Cayley - Hamilton theorem: Every square matrix satisfies it's own characteristic equation ie if the characterstic equation of the n^{th} order square matrix A is

$$|A-\lambda I| = (-1)^n \lambda^n + K_1 \lambda^{n-1} + \dots + K_n = 0$$

then
$$(-1)^n A^n + K_1 A^{n-1} + \dots + K_n = 0$$

Prob: If $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ then $A^{-1} = ?$

Sol:
$$A^2 + 7A + 6I = 0 \Rightarrow A + 7I + 6A^{-1} = 0$$
$$\Rightarrow A^{-1} = \frac{1}{6}(-A - 7I)$$

Prob If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ then $A^8 =$

Sol:
$$\lambda^2 + 0\lambda - 5 = 0 \Rightarrow \lambda^2 = 5I$$
$$A^2 = 5I \Rightarrow A^8 = 625I = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}$$

Prob: If $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ then $A^{-1} =$

Sol:
$$\lambda = 1, 1, 1 \Rightarrow (\lambda - 1)^3 = 0$$
$$(A-1)^3 = 0$$
$$A^3 - 1 + 3A - 3A^2 = 0$$
$$A^2 - A^{-1} + 3I - 3A = 0 \Rightarrow A^{-1} = A^2 - 3A + 3I$$

* By cayley - hamilton theorem : $A^2 - A[Tr(A)] + |A| \cdot I = 0$

* For 3x3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ characterstic equation is

$$\lambda^3 - \lambda^2 [Tr(A)] + \lambda \left\{ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \right\} - |A| = 0$$

Method to find higher powers of A:

Let $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} \Rightarrow \lambda^2 - (-3)\lambda + 2 = 0$
$$\Rightarrow \lambda = -1, -2$$

If λ values are not repeated then $\lambda^n = a\lambda + b \rightarrow (1)$

for $\lambda = -1$ $\quad (-1)^n = -a + b \rightarrow (2)$

for $\lambda = -2$ $\quad (-2)^n = -2a + b \rightarrow (3)$

$\therefore b = (-1)^n + a$

$= (-1)^n + (-1)^n - (-2)^n = 2(-1)^n - (-2)^n$

put a & b in eq.(1) $\Rightarrow \lambda^n = [(-1)^n - (-2)^n]\lambda + [2(-1)^n - (-2)^n]$

by C-H theorem $A^n = [(-1)^n - (-2)^n]A + [2(-1)^n - (-2)^n]I$

Ex: put $n=9$ $\quad A^9 = 511A + 510I$

Properties:

Idompotent $A^2 = A$

Involutory $A^2 = I$

Nilpotent $A^n = 0$, n - index

Hermetian $A = \bar{A}^T = A^\theta = A^*$

skew hermetian $A = \bar{A}^T = -A^\theta = -A^*$

unitary $U \cdot U^\theta = I \Rightarrow U^{-1} = U^\theta$

Prob: The matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ -1 & -2 & -5 \end{bmatrix}$ is

$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Nilpotent

Prob: The matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$ is

$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \Rightarrow$ Involutory

Prob: $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

$A^2 = I \Rightarrow$ Involutory $\qquad \Rightarrow A \cdot A = I \rightarrow A \cdot A^T = I$

$A^T = A \Rightarrow$ Symmetric $\qquad \Rightarrow$ orthogonal.

LINEAR ALGEBRA

Matrix: A matrix is a rectangular array of numbers or functions arranged in 'm' rows and 'n' columns such that each row has same no. of elements (n) and each column has same no. of elements (m).

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

a_{ij} denotes element in i^{th} row and j^{th} column.

Row matrix: Matrix having single row. Ex: $[2\ 4\ 6\ 8]$

Column matrix: Matrix having single column. Ex: $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

Square matrix: Matrix having equal no. of rows and columns.

Ex: $\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$

* The elements along diagnol of a square matrix (a_{ij}, $i=j$) are called leading or principle diagonal elements.

* The sum of diagonal elements of a square matrix A is called Trace of A.

Diagonal matrix: A square matrix, except the leading diagonal elements are equal to zero is called diagonal matrix.

