaginary or zero.

Proof Let A be a skew Hermitian matrix and λ be a characteristic root of A.

$$\therefore \quad |A - \lambda I| = 0$$
$$\therefore \quad |iA - i\lambda I| = 0$$
$\therefore$ $i\lambda$ is a characteristic root of iA.

Since A is skew Hermitian iA is Hermitian (refer result theorem 7.14)

$\therefore$ By theorem 7.32 $i\lambda$ is real. Hence λ is purely imaginary or zero.

Corollary. The characteristic roots of a real skew symmetric matrix are either purely imaginary or zero.

Proof. We know that any real skew symmetric matrix is skew Hermitian.

Hence the result follows from the above property.

Property 10. Let λ be a characteristic root of an unitary matrix A. Then $|\lambda| = 1$. (i.e) the characteristic roots of a unitary matrix are all the unit modulus.

Proof. Let λ be a characteristic root of an unitary matrix A and X be a characteristic vector corresponding to λ.

$$\therefore \quad AX = \lambda X \ldots\ldots \qquad (1)$$

Taking conjugate and transpose in (1) we get

$$(\overline{AX})^T = (\overline{\lambda X})^T.$$
$$\therefore \quad \overline{X}^T \overline{A}^T = \overline{\lambda}\,\overline{X}^T \ldots\ldots \qquad ($$

Multiplying (1) and (2) we get

$$(\overline{X^T A^T})(AX) = (\overline{\lambda}\,\overline{X}^T)(\lambda X)$$
$$\therefore \quad \overline{X}^T(\overline{A}^T A)X = \overline{\lambda}\lambda(\overline{X}^T X)$$

Now, since A is an unitary matrix $\overline{A}^T A = I$.

Hence $\overline{X}^T X = (\overline{\lambda}\lambda)\overline{X}^T X$

Since X is non-zero vector $\overline{X}^T$ is also non-ze vector and $\overline{X}^T X = |x_1|^2 + |x_2|^2 + \ldots\ldots + |x_n|^2 \neq$ we get $\lambda\overline{\lambda} = 1$.

Hence $|\lambda|^2 = 1$. Hence $|\lambda| = 1$.

Corollary. Let λ be a characteristic root of orthogonal matrix A. Then $|\lambda| = 1$.

Since any orthogonal matrix is unitary the res follows from property 10.

Property 11. Zero is an eigen value of A if and or if A is a singular matrix.

Proof. The eigen values of A are the roots of characteristic equation $|A - \lambda I| = 0$. Now, 0 is eigen value of $A \Leftrightarrow |A - 0I| = 0$

$$\Leftrightarrow |A| = 0$$
$$\Leftrightarrow A \text{ is a singular matrix}$$

Property 12. If A and B are two square matrice the same order then AB and BA have the same ei values.

Solution. Let λ be an eigen value of AB and X an eigen vector corresponding to λ.

$$\therefore \quad (AB)X = \lambda X$$
$$\therefore \quad B(AB)X = B(\lambda X) = \lambda(BX).$$
$$\therefore \quad (BA)(BX) = \lambda(BX)$$
$$\therefore \quad (BA)Y = \lambda Y \quad \text{where} \quad Y = BX.$$

Hence λ is an eigen value of BA.

Also BX is the corresponding eigen vector.

Property 13. If P and A are $n \times n$ matrices and P is a nonsingular matrix then A and $P^{-1}AP$ have the same eigen values.

Proof. Let $B = P^{-1}AP$.

To prove A and B have same eigen values, it is enough to prove that the characteristic polynomials of A and B are the same.

Now
$$\begin{aligned}|B - \lambda I| &= |P^{-1}AP - \lambda I| \\ &= |P^{-1}AP - P^{-1}(\lambda I)P| \\ &= |P^{-1}(A - \lambda I)P| \\ &= |P^{-1}||A - \lambda I||P| \\ &= |P^{-1}||P||A - \lambda I| \\ &= |P^{-1}P||A - \lambda I| \\ &= |I||A - \lambda I| \\ &= |A - I\lambda|.\end{aligned}$$

$\therefore$ The characteristic equations of A and $P^{-1}AP$ are the same.

