* Differentiability :—

If a function $f(z)$ is single valued in a domain D, then the derivative of $f(z)$ at $z=z_0$ is denoted by $f'(z_0)$ and is defined as $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$

provided that the limit exists. In this case we say that $f(z)$ is differentiable at $z=z_0$.

(or)

A function 'f' is said to be differentiable at z, if

$\lim_{h \to 0} \frac{f(z+h)-f(z)}{h}$ exists and is denoted by $f'(z)$.

i.e. $f'(z) = \lim_{h \to 0} \frac{f(z+h)-f(z)}{h}$

Note :— 'h' approaches '0' through points in the plane, not just along the real axis.

→ Every differentiable function $f(z)$ is continuous But the converse is not true.

Proof : Let $f(z)$ be differentiable at $z=z_0$.

$\therefore f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$

Now we have

$$f(z)-f(z_0) = \frac{f(z)-f(z_0)}{z-z_0}(z-z_0)$$

$$\Rightarrow \lim_{z \to z_0}[f(z)-f(z_0)] = \lim_{z \to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0} \times (z-z_0)\right]$$

$$= \lim_{z \to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}\right] \lim_{z \to z_0}(z-z_0)$$

$$= f'(z_0).(0)$$

$$= 0.$$

$$\Rightarrow \lim_{z \to z_0}(f(z)-f(z_0)) = 0$$

$$\Rightarrow \lim_{z \to z_0} f(z) = \lim_{z \to z_0} f(z_0)$$

$$= f(z_0)$$

$\therefore f(z)$ is continuous at $z=z_0$

Converse: For example

$f(z) = |z|$

$= \sqrt{x^2+y^2}$ is continuous at $(0,0)$

but not differentiable at $(0,0)$.

Since:

$\lim_{z \to 0} f(z) = 0 = f(0)$

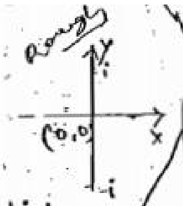
$\therefore f(z)$ is continuous at $(0,0)$.

But $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z \to 0} \frac{|z|}{z}$

$$= \lim_{(x,y) \to (0,0)} \frac{\sqrt{x^2+y^2}}{x+iy}$$

$$= \lim_{(x,y) \to (0,0)} \frac{|x+iy|}{x+iy}$$

Rough



$\frac{|iy|}{iy} = \frac{\sqrt{y^2}}{iy} = \frac{1}{i} = -i$

$$= \begin{cases} 1 & \text{for +ve real values of } z \\ -1 & \text{for -ve real values of } z \\ -i & \text{for values of } z \text{ on the +ve imaginary axis.} \\ i & \text{for values of } z \text{ on the -ve imaginary axis} \end{cases}$$

$\therefore f'(0)$ does not exist.

(or)

Since $\lim_{h\to 0} f(z+h) = \lim_{h\to 0} |z+h|$

the value of the limit at $z=0$ is $\lim_{h\to 0} |0+h| = \lim_{h\to 0} |h| = 0$

Hence $|z|$ is continuous at the origin.

But $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{|z+h|-|z|}{h}$

$$= \lim_{h\to 0} \frac{|0+h|-0}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h} \quad \text{——(i)}$$

Let $h = h_1 + ih_2$ and let $h \to 0$ along the real axis then $h_2 = 0$ & $h_1 \to 0$.

$\therefore \lim_{h\to 0} \frac{|h|}{h} = \lim_{h_1\to 0} \frac{|h_1|}{h_1} = \pm 1$.

If $h \to 0$ along the imaginary axis, $h_1 = 0$ and $h_2 \to 0$ then

$\lim_{h\to 0} \frac{|h|}{h} = \lim_{h_2\to 0} \frac{|ih_2|}{h_2} = \pm i$

$\therefore$ the function is behaving different along the different paths..

$\Longrightarrow$ the function is not differentiable at the origin.

Note:- The rules of differentiation of real functions are also valid for complex function.

$\rightarrow$ Show that $f(z) = z^2$ is differentiable every where.

Sol^n: $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$

$$= \lim_{h\to 0} \frac{(z+h)^2 - z^2}{h}$$

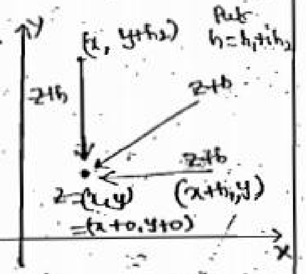
$$= \lim_{h\to 0} \frac{\cancel{z^2} + 2zh + h^2 - \cancel{z^2}}{h}$$

$$= \lim_{h\to 0} (2z+h)$$

$$= 2z$$

whatever be the path along which $h \to 0$, the limit exist and z^2 is defined everywhere.

$\therefore$ The function is differentiable Every where.



$\rightarrow$ where is $|z|$ differentiable?