Ex: $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Scalar matrix: A diagonal matrix, whose leading diagonal elements are equal is called Scalar matrix

Ex: $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Unit matrix: A diagonal matrix, whose leading diagonal elements all equal to '1' is called unit matrix

<u>cofactor</u>: co. factor of any element 'a_{ij}' in a matrix is equal to $(-1)^{i+j} \cdot m_{ij}$. where 'm_{ij}' is minor of 'a_{ij}'.

Ex: $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then co.factor of b_3 i.e $B_3 = (-1)^{3+2} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

<u>Determinant</u>: Determinant of a matrix is defined as sum of product of elements of any row or column with corresponding co. factor.

$$\Delta = a_1 A_1 + b_1 B_1 + c_1 C_1$$

$$= a_1 (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 (-1)^{1+3} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$$

Tips for GATE:

* while calculating determinant, always select a row or column with more number of elements equal to '0'

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 5 & 0 & 6 \end{vmatrix} = 4[6-15] = -36$ [select 2^{nd} row]

* Try to remember co.factor signs $(-1)^{i+j}$ for each element as

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}_{3\times3} \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}_{4\times4}$$

<u>Properties of determinants</u>:

1. Determinant remains unaltered by changing its rows into columns and columns into rows. $|A| = |A^T|$

2. If two parellel lines of determinant are interchanged, then the sign of determinant changes (same numerical value)

3. If two parellel lines are identical then det = 0
   If all the elements in a row or column are zeros then det = 0

*4. If each element of a row or column is multiplied by

3

$$\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$|kA| = k^n |A|$

5. The determinant of upper or lower triangular matrix is equal to product of leading diagonal elements.

Ex: $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 1[18-0] = 18$
$= \underbrace{1\times 3\times 6}_{\text{diagonal elements}}$

6. If the elements of determinant Δ are function of 'x' and if 'k' parellel lines becomes equal when $x=a$ then $(x-a)^{k-1}$ is a factor of Δ.

Ex: $\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = (a-b)(a-c)(b-c)$ by putting $a=b, a=c, b=c$.

**7. Product of determinants:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} a_1l_1+b_1m_1+c_1n_1 & a_1l_2+b_1m_2+c_1n_2 & a_1l_3+b_1m_3+c_1n_3 \\ a_2l_1+b_2m_1+c_2n_1 & a_2l_2+b_2m_2+c_2n_2 & a_2l_3+b_2m_3+c_2n_3 \\ a_3l_1+b_3m_1+c_3n_1 & a_3l_2+b_3m_2+c_3n_2 & a_3l_3+b_3m_3+c_3n_3 \end{vmatrix}$$

8. $|\text{co.factor matrix of } A| = |A|^{n-1}$ n = order

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ co.factor matrix $= \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

$|A| = 4-6 = -2$ $|\text{co.factor matrix}| = 4-6 = -2$

9. If $|A| = 0$, then A is called Singular matrix.

10. If $|A| \neq 0$, then A is called non-Singular matrix.

<u>Adjoint</u> <u>matrix</u>: Transpose of the co.factor matrix.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ co.factor matrix $= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$\text{Adj}[A] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$|adj A| = \big| |A| \cdot A^{-1} \big|$
$= |A|^n |A^{-1}|$
$= |A|^n |A|^{-1} = |A|^{n-1}$

* $A\,(Adj\,A) = (\det A)\,I$
* $\det(Adj\,A) = (\det A)^{n-1}$ n = order

<u>Inverse of square matrix</u>: A matrix "B" is said to be inverse of a non-singular matrix "A" if $AB = BA = I$.

$$A^{-1} = \frac{Adj[A]}{|A|} \qquad A \cdot A^{-1} = I$$

$(Adj[A])^{-1} = \frac{A}{|A|}$

* $adj(adj[A]) = |A|^{n-2} \cdot A$

we know
$A \cdot adj[A] = |A| \cdot I$
replace A by $adj[A]$
$adj A\,[adj(adj A)] = |Adj A| \cdot I$
$A^{-1}|A|\,[adj(adj A)] = |Adj A|\,I$
$adj(adj A) = \frac{|A|^{n-1}}{|A|} \cdot A$
$= |A|^{n-2} \cdot A$

Ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

<u>Properties</u>:

* $(AB)^{-1} = B^{-1} \cdot A^{-1}$
* If $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} 1/d_1 & 0 & 0 \\ 0 & 1/d_2 & 0 \\ 0 & 0 & 1/d_3 \end{bmatrix}$
* If a non-singular matrix A is symmetric then A^{-1} is also symmetric.