Property 14. If λ is a characteristic root of A then $f(\lambda)$ is a characteristic root of the matrix $f(A)$ where $f(x)$ is any polynomial.

Proof. Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ where $a_0 \neq 0$ and $a_1, a_2, \ldots, a_n$ are all real numbers.

$$\therefore \quad f(A) = a_0A^n + a_1A^{n-1} + \ldots + a_{n-1}A + a_nI.$$

Since λ is a characteristic root of A, λ^n is a characteristic root of A^n for any positive integer n (refer Property 6)

$$\begin{aligned}\therefore \quad A^nX &= \lambda^nX \\ A^{n-1}X &= \lambda^{n-1}X \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ AX &= \lambda X\end{aligned}$$

$$\begin{aligned}\therefore \quad a_0A^nX &= a_0\lambda^nX \\ a_1A^{n-1}X &= a_1\lambda^{n-1}X \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ a_{n-1}AX &= a_{n-1}\lambda X\end{aligned}$$

Adding the above equations we have

$$a_0A^nX + a_1A^{n-1}X + \ldots + a_{n-1}AX = a_0\lambda^nX + a_1\lambda^{n-1}X + \ldots + a_{n-1}\lambda X$$

$$\therefore \quad (a_0A^n + a_1A^{n-1} + \ldots + a_{n-1}A)X = (a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda)X$$

$$\therefore \quad (a_0A^n + a_1A^{n-1} + \ldots + a_{n-1}A + a_nI)X = (a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n)X$$

$$\therefore \quad f(A)X = f(\lambda)X$$

Hence $f(\lambda)$ is a characteristic root of $f(A)$.

Solved Problems

Problem 1. If X_1, X_2 are eigen vectors correspon[ding] to an eigen value λ then $aX_1 + bX_2$ (a, b non-[zero] scalars) is also an eigen vector corresponding to λ

Solution. Since X_1 and X_2 are given vectors co[rre]sponding to λ, we have

$$AX_1 = \lambda X_1 \text{ and } AX_2 = \lambda X_2.$$

Hence $A(aX_1) = \lambda(aX_1)$ and $A(bX_2) = \lambda(bX_2)$

$$\therefore \quad A(aX_1 + bX_2) = \lambda(aX_1 + bX_2).$$

$\therefore$ $aX_1 + bX_2$ is an eigen vector corresponding t[o λ]

Problem 2. If the eigen values of

$$A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$$ are 2, 2, 3 find the ei[gen] values of A^{-1} and A^2.

Solution. Since 0 is not an eigen value of A, A i[s a] non singular matrix and hence A^{-1} exists.

Eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$ and eigen valu[es] of A^2 are $2^2, 2^2, 3^2$.

Problem 3. Find the eigen values of A^5 when

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$$

Solution. The characteristic equation of A is obviously $(3-\lambda)(4-\lambda)(1-\lambda) = 0$.

Hence the eigen values of A are 3, 4, 1.

$\therefore$ The eigen values of A^5 are $3^5, 4^5, 1^5$.

Problem 4. Find the sum and product of the eigen values of the matrix $\begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$ without actually finding the eigen values.

Solution. Let $A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

Sum of the eigen values = trace of$A = 3+(-2)+3 = 4$.

Product of the eigen values = $|A|$.

Now, $$|A| = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(-6+4) + 4(3-4) - 4(-1+2)$$

$$= -6 - 4 - 4 = -14.$$

$\therefore$ Product of the eigen values $= -14$.

Problem 5. Find the characteristic roots of the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Solution. Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

The characteristic equation of A is given by $|A - \lambda I| = 0$.

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = 0.$$

$\therefore$ $(\cos\theta - \lambda)^2 - \sin^2\theta = 0.$

$\therefore$ $(\cos\theta - \lambda - \sin\theta)(\cos\theta - \lambda + \sin\theta) = 0.$

$[\lambda - (\cos\theta - \sin\theta)][\lambda - (\cos\theta + \sin\theta)] = 0.$

$\therefore$ The two characteristic roots, (the two eigen values) of the matrix are $(\cos\theta - \sin\theta)$ and $(\cos\theta + \sin\theta)$.