Sol'n: Let $f(z) = |z|$

Now $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$

$$= \lim_{h\to 0} \frac{|z+h|-|z|}{h}$$

$$= \underset{h\to 0}{lt} \frac{(|z+h|-|z|)(|z+h|+|z|)}{h(|z+h|+|z|)}$$

$$= \underset{h\to 0}{lt} \frac{1}{|z+h|+|z|} \; \underset{h\to 0}{lt} \frac{(z+h)(\overline{z+h})-z\bar{z}}{h}$$

$$= \frac{1}{|z+0|+|z|} \; \underset{h\to 0}{lt} \frac{(z+h)(\bar{z}+\bar{h})-z\bar{z}}{h}$$

$$= \frac{1}{2|z|} \underset{h\to 0}{lt} \frac{\cancel{z\bar{z}}+z\bar{h}+h\bar{z}+h\bar{h}-\cancel{z\bar{z}}}{h}$$

$$= \frac{1}{2|z|} \underset{h\to 0}{lt} \left(\bar{z}+\bar{h}+z\frac{\bar{h}}{h}\right) \quad \text{——} (1)$$

Let us approach z along a line parallel to x-axis.

$\therefore$ h is a real number.

$\therefore$ $\bar{h}=h$. Then from (1),

$$f'(z) = \underset{h\to 0}{lt} \frac{f(z+h)-f(z)}{h}$$

$$= \frac{1}{2|z|} \underset{h\to 0}{lt} \; \bar{z}+h+\frac{z\cancel{h}}{\cancel{h}}$$

$$= \frac{1}{2|z|} \underset{h\to 0}{lt} (\bar{z}+h+z)$$

$$= \frac{1}{2|z|} \underset{h\to 0}{lt} (2x+h) \quad (\because z+\bar{z}=2x)$$

$$= \frac{\cancel{2}x}{\cancel{2}|z|} = \frac{x}{|z|} \quad \text{——} (2)$$

Again let us approach z along a line parallel to y-axis.

$\therefore$ h is purely imaginary.

$\therefore$ $h = ih_1$

Then from (1),

$$f'(z) = \underset{h\to 0}{lt} \frac{f(z+h)-f(z)}{h}$$

$$= \frac{1}{2|z|} \underset{h_1\to 0}{lt} \left(\bar{z}-ih_1+z\left(\frac{-ih_1}{ih_1}\right)\right)$$

$$= \frac{1}{2|z|} \underset{h_1\to 0}{lt} (\bar{z}-ih_1-z)$$

$$= \frac{1}{2|z|} \underset{h_1\to 0}{lt} (-2iy-ih_1)$$

$$= \frac{1}{2|z|}(-2iy) = \frac{-iy}{|z|} \quad \text{——} (3)$$

(2) & (3) are unequal if $x\neq 0$ and $y\neq 0$.

$\therefore$ $|z|$ does not have a derivative if $z\neq 0$.

Now if $z=0$, $\underset{h\to 0}{lt} \frac{f(0+h)-f(0)}{h} = \underset{h\to 0}{lt} \frac{|h|}{h}$

$$\Rightarrow \underset{h\to 0}{lt} \frac{f(z+h)-f(z)}{h} = \underset{h\to 0}{lt} \frac{|h|}{h}$$

$$\Rightarrow \underset{h\to 0}{lt} \frac{f(z+h)-f(z)}{h} = \begin{cases} \pm 1 & \text{if approached parallel to x-axis} \\ \pm i & \text{if approached parallel to y-axis.} \end{cases}$$

$\therefore$ the limit would not exist even now.

$\therefore$ $f(z)=|z|$ is no where differentiable.

Equating the real & imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

i.e. $u_x = v_y$, $v_x = -u_y$ or $u_y = -v_x$

these equations are known as Cauchy-Riemann Equations.

Note: from ②, we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial}{\partial x}(u+iv)$$

$$= \frac{\partial}{\partial x} f = \frac{\partial f}{\partial x}$$

Similarly from ③,

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$= -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= -i\frac{\partial}{\partial y}(u+iv) = -i\frac{\partial f}{\partial y}$$

From these we get,

$$\boxed{f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}}$$

this equation provides a method for calculating the derivative, if the derivative is known to exist.

Ex:- $f(z) = z^2$

$$= (x+iy)^2 = x^2 - y^2 + i(2xy)$$

is everywhere differentiable.

Sothat $f'(z) = \frac{\partial f}{\partial x}$

$$= 2x + i(2y)$$

$$= 2(x+iy) = 2z$$

→ **Problems:**

(1) Consider $f(z) = \bar{z}^2$

$$= (x-iy)^2$$

$$= x^2 - y^2 - 2ixy$$

$\therefore u(x,y) = x^2 - y^2$, $v(x,y) = -2xy$

$\therefore \frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial x} = -2y$

$\frac{\partial u}{\partial y} = -2y \quad \frac{\partial v}{\partial y} = -2x$

In general $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

But at $(0,0)$ these conditions are satisfied.

$\therefore$ The Cauchy-Riemann (CR) Conditions are not satisfied except at $(0,0)$.