* Every odd order symmetric matrix is singular i.e $|A| = 0 \Rightarrow A^{-1}$ does not exist for that

<u>Problems</u>

1. consider matrix $X_{4\times3}$, $Y_{4\times3}$, $P_{2\times3}$ then order of $[(P.(X^TY)^{-1})P^T]^T$

$$[(P_{2\times3} \cdot (X^T_{4\times3} \cdot Y_{4\times3})^{-1}) . P^T_{2\times3}]^T = [(P_{2\times3} \cdot (X_{3\times4} \cdot Y_{4\times3})^{-1}) P_{3\times2}]^T$$
$$= [(P_{2\times3} \cdot (3\times3)^{-1} \cdot P_{3\times2}]^T$$
$$= [P_{2\times3} \cdot P_{3\times2}]^T = (2\times2)^T = (2\times2)$$

Tip for GATE:

Determinant calculation [for matrix with less no. of zero elements]

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, (Applicable for 3x3)

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$|A| = a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12} \cdot a_{23}a_{31}$
$\quad - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$

Ex: Find inverse of $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$A = \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$R_1 \to R_1 - 3R_3 \sim \left[\begin{array}{ccc:ccc} 1 & 3 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$R_1 \to R_1 - 3R_2 \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$\hookrightarrow A^{-1}$

<u>Normal form of a matrix</u>: Every non-zero matrix of rank r can be reduced by sequence of elementary transformations to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ called the normal form of A.

For every matrix of rank 'r', there exist non-singular matrix P and Q such that. $PAQ = I$.

Ex: for matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ find non singular matrix P&Q such that PAQ is in normal form. Hence find the rank of

Sol:

$$A = IAI$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every row transformation $\to$ Same transformation in pre-factor matrix

Every column transformation $\to$ Same transformation in post factor matrix

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

Row echelon form: A matrix 'A' is said to be in row echelon form if

(i) Zero rows should occupy last rows, if any

(ii) The no. of zeros before a non zero element of each row is less than the no. of such zero's before the non zero element of the next row.

Ex: $\begin{bmatrix} 0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_4 & a_5 & 0 \\ 0 & 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

* Rank of a matrix = No. of non-zero rows in row echelon form
* Non-zero rows are called linearly independent rows/vector.
* To reduce any matrix into row echelon form, we should use only row transformations.

Ex: $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$ Find k if (i) Rank(A) = 2
(ii) Rank(A) = 3

$R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$

$\begin{matrix} R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & k-1 & -1 \end{bmatrix}$

$\begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_3 - R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k+2 & 0 \end{bmatrix} \Rightarrow$ For $\rho(A) = 2 \to k = -2$
For $\rho(A) = 3 \to k \neq -2$

* $\rho(O_{n\times n}) = 0$ * $\rho(I_{n\times n}) = n$
* $\rho\{adj(I_{n\times n})\} = n$ * $\rho(A) = \rho(A^T)$
* $\rho(A \pm B) \leq \rho(A) + \rho(B)$ * $\rho(A-B) \leq \rho(A) - \rho(B)$
* $\rho(AB) \geq \rho(A) + \rho(B) - n$ if A&B are $n\times n$ matrices

** $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

Rank of a Matrix: A matrix is said to be of rank r when

(i) it has atleast one non-zero minor of order 'r'

(ii) Every minor of order higher than r vanishes.

Rank of A is denoted by $\rho(A)$. Rank of null matrix = 0

Elementary transformations of a matrix:

1. Interchange of any rows / Columns
2. Multiplication of any row / column by a non zero number.
3. Addition of constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Elementary transformations do not change order or rank of the matrix.