Problem 6. Find the characteristic roots of the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$$

Solution. The characteristic equation of A is given by $|A - \lambda I| = 0$.

(i.e.) $$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ -\sin\theta & -\cos\theta - \lambda \end{vmatrix} = 0$$

$\therefore$ $-(\cos^2\theta - \lambda^2) - \sin^2\theta = 0.$

$\therefore$ $\lambda^2 - (\cos^2\theta + \sin^2\theta) = 0.$

$\therefore$ $\lambda^2 - 1 = 0.$

$\therefore$ The characteristic roots are 1 and -1.

Problem 7. Find the sum and product of the eigen values of the matrix

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ without finding the roots of the characteristic equation.

Solution. Sum of the eigen values of $A =$ trace of $A = a_{11} + a_{22}$.

Product of the eigen values of $A = |A| = a_{11}a_{22} - a_{12}a_{21}$.

Problem 8. Verify the statement that the sum of the elements in the diagonal of a matrix is the sum of the eigen values of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution. The characteristic equation of A is $|A - \lambda I| = 0$.

(i.e.) $$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0.$$

(i.e.) $(-2-\lambda)[(1-\lambda)(-\lambda) - 12] - 2[-2\lambda - 6]$
$- 3[-4 + (1-\lambda)] = 0.$

(i.e.) $(-2-\lambda)(\lambda^2 - \lambda - 12) + 4(\lambda + 3)$

$$+3(\lambda+3)=0.$$

(i.e.) $-2\lambda^2+2\lambda+24-\lambda^3+\lambda^2$
$$+12\lambda+4\lambda+12+3\lambda+9=0.$$

(i.e.) $-\lambda^3-\lambda^2+21\lambda+45=0.$

(i.e.) $\lambda^3+\lambda^2-21\lambda-45=0.$

This is a cubic equation in λ and hence it has 3 roots and the three roots are the three eigen values of the matrix.

The sum of the eigen values $=-\left(\dfrac{\text{coefficient of } \lambda^2}{\text{coefficient of } \lambda^3}\right)$

$$=-1.$$

The sum of the elements on the diagonal of the matrix

$$A=-2+1+0=-1.$$

Hence the result.

Problem 9. The product of two eigen values of the matrix $A=\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. Find the third eigen value. What is the sum of the eigen values of A?

Solution. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

Given, product of 2 eigen values (say) λ_1, λ_2 is 16.

$$\therefore \quad \lambda_1\lambda_2=16$$

We know that the product of the eigen values is $|A|$.

(i.e) $\lambda_1\lambda_2\lambda_3=\begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

(i.e) $16\lambda_3=6(9-1)+2(-6+2)+2(2-6)$

$$=48-8-8$$

$$=32$$

$$\therefore \quad \lambda_3=2$$

$\therefore$ The third eigen value is 2.

Also we know that the sum of the eigen values of

$$A=\text{trace of } A=6+3+3=12$$

Problem 10. The product of two eigen values of the matrix $A=\begin{pmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{pmatrix}$ is -12. Find the eigen values of A.

Solution. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A. Given product of 2 eigen values, say, λ_1 and λ_2 is -12.

$$\therefore \quad \lambda_1\lambda_2=-12 \qquad \ldots (1)$$

We know that the product of the eigen values is $|A|$.

$$\therefore \quad \lambda_1\lambda_2\lambda_3=\begin{vmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{vmatrix}$$

i.e $\quad 12\lambda_3=-12$

$$\therefore \quad \lambda_3=1 \qquad \ldots (2)$$

Also we know sum of the eigen values $=$ Trace of A.

$$\therefore \quad \lambda_1+\lambda_2+\lambda_3=2+1-3=0$$

$$\therefore \quad \lambda_1+\lambda_2=-1 \quad \text{(using (2))} \qquad \ldots (3)$$

Using (3) in (1) we get $\lambda_1(-1-\lambda_1)=-12$

$$\lambda_1^2+\lambda_1-12=0$$

$$(\lambda_1+4)(\lambda_1-3)=0$$

$$\therefore \lambda_1=3 \quad \text{or} \quad -4$$

Putting $\lambda_1=3$ in (1) we get $\lambda_2=-4$. Or putting $\lambda_1=-4$ in (1) we get $\lambda_2=3$.