$\therefore$ ~~the function is not differentiable except at (0,0).~~

(2) Consider $f(z) = z^2$

$$= (x+iy)^2$$

$$= x^2 - y^2 + 2ixy$$

$\therefore u(x,y) = x^2 - y^2$ & $v(x,y) = 2xy$

$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial x} = 2y$

$\frac{\partial u}{\partial y} = -2y \quad \frac{\partial v}{\partial y} = 2x$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\therefore$ Cauchy-Riemann-conditions are satisfied every where.

Now let us check whether the function is differentiable (or) not

For that consider

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h\to 0} \frac{(z+h)^2 - z^2}{h}$$

$$= \lim_{(a+ib)\to 0} \frac{(x+a)^2-(y+b)^2+2i(x+a)(y+b)-x^2+y^2-2ixy}{a+ib}$$

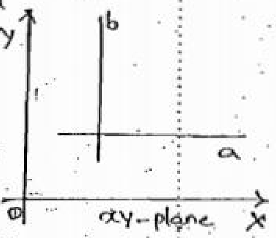
where $h = a+ib$

Let $h \to 0$ along the real axis of ab-plane, then $b=0$ and $a \to 0$

$$f'(z) = \lim_{a\to 0} \frac{2ax+a^2+2i(ay)}{a}$$

$$= 2(x+iy)$$

$$= 2z$$



Let $h \to 0$ along the Imaginary axis of the ab-plane

$\therefore a = 0$ and $b \to 0$

$$\therefore f'(z) = \lim_{b\to 0} \frac{-2by - b^2 + 2i(xb)}{ib}$$

$$= \lim_{b\to 0} \frac{-2y + 2ix}{i} = 2z$$

Since the limit is not depending on the value 'h' (or) a,b ;

We say that the function is differentiable for all values of z.

Note!- The Cauchy - Riemann Conditions are necessary Condition only. That is when a function f(z) is differentiable then Cauchy-Riemann Conditions are satisfied. But they are not sufficient conditions i.e. even though Cauchy-Riemann conditions are satisfied, the function may not be differentiable at that point. This Can be Verified in the following example.

$$\text{Let } f(z) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$

Let us observe that the behaviour of the function at the origin.

Let us verify Cauchy-Riemann Conditions at origin.

From the given function we get

$u = \dfrac{xy^2}{x^2+y^2}$ $\quad v = 0$ when $z \neq 0$

$u = 0$ $\quad v = 0$ when $z = 0$

$$\therefore \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

$$\text{Now } \left.\frac{\partial u}{\partial x}\right|_{z=0} = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$\left.\frac{\partial u}{\partial y}\right|_{z=0} = \lim_{h\to 0} \frac{u(0,k)-u(0,0)}{h}$$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \;\&\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

$\therefore$ Cauchy Riemann Conditions are satisfied at the origin.

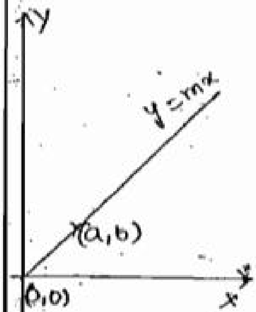
Now let us check the differenbility at the origin.

We say that the function is differentiable at the origin if

$\therefore \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ exists

Now let $h=a+ib$ and $h\to 0$ along a line $y=mx$ of xy-plane then

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)}{h}$$



$$= \lim_{h\to 0}\frac{f(a+ib)}{a+ib} \quad \text{i.e. } (a+ib)\to 0$$

$$= \lim_{(a+ib)\to 0}\frac{ab^2}{a^2+b^2}\cdot\frac{1}{a+ib}$$

$$= \lim_{a\to 0}\frac{a(ma)^2}{a^2+b^2}\cdot\frac{1}{a+ima}$$

$$= \lim_{a\to 0}\frac{m^2a^3}{a^2(1+m^2)}\cdot\frac{1}{a(1+im)}$$

$$= \lim_{a\to 0}\frac{m^2}{(1+im)(1+m^2)}$$

$$= \frac{m^2}{(1+im)(1+m^2)}$$

$\therefore$ the value of the limit dependens upon the value of m

$\therefore$ the derivative of $f(z)$ at $z=0$ doesnot exist.

$\therefore$ The function is not differentiable at the origin.

Note:- If the Cauchy-Riemann equations are not satisfied then the function is nowhere differentiable.

problems:

→ Determine where the following functions satisfy the Cauchy-Riemann equations and where the functions are differentiable $(0,0)$

(a) $f(z) = x^2-y^2$ (b) $f(z) = 2xyi$ differentiable at (0)

(c) $f(z) = z.\,Re(z)$ $(0,0)$ diff at $(0,0)$ (d) $f(z) = z|z|$

(e) $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ; z\neq 0 \\ 0 & ; z=0 \end{cases}$ CR-satis $(0,0)$ but not diff at $(0,0)$

Sol^n: (a) $f(z) = x^2-y^2$

Comparing with $f(z) = u(x,y)+iv(x,y)$

$u(x,y) = x^2-y^2$; $v(x,y)=0$

$\dfrac{\partial u}{\partial x} = 2x \qquad \dfrac{\partial v}{\partial x} = 0$

$\dfrac{\partial u}{\partial y} = -2y \qquad \dfrac{\partial v}{\partial y} = 0$

$\therefore \dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial v}{\partial x} \neq -\dfrac{\partial u}{\partial y}$

But at $(0,0)$ these conditions are satisfied.