Ex: Find rank of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$R_2 - R_1 \rightarrow R_2$
$R_3 - 2R_1 \rightarrow R_3$ $\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} = A$ (say)

$|A| = 0 \Rightarrow$ rank $\neq 3$

2nd order minor $\begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix} = -8 \neq 0$ $\therefore$ rank = 2

Gauss - Jordan Method to find Inverse:

1. write two matrices A & I side by side
2. Reduce A to I by performing same row operations on both. Then other matrix represent A^{-1}

Reducing A to I:

1. using R_1, make 1st element in $R_2, R_3 \ldots$ to zero
2. using R_2, make 2nd element in $R_3, R_4 \ldots$ to zero continue like this
3. using R_n make last element in $R_{n-1}, R_{n-2} \ldots$ to zero
4. using R_{n-1} make last-1 element in $R_{n-2}, R_{n-3} \ldots$ to zero

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Ex: Find inverse of $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$A = \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - 3R_3$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 3 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - 3R_2$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$\hookrightarrow A^{-1}$

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$$A = IAI$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every row transformation $\rightarrow$ Same transformation in pre-factor matrix

Every column transformation $\rightarrow$ Same transformation in post factor matrix

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

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Ex: $\begin{bmatrix} 0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_4 & a_5 & 0 \\ 0 & 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Dr.S. Anand

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$R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$ $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & k-1 & -1 \end{bmatrix}$

$R_3 \to R_3 - R_2$, $R_4 \to R_3 - R_2$ $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k+2 & 0 \end{bmatrix}$ $\Rightarrow$ For $\rho(A) = 2 \to k = -2$; For $\rho(A) = 3 \to k \neq -2$

* $\rho(O_{n\times n}) = 0$
* $\rho(I_{n\times n}) = n$
* $\rho\{\operatorname{adj}(I_{n\times n})\} = n$
* $\rho(A) = \rho(A^T)$
* $\rho(A \pm B) \leq \rho(A) + \rho(B)$
* $\rho(A - B) \leq \rho(A) - \rho(B)$
* $\rho(AB) \geq \rho(A) + \rho(B) - n$ if A&B are $n\times n$ matrices

** $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

Ex: Find inverse of $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$A = \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$$\sim \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - 3R_3$
$$\sim \left[\begin{array}{ccc:ccc} 1 & 3 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - 3R_2$
$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$\hookrightarrow A^{-1}$

<u>Normal form of a matrix</u>: Every non-zero matrix of rank 'r' can be reduced by sequence of elementary transformations to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ called the normal form of A.

For every matrix of rank 'r', there exist non-singular matrix P and Q such that. PAQ = I.

Ex: For matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ find non singular matrix P & Q such that PAQ is in normal form. Hence find the rank of

Sol:

$$A = IAI$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every row transformation → Same transformation in pre-factor matrix

Every column transformation → Same transformation in post factor matrix

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

Rank of a Matrix: A matrix is said to be of rank r when

(i) it has atleast one non-zero minor of order 'r'

(ii) Every minor of order higher than r vanishes.

Rank of A is denoted by $\rho(A)$. Rank of null matrix = 0

Elementary transformations of a matrix:

1. Interchange of any rows / columns
2. Multiplication of any row / column by a non zero number.
3. Addition of constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Elementary transformations do not change order or rank of the matrix.

Ex: Find rank of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$$\begin{matrix} R_2 - R_1 \rightarrow R_2 \\ R_3 - 2R_1 \rightarrow R_3 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} = A \text{ (say)}$$

$$|A| = 0 \Rightarrow \text{rank} \neq 3$$

2^{nd} order minor $\begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix} = -8 \neq 0 \qquad \therefore \text{rank} = 2$

Gauss - Jordan Method to find Inverse:

1. write two matrices A & I side by side
2. Reduce A to I by performing same row operations on both. Then other matrix represent A^{-1}

Reducing A to I:

1. using R_1, make 1^{st} element in $R_2, R_3 \ldots$ to zero
2. using R_2, make 2^{nd} element in $R_3, R_4 \ldots$ to zero continue like this
3. using R_n make last element in $R_{n-1}, R_{n-2} \ldots$ to zero
4. using R_{n-1} make last-1 element in $R_{n-2}, R_{n-3} \ldots$ to zero