Thus the three eigen values are $3, -4, 1$.

Problem 11. Find the sum of the squares of the eigen values of $A=\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$

Solution. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

We know that $\lambda_1^2, \lambda_2^2, \lambda_3^2$ are the eigen values of A^2.

$$\therefore A^2 = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 5 & 38 \\ 0 & 4 & 42 \\ 0 & 0 & 25 \end{pmatrix}$$

$\therefore$ Sum of the eigen values of $A^2 =$ Trace of A^2

$= 9 + 4 + 25$

(i.e) $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 38$

$\therefore$ Sum of the squares of the eigen values of $A = 38$.

Problem 12. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution. The characteristic equation of A is

$$|A - \lambda I| = 0.$$

$$\therefore \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0.$$

$\therefore (1-\lambda)[(5-\lambda)(1-\lambda) - 1] - [(1-\lambda) - 3]$
$+ 3[1 - 3(5-\lambda)] = 0.$

$(1-\lambda)(\lambda^2 - 6\lambda + 4) + (\lambda + 2) + 3(3\lambda - 14) = 0.$

$\lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + \lambda + 2$
$+ 9\lambda - 42 = 0.$

$\therefore -\lambda^3 + 7\lambda^2 - 36 = 0$. Hence $\lambda^3 - 7\lambda^2 + 36 = 0$.

$\therefore (\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0.$

Hence $(\lambda + 2)(\lambda - 6)(\lambda - 3) = 0.$

$\therefore \lambda = -2, 3, 6$ are the three eigen values.

Case (i) Eigen vector corresponding to $\lambda = -2$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector corresponding to $\lambda = -2$.

Hence $AX = -2X$.

(i.e.) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ -2x_2 \\ -2x_3 \end{bmatrix}$

$\therefore x_1 + x_2 + 3x_3 = -2x_1$

$x_1 + 5x_2 + x_3 = -2x_2$

$3x_1 + x_2 + x_3 = -2x_3$

$\therefore 3x_1 + x_2 + 3x_3 = 0$ ... (1)

$x_1 + 7x_2 + x_3 = 0$ ... (2)

$3x_1 + x_2 + 3x_3 = 0$ ... (3)

Clearly this system of three equations reduces to two equations only. From (1) and (2) we get

$\therefore \quad x_1 = -2k;\ x_2 = 0;\ x_3 = 2k.$

$\therefore$ It has only one independent solution and can be obtained by giving any value to k say $k = 1$.

$\therefore$ $(-2, 0, 2)$ is an eigen vector corresponding to $\lambda = -2$.

Case (ii) Eigen vector corresponding to $\lambda = 3$.

Then $AX = 3X$ gives

$-2x_1 + x_2 + 3x_3 = 0$
$x_1 + 2x_2 + x_3 = 0$
$3x_1 + x_2 - 2x_3 = 0.$

Taking the first 2 equations we get

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} = k(\text{say}).$$

$\therefore \quad x_1 = -k;\ x_2 = k;\ x_3 = -k.$

Taking $k = 1$ (say) $(-1, 1, -1)$ is an eigen vector corresponding to $\lambda = 3$.

Case (iii) Eigen vector corresponding to $\lambda = 6$.

We have $AX = 6X$.

Hence $\quad -5x_1 + x_2 + 3x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$3x_1 + x_2 - 5x_3 = 0.$

Taking the first two equations we get

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} = k.$$

$\therefore$ $x_1 = k; x_2 = 2k; x_3 = k$. It satisfies the third equation also.

Taking $k = 1$ (say) $(1, 2, 1)$ is an eigen vector corresponding to $\lambda = 6$.