$\therefore$ Cauchy-Riemann equations are not satisfied except at $(0,0)$.

Now let us check the differentibility at $(0,0)$.

We say that the function is differentiable at the origin if $\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$ exists.

Let $h=a+ib$ and $h\to 0$ along a line $y=mx$ of xy-plane,

then $\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h} = \lim\limits_{(a+ib)\to 0}\dfrac{f(a+ib)-0}{a+ib}$

$$= \lim_{(a+ib)\to 0}\frac{a^2-b^2}{a+ib}$$

$$= \underset{a\to 0}{Lt} \frac{a^2-(ma)^2}{a+i(ma)}$$

$$= \underset{a\to 0}{Lt} \frac{a^2(1-m^2)}{a(1+im)}$$

$$= 0$$

$\therefore$ At $(0,0)$ the given function $f(z) = x^2 - y^2$ is differentiable.

(d) $f(z) = z|z|$,

Sol^n: $f(z) = (x+iy)\sqrt{x^2+y^2}$

$$= x\sqrt{x^2+y^2} + iy\sqrt{x^2+y^2}$$

$u(x,y) = x\sqrt{x^2+y^2}$; $v(x,y) = y\sqrt{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}} \qquad \frac{\partial v}{\partial x} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{xy}{\sqrt{x^2+y^2}} \qquad \frac{\partial v}{\partial y} = \frac{x^2+2y^2}{\sqrt{x^2+y^2}}$$

$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}$ at $(x,y) \neq (0,0)$.

Now let us check at the origin:

At $z=0$:

$$\frac{\partial u}{\partial x} = \underset{x\to 0}{Lt} \frac{u(x,0)-u(0,0)}{x}$$

$$= \underset{x\to 0}{Lt} \frac{x\sqrt{x^2}}{x} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y\to 0}{Lt} \frac{u(0,y)-u(0,0)}{y}$$

$$= \underset{y\to 0}{Lt} \frac{0-0}{y} = 0$$

and $$\frac{\partial v}{\partial x} = \underset{x\to 0}{Lt} \frac{v(x,0)-v(0,0)}{x}$$

$$= \underset{x\to 0}{Lt} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \underset{y\to 0}{Lt} \frac{v(0,y)-v(0,0)}{y}$$

$$= 0$$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$.

$\therefore f(z)$ satisfies the Cauchy-Riemann equations at the origin.

Let us check the differentiability at $(0,0)$

Now $$f'(0) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z-0}$$

$$= \underset{z\to 0}{Lt} \frac{z|z|}{z} = \underset{z\to 0}{Lt} |z|$$

$$= \underset{(x,y)\to(0,0)}{Lt} \sqrt{x^2+y^2}$$

Along any path $f'(0) = 0$

$\therefore f(z)$ is differentiable at $(0,0)$.

* Analytic Function :-

— Consider a single valued function $f(z)$ in a domain D.

The function $f(z)$ is said to be analytic at a point $z = z_0$ if it is differentiable every where in the neighbourhood of z_0 (i.e. if there exists a neighbourhood $|z-z_0| < \delta$ at all points of which $f'(z)$ exists).

— Thus analyticity is a region based property.

— A function $f(z)$ is analytic in a domain D if it is analytic at every point in the domain.

— A function $f(z)$ is analytic at every point in the complex plane is called an entire function.

— If $f'(z)$ exists at every point of a domain D except at a finite number of exceptional points, then $f(z)$ is said to be analytic in D and is reffered to as analytic function in D. These exceptional points are called singular points (or) singularities of the function.

— If $f'(z)$ exists at every point of D, then we say that $f(z)$ is regular in D.

— The terms regular and holomorphic are also sometimes used as synonyms for analytic.

* Singular Point :-

A point, $z = z_0$ is said to be a singular point of a function $f(z)$ if $f'(z_0)$ does not exist.

Examples :

→ The function $f(z) = |z|^2$

$= x^2+y^2$ is differentiable only at origin but not differentiable at any other point. So it is not analytic at any other point.

Sol'n: Consider $\dfrac{f(z+h)-f(z)}{h} = \dfrac{|z+h|^2-|z|^2}{h}$

Let $h = a+ib$ then $\dfrac{|(x+a)+i(y+b)|^2 - |x+iy|^2}{a+ib}$

$\therefore \lim\limits_{h\to 0} \dfrac{f(z+h)-f(z)}{h} = \lim\limits_{h\to 0} \dfrac{(x+a)^2+(y+b)^2-(x^2+y^2)}{a+ib}$

$$= \lim_{(a+ib)\to 0} \frac{2ax+a^2+2by+b^2}{a+ib} \quad \text{--- (1)}$$

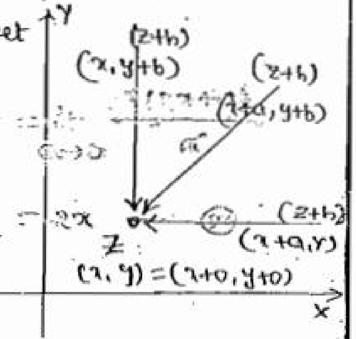
Let the point $z+h$ tends to z along a line parallel to real axis then as $h\to 0$, $b=0$ & $a\to 0$.

$\therefore$ from (1), we get

$\lim\limits_{h\to 0} \dfrac{f(z+h)-f(z)}{h}$

$= \lim\limits_{a\to 0} \dfrac{a(2x+a)}{a}$

$= 2x$ --- (2)



Similarly let us suppose that $z+h$ tends to z along a line parallel to imaginary axis, then as $h\to 0$, $a=0$ & $b\to 0$.

$\therefore$ from (1), we get

$$f'(z) = \lim_{b\to 0} \frac{b(2y+b)}{ib} = -i2y \quad \text{--- (3)}$$

Comparing (2) & (3), we say that the given function $|z|^2$ is not differentiable when $x\neq 0$, $y\neq 0$, i.e. other than origin. At the origin, we have.

Prove that the function f defined by $f(z)=\begin{cases}\frac{z^5}{|z|^4}, & z\neq 0\\ 0, & z=0\end{cases}$

is not differentiable at $z=0$.

Sol^n: We say that $f(z)$ is not differentiable at the origin if

$\underset{h\to 0}{\mathrm{Lt}}\ \frac{f(0+h)-f(0)}{h}$ does not exist.

Let $h=a+ib$

then $\underset{h\to 0}{\mathrm{Lt}}\ \frac{f(h)-f(0)}{h}=\underset{(a+ib)\to 0}{\mathrm{Lt}}\ \frac{f(a+ib)}{a+ib}$

$$=\underset{(a+ib)\to 0}{\mathrm{Lt}}\ \frac{(a+ib)^5}{|a+ib|^4}\times\frac{1}{(a+ib)}$$

$$=\underset{(a+ib)\to 0}{\mathrm{Lt}}\ \frac{(a+ib)^4}{|a+ib|^4}$$

$$=\left[\underset{(a+ib)\to 0}{\mathrm{Lt}}\ \frac{a+ib}{|a+ib|}\right]^4 \quad \text{——— (1)}$$

Let $h\to 0$ along the real axis of ab-plane then $b=0$ and $a\to 0$.

$$\therefore f'(0)=\left(\underset{a\to 0}{\mathrm{Lt}}\ \frac{a}{|a|}\right)^4$$

$$=1.$$

Let $h\to 0$ along the imaginary axis of ab-plane then $a=0$ and $b\to 0$.

$$\therefore f'(0)=\left[\underset{b\to 0}{\mathrm{Lt}}\ \frac{ib}{|ib|}\right]^4$$

$$=\left[\underset{b\to 0}{\mathrm{Lt}}\ \frac{ib}{b}\right]^4=1.$$

Let $h\to 0$ along a line $y=mx$ of xy-plane then from (1).

$$f'(0)=\left[\underset{a\to 0}{\mathrm{Lt}}\ \frac{a+ima}{\sqrt{a^2+m^2a^2}}\right]^4$$

$$=\left[\underset{a\to 0}{\mathrm{Lt}}\ \frac{1+im}{\sqrt{1+m^2}}\right]^4=\frac{1+i}{\sqrt{1+}}$$

$\therefore$ The value of the limit depe upon the value of 'm'.

$\therefore$ The derivative of $f(z)$ at $z=0$ c not exist.

$\therefore$ The function is not differentiable c origin.

(or)

Now $f'(0)=\underset{z\to 0}{\mathrm{Lt}}\ \frac{f(z)-f(0)}{z-0}$

$$=\underset{z\to 0}{\mathrm{Lt}}\ \frac{z^5}{|z|^4}\times\frac{1}{z}$$

$$=\underset{z\to 0}{\mathrm{Lt}}\ \frac{z^4}{|z|^4}=\left[\underset{z\to 0}{\mathrm{Lt}}\ \frac{z}{|z|}\right]$$

$$=\left[\underset{(x+iy)\to 0}{\mathrm{Lt}}\ \frac{x+iy}{\sqrt{x^2+y^2}}\right]^4$$

$z\to 0$ along a line $y=mx$ of xy.

then from (1)

$$f'(0)=\left[\underset{x\to 0}{\mathrm{Lt}}\ \frac{x+imx}{\sqrt{x^2+m^2x^2}}\right]^4=\left[\underset{x\to 0}{\mathrm{Lt}}\ \frac{1+im}{\sqrt{1+m^2}}\right]$$

$$=\left[\frac{1+im}{1+m^2}\right]^4$$

$\therefore$ The value of the limit dep upon the value of m

$\therefore f'(z)$ does not exist at $z=0$

Problems:

(1) $f(z)=e^z$. Is entire function?