Problem 13. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

Solution. The characteristic equation of A is $|A - \lambda I| = 0$.

$$\therefore \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0.$$

$\therefore$ $(6-\lambda)[(3-\lambda)^2 - 1] + 2[(2\lambda - 6) + 2]$
$+ 2(2 - 6 + 2\lambda) = 0.$

$\therefore$ $(6-\lambda)(8 + \lambda^2 - 6\lambda) + 4\lambda - 8 + 4\lambda - 8 = 0.$

$\therefore$ $48 + 6\lambda^2 - 36\lambda - 8\lambda - \lambda^3 + 6\lambda^2 + 8\lambda - 16 = 0.$

$\therefore$ $-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0.$

Hence $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0.$

$\therefore$ $(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0.$

$\therefore$ The eigen values are 2, 2, 8.

We now find the eigen vectors.

Case (i) $\lambda = 2$.

The eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is got from $AX = 2X$.

$$\therefore \quad 6x_1 - 2x_2 + 2x_3 = 2x_1$$
$$-2x_1 + 3x_2 - x_3 = 2x_2$$
$$2x_1 - x_2 + 3x_3 = 2x_3.$$
$$\therefore \quad 4x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 + x_2 - x_3 = 0$$
$$2x_1 - x_2 + x_3 = 0.$$

The above three equations are equivalent to the single equation $2x_1 - x_2 + x_3 = 0$.

The independent eigen vectors can be obtained by giving arbitrary values to any two of the unknowns x_1, x_2, x_3.

Giving $x_1 = 1; x_2 = 2$ we get $x_3 = 0$.

Giving $x_1 = 3; x_2 = 4$ we get $x_3 = -2$.

$\therefore$ Two independent vectors corresponding to $\lambda = 2$ are $(1, 2, 0)$ and $(3, 4, -2)$.

Case (ii) $\lambda = 8$.

The eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is got from $AX = 8X$.

$$\therefore \quad -2x_1 - 2x_2 + 2x_3 = 0 \quad \ldots (1)$$
$$-2x_1 - 5x_2 - x_3 = 0 \quad \ldots (2)$$
$$2x_1 - x_2 - 5x_3 = 0 \quad \ldots (3)$$

From (1) and (2) we get

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6} = k \text{ (say)}.$$

$\therefore$ $x_1 = 2k; x_2 = -k; x_3 = k$.

Giving $k = 1$ we get an eigen vector corresponding to 8 as $(2, -1, 1)$.

Problem 14. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}.$$

Solution. The characteristic equation of A is $|A - \lambda I| = 0$.

(i.e.) $\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0.$

$\therefore (2-\lambda)[-(1-\lambda)(1+\lambda)-3]$
$+2[-(1+\lambda)-1]+2[3-(1-\lambda)]=0$

$\therefore (2-\lambda)(\lambda^2-4)-2(2+\lambda)+2(2+\lambda)=0$

$\therefore 2\lambda^2-8-\lambda^3+4\lambda-4-2\lambda+4+2\lambda=0$

$\therefore -\lambda^3+2\lambda^2+4\lambda-8=0$

Hence $\lambda^3-2\lambda^2-4\lambda+8=0$

$\therefore (\lambda-2)(\lambda^2-4)=0$

Hence $(\lambda-2)(\lambda-2)(\lambda+2)=0$

$\therefore \lambda = 2, 2, -2$ are the three eigen values.

Case (i) $\lambda = 2$.

Let $X = (x_1, x_2, x_3)$ be an eigen vector corresponding to $\lambda = 2$, X is got from $AX = 2X$.

(i.e.) $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}.$

$\therefore$ The eigen vector corresponding to $\lambda = 2$ is given by the equations

$$2x_1 - 2x_2 + 2x_3 = 2x_1$$
$$x_1 + x_2 + x_3 = 2x_2$$
$$x_1 + 3x_2 - x_3 = 2x_3$$

(i.e.) $-x_2 + x_3 = 0$ ... (1)

$x_1 - x_2 + x_3 = 0$ ... (2)

$x_1 + 3x_2 - 3x_3 = 0$ ... (3)

Taking (1) and (2) we get $\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1} = k$ (say).

$\therefore \quad x_1 = 0; x_2 = k; x_3 = k.$

Taking $k = 1$, we get $(0, 1, 1)$ as an eigen vector corresponding to $\lambda = 2$.

Case (ii) $\lambda = -2$.