Sol^n: $f(z)=e^z$

$$=e^{x+iy}$$

$= e^x \cdot e^{iy}$

$= e^x (\cos y + i \sin y)$

$= e^x \cos y + i e^x \sin y$

Comparing with $f(z) = u(x,y) + iv(x,y)$

$u(x,y) = e^x \cos y$; $v(x,y) = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y$; $\frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y$; $\frac{\partial v}{\partial y} = e^x \cos y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ Cauchy - Riemann conditions are satisfied.

Since $e^x \cos y$ & $e^x \sin y$ are always continuous everywhere.

$\therefore$ The partial derivatives are continuous everywhere.

$\therefore f(z)$ is an analytic everywhere.

Place restriction on the constants a, b & C so that the following functions are entire

(a) $f(z) = x + ay - i(bx + cy)$

(b) $f(z) = ax^2 - by^2 + icxy$

(c) $f(z) = e^x \cos ay \pm ie^x \sin(y+b) + c$

(d) $f(z) = a(x^2+y^2) + ibxy + c$

Sol^n :- (a) $f(z) = x + ay - i(bx + cy)$

Since $f(z)$ is entire function

$\therefore f(z)$ is analytic

$\therefore f(z)$ has continuous partial derivatives and satisfy Cauchy-Riemann equations.

Now $u(x,y) = x + ay$; $v(x,y) = -(bx+cy)$

$\frac{\partial u}{\partial x} = 1$; $\frac{\partial v}{\partial x} = -b$

$\frac{\partial u}{\partial y} = a$; $\frac{\partial v}{\partial y} = -c$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 1 = -c$
$\Rightarrow \boxed{c = -1}$

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow a = b$

(c) $f(z) = e^x \cos ay + ie^x \sin(y+b) + c$

Since $f(z)$ is entire function

$\therefore f(z)$ is analytic.

$\therefore f(z)$ has continuous partial derivatives & satisfies Cauchy - Riemann conditions

Now $u(x,y) = e^x \cos ay + c$;
$v(x,y) = e^x \sin(y+b)$

$\frac{\partial u}{\partial x} = e^x \cos ay + 0$; $\frac{\partial v}{\partial x} = e^x \sin(y+b)$

$\frac{\partial u}{\partial y} = -ae^x \sin ay$; $\frac{\partial v}{\partial y} = e^x \cos(y+b)$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow e^x \cos ay = e^x \cos(y+b)$

$\Rightarrow \cos ay = \cos(y+b)$

$\Rightarrow a = 1, b = 2k\pi, k = 1, 2 \ldots$

and c is any complex number.

* Problems Related to the test of Analyticity of a function:

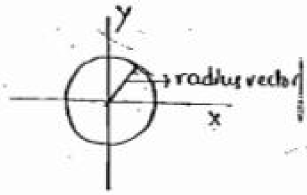
1) $f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2} & ; z \neq 0 \\ 0 & ; z=0 \end{cases}$

show that $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not $z \to 0$ in any manner.

i.e. f is not differentiable at $z=0$.

Sol'n: $\dfrac{f(z)-f(0)}{z} = \dfrac{f(z)-0}{z}$

$= \dfrac{f(z)}{z}$



$= \dfrac{x^3y(y-ix)}{(x^6+y^2)z}$

$= \dfrac{-ix^3y\cancel{(x+iy)}}{(x^6+y^2)\cancel{z}}$

$= \dfrac{-ix^3y}{x^6+y^2}$

Along the path $y=mx$ (radius vector)

$\lim_{z\to 0} \dfrac{f(z)-f(0)}{z} = \lim_{x\to 0} \dfrac{-ix^3(mx)}{x^6+(mx)^2}$

$= \lim_{x\to 0} \dfrac{-ix^4m}{x^2(x^4+m^2)}$

$= \lim_{x\to 0} \dfrac{-ix^2m}{x^4+m^2}$

$= 0$

$\therefore \dfrac{f(z)-f(0)}{z} \longrightarrow 0$ as $z \longrightarrow 0$.

Now along the path $y=x^3$.

$\lim_{z\to 0} \dfrac{f(z)-f(0)}{z} = \lim_{x\to 0} \dfrac{-ix^3y^3}{x^6+(x^3)^2}$

$= \lim_{x\to 0} \dfrac{-i}{2} = -\dfrac{i}{2} \neq 0.$

$\therefore \lim_{z\to 0} \dfrac{f(z)-f(0)}{z} \neq 0.$

along any path except radius vector

→ show that the function $f(z)=\sqrt{|xy|}$ is not analytic at $(0,0)$, although Cauchy-Riemann are satisfied at the point.

Sol'n: Given that $f(z)=\sqrt{|xy|}$

Here $u(x,y) = \sqrt{|xy|}$; $v(x,y)=0$.

At the point $(0,0)$

$\dfrac{\partial u}{\partial x} = \lim_{x\to 0} \dfrac{u(x,0)-u(0,0)}{x}$

$= \lim_{x\to 0} \dfrac{0-0}{x} = 0$

$\dfrac{\partial u}{\partial y} = \lim_{y\to 0} \dfrac{u(0,y)-u(0,0)}{y} = 0$

$\dfrac{\partial v}{\partial x} = \lim_{x\to 0} \dfrac{v(x,0)-v(0,0)}{x} = 0$

$\dfrac{\partial v}{\partial y} = 0.$

$\therefore$ Cauchy-Riemann equations are satisfied at the point $(0,0)$.

Again $f'(0) = \lim_{z\to 0} \dfrac{f(z)-f(0)}{z-0}$

$= \lim_{(x,y)\to(0,0)} \dfrac{\sqrt{|xy|}-0}{(x+iy)}$

Let $(x,y) \longrightarrow (0,0)$ along $y=mx$

$\therefore f'(0) = \lim_{x\to 0} \dfrac{\sqrt{|mx^2|}}{x+imx}$

$$= \lim_{x\to 0} \left(\frac{\sqrt{|m|}}{1+im}\right)$$

$$= \frac{\sqrt{|m|}}{1+im}$$

which depends on m.

$\therefore f'(0)$ doesnot exist.

$\therefore f(z)$ is not analytic at $(0,0)$

$\boxed{1998}$ Prove that the function

$f(z) = u+iv$ where

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} ; & z \neq 0 \\ 0 ; & z = 0 \end{cases}$$

is Continuous and Cauchy-Riemann equations are satisfied at the origin, yet $f'(z)$ doesnot exist at $z=0$.

Soln:- $f(z) = u+iv$

$$\Rightarrow u+iv = f(z)$$

$$= \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} ; z \neq 0.$$

$$= \frac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2} ; z \neq 0.$$

$$\Rightarrow u = \frac{x^3-y^3}{x^2+y^2}, \quad v = \frac{x^3+y^3}{x^2+y^2}$$

where $x \neq 0$, $y \neq 0$.

(I) To prove that $f(z)$ is continuous everywhere:

When $z \neq 0$, u & v both are

doit e by

$\varepsilon - \delta$ method.

rational functions of x & y with non-zero denominators.

$\therefore u$ & v are continuous at all those points for which $z \neq 0$.

$\therefore f(z)$ is continuous at $z \neq 0$.

At the origin:

$u=0$, $v=0$ ($\because f(0)=0$).

$\therefore u$ & v are both Continuous at the origin.

$\therefore f(z)$ is Continuous at $(0,0)$

$\therefore f(z)$ is Continuous everywhere

(II) To show that Cauchy-Riemann Equations are satisfied at $z=0$;

Since $f(0)=0$

$$\Rightarrow u(0,0) + iv(0,0) = 0.$$

$$\Rightarrow u(0,0) = 0 = v(0,0)$$

Now $$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{x^2} - 0}{x}$$

$$= \lim_{x\to 0} (1) = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{-y-0}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{y-0}{y} = 1$$

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{|0+h|^2 - |0|^2}{h}$$

$$= \lim_{h\to 0} \frac{|h|^2}{h}$$

Now as $h\to 0$ along real axis, h is any real number say h_1.

Then $\lim_{h_1\to 0} \frac{|h_1|^2}{h_1} = \lim_{h_1\to 0} \frac{h_1^2}{h_1} = 0$

As $h\to 0$ along imaginary axis.

$\cdot$ h is an imaginary number

say $h = ih_2$

then $\lim_{h\to 0} \frac{|h|^2}{h} = \lim_{h_2\to 0} \frac{|ih_2|^2}{h_2}$

$$= \lim_{h_2\to 0} \frac{h_2^2}{h_2}$$

$$= 0$$

$\therefore$ At origin the given function $|z|^2$ is differentiable.

Ex(2): The function $f(z) = x^2y^2$ is differentiable at all points on each of the coordinate axis, but is still no where analytic.

Ex(3): All polynomials are entire functions and $f(z) = \frac{1}{1-z}$ is analytic anywhere except at $z=1$.

* Now we can state the necessary and sufficient condition for a function to be analytic in a domain D as below:

— Necessary and Sufficient Conditions:

Let $f(z) = u(x,y) + iv(x,y)$ be defined in a domain D with $u(x,y)$ and $v(x,y)$ having continuous partial derivatives throughout D. then the necessary and sufficient condition for a function $f(z)$ to be analytic in D is the satisfaction of Cauchy - Riemann Conditions.

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

→ Show that an analytic $g(z)$ is independent of $\bar{z}$.