Corresponding to $\lambda = -2$ we have $AX = -2X$.

$$\therefore \quad 2x_1 - 2x_2 + 2x_3 = -2x_1$$
$$x_1 + x_2 + x_3 = -2x_2$$
$$x_1 + 3x_2 - x_3 = -2x_3$$
$$\therefore \quad 2x_1 - x_2 + x_3 = 0$$
$$x_1 + 3x_2 + x_3 = 0$$
$$x_1 + 3x_2 + x_3 = 0$$

$\therefore$ Taking the first two equations we get,

$$\frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7} = k \text{ (say)}.$$

$\therefore \quad x_1 = -4k; x_2 = -k; x_3 = 7k.$

Taking $k = 1$ we get $(-4, -1, 7)$ as an eigen vector corresponding to the eigen value $\lambda = -2$.

Exercises

1. For each of the following matrices find the characteristic vectors corresponding to each characteristic root.

(a) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & i & 0 & 0 \\ 2 & 1/2 & -i & 0 \\ 1/3 & -i & \pi & -1 \end{bmatrix}$

2. For what value of k is 3 a characteristic root of $\begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & -k \\ 3 & k & -1 \end{bmatrix}$

3. Find the characteristic roots and the corresponding characteristic vectors of

$A^3 + A^2 + A + I$ if $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

4. Prove that if λ is a characteristic root of the matrix A then $\lambda - k$ is a characteristic root of $A - kI$.
5. Show that the characteristic root of a triangular matrix are just the diagonal elements of the matrix.
6. Show that every orthogonal matrix of odd order should have 1 or -1 as a characteristic root.
7. Show that if λ is a characteristic root of an unitary matrix then so is $\frac{1}{\lambda}$.

Answers.

1. (a) 0, 3, 15. The corresponding characteristic vectors are $(1, 2, 2)$; $(2, -1, -2)$; $(2, -2, 1)$.
   (b) 1, 1, 5; The characteristic vectors corresponding to 1 is a linear combination of $(2, -1, 0)$ and $(1, 0, -1)$. A characteristic vector corresponding to 5 is $(1, 1, 1)$.
   (c) 1, 1, 1. A characteristic vector is $(1, 0, 0)$.
2. $k = 2$.

Revision questions on chapter 7

Determine which of the following statements are true and which are false.

1. The set of all $n \times n$ matrices over a field F is a ring w.r.t matrix addition and matrix multiplication.
2. The set of all $m \times n$ matrices over a field F is a vector space over F w.r.t matrix addition and scalar multiplication.
3. Matrix multiplication is commutative.
4. If A is an $m \times n$ matrix, A^T is also an $m \times n$ matrix.
5. The transpose of a row matrix is a column matrix.
6. The transpose of a lower triangular matrix is an upper triangular matrix.
7. A matrix A is symmetric iff $A = A^T$.
8. For any square matrix A, $A + A^T$ and AA^T are symmetric.
9. For any square matrix A, $A - A^T$ is skew symmetric.
10. Any symmetric matrix is Hermitian.
11. Any Hermitian matrix over **R** is symmetric.
12. Any orthogonal matrix is non-singular.
13. The product of two non-singular matrices is non-singular.
14. The product of a singular matrix and a non-singular matrix is singular.
15. For any square matrix A, $A(adjA) = |A|I$.
16. (15) is true only for non-singular matrices.
17. For any singular matrix A we can find a matrix B such that $AB = \mathbf{0}$.
18. Any matrix can be reduced to a canonical form by elementary transformations.
19. Any matrix has a rank.
20. Rank is defined only for square matrices.
21. Any non-singular matrix of order n has rank n.
22. The degree of the characteristic polynomial of a square matrix A is equal to the order of the matrix.
23. Every square matrix satisfies its characteristic equation.
24. The characteristic vectors corresponding to a characteristic root form a vector space.

Answers.

1. T	2. T	3. F	4. F	5. T	6. T	7. T
8. T	9. T	10. F	11. T	12. T	13. T	14. T
15. T	16. F	17. T	18. T	19. T	20. F	21. T
22. T	23. T	24. T				

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