Sol'n:- Let $z = x+iy$ then

$x = \frac{z+\bar{z}}{2}$ & $y = \frac{z-\bar{z}}{2i}$

$$\therefore \frac{\partial g}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(g(z)).$$

$$= \frac{\partial}{\partial \bar{z}}(g(x+iy))$$

$$= \frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + i\frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial g}{\partial x}\left(\tfrac{1}{2}\right) + \frac{\partial g}{\partial y}\left(-\tfrac{1}{2i}\right)$$

$$= \tfrac{1}{2}\left[\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right]$$

Since $g(z)$ is analytic

$\therefore$ by Cauchy - Riemann Conditions, we have

$$\frac{\partial g}{\partial \bar{z}} = \tfrac{1}{2}\left(-i\frac{\partial g}{\partial y} + i\frac{\partial g}{\partial y}\right)$$

$$= 0. \qquad \left(\because f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}\right)$$

$\therefore$ then analytic function $g(z)$ is independent of $\bar{z}$.

* Cauchy - Riemann Equations:

Let $f(z) = u(x,y) + iv(x,y)$ and $f'(z)$ exists at $z = x+iy$. Then the first order partial derivatives of u and v exist at (x,y) and they must satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ there.

Also $f'(z) = u_x + iv_x$, where these partial derivatives are evaluated at (x,y).

Proof :- It is given that $f'(z)$ exists at z

$\therefore f'(z) = \lim_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$ exists.

Since $f(z) = u(x,y) + iv(x,y)$

$\Rightarrow f(x+iy) = u(x,y) + iv(x,y)$

Now we can write

$$\frac{f(z+h) - f(z)}{h} = \frac{f((x+a)+i(y+b)) - f(x+iy)}{a+ib}$$

where $h = a+ib$

$$= \frac{u(x+a, y+b) + iv(x+a, y+b) - u(x,y) - iv(x,y)}{a+ib}$$

$$= \frac{u(x+a, y+b) - u(x,y)}{a+ib} + i\frac{v(x+a, y+b) - v(x,y)}{a+ib} \quad \text{--- (1)}$$

Let us suppose that $h \to 0$ along the real axis then $b=0$ and $a \to 0$. (along a line ll lel to x-axis)

$$\therefore \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{a \to 0} \frac{u(x+a,y) - u(x,y)}{a} + i \lim_{a \to 0} \frac{v(x+a,y) - v(x,y)}{a}$$

Since the limit on the L.H.S exists, the limits on the R.H.S must also exist.

In addition to this we can observe that the limits on R.H.S are nothing but partial derivatives of u & v with respect to x.

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= u_x + iv_x \quad \text{--- (2)}$$

Now let us suppose that $h \to 0$ along a line parallel to the imaginary axis through the point z.

Then $a = 0$ and $b \to 0$.

$\therefore$ from (1),

$$f'(z) = \lim_{b \to 0} \frac{u(x,y+b) - u(x,y)}{ib} + i\lim_{b \to 0} \left(\frac{v(x,y+b) - v(x,y)}{ib} \right)$$

Since the limit on L.H.S. exists, the limits on R.H.S also exist. The limits on R.H.S are partial derivatives of u & v with respect to y.

$$\therefore f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{--- (3)}$$

Now comparing (2) & (3), we get

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

At $z=0$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0}\left(\frac{e^{-x^{-4}}}{x}\right) = \lim_{x\to 0}\left(\frac{e^{-1/x^4}}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{x e^{1/x^4}}\right)$$

$$= \lim_{x\to 0}\left[\frac{1}{x}\left(\frac{1}{1+\frac{1}{x^4}+\left(\frac{1}{x^4}\right)^2+\cdots}\right)\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{x+\frac{1}{x^3}+\frac{1}{x^7}+\cdots}\right]$$

$$=0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}$$

$$= \lim_{y\to 0}\frac{e^{-y^{-4}}}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y} = 0$$

$\therefore$ Cauchy–Riemann equations are satisfied at $z=0$.

(II) To show that $f(z)$ is not analytic at $z=0$.

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} e^{-z^{-4}}$$

Let $z\to 0$ along the path

$$z = re^{i\pi/4}$$

$\therefore r\to 0$ as $z\to 0$.

$$\therefore \lim_{z\to 0} f(z) = \lim_{r\to 0} e^{-[re^{i\pi/4}]^{-4}}$$

$$= \lim_{r\to 0} e^{-\left[r^{-4}\left(e^{i\pi/4}\right)^{-4}\right]}$$

$$= \lim_{r\to 0} e^{r^{-4}}$$

$$= \lim_{r\to 0} e^{1/r^4} = \infty$$

$\therefore \lim_{z\to 0} f(z)$ doesnot exist.

$\therefore f(z)$ is not continuous at $z=0$.

$\therefore f(z)$ is not differentiable at $z=0$

$\therefore f(z)$ is not analytic at $z=0$.

(III) Explanation

The function $f(z)$ is analytic at $z=0$ if (i) Cauchy–Riemann equations satisfied at $z=0$

(ii) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all continuous at $z=0$.

Here the first one is satisfied and the second one is not satisfied.

$\therefore$ The function is not analytic at $z=0$

DR.S.ANAND
DEPT.OF MATHS
ST.THERESA'S
COLLEGE