Queens College of Arts and Science for Women
Punalkulam

DEPARTMENT OF MATHEMATICS

SUBJECT :DIFFENENTIAL EQUATIONS AND LAPLACE TRANSFORMS

TOPIC : LAPLACE TRANSFORMS

SUB CODE : 16SCCMM3

Unit - v

STAFF NAME : Mrs . G.Suguna

UNIT –V

LAPLACE TRANSFORMS

Definition :

If a function f(t) is defined for all positive variable t and if equal to F(s), then F(s) is called the Laplace transform of f(t) and it is denoted by L$\{f(t)\}$.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Theorems :

1.<u>Linear Property</u>

$$L\{f(t) + \varphi(t)\} = L\{f(t)\} + L\{\varphi(t)\}$$

Proof :

$$\text{w.k.t } L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

$$L\{f(t) + \varphi(t)\} = \int_0^\infty e^{-st} [f(t) + \varphi(t)]dt$$

$$= \int_0^\infty e^{-st} f(t)dt + \int_0^\infty e^{-st} \varphi(t)dt$$

$$= L\{f(t)\} + L\{\varphi(t)\}.$$

2. <u>Scale Property</u>

$$L\{cf(t)\} = cL\{f(t)\} \text{ where c is constant}$$

Proof:

$$\text{w.k.t } L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

$$L\{cf(t)\} = \int_0^\infty e^{-st} [cf(t)]dt$$

$$= c\int_0^\infty e^{-st} f(t)dt$$

$$= c\, L\{f(t)\}$$

3. L{$f'(t)$} = s L{$f(t)$} – f(0).

Proof :

w.k.t L{$f(t)$} = $\int_0^{\infty} e^{-st} f(t)dt$

L{$f'(t)$} = $\int_0^{\infty} e^{-st} f'(t)dt$

Using integration by parts

$u = e^{-st}$ $\quad$ $dv = f'(t)\, dt$

$du = e^{-st}(-s)$ $\quad$ $\int dv = \int f'(t)\, dt$

$du = -se^{-st}dt$ $\quad$ $v = f(t)$

= $[\, e^{-st} f(t)]_0^{\infty}$ - $\int_0^{\infty} (-se^{-st})\, f(t)dt$

= $[\, e^{-\infty} f(\infty) - f(0)]$ + s $\int_0^{\infty} e^{-st} f(t)dt$

= - f(0) +s L{$f(t)$}

= s L{$f(t)$} – f(0).

4. L{$f''(t)$} = s^2L{$f(t)$} – sf(0)- f'(0)

Proof :

L{$f''(t)$} = L{$F'(t)$}

Where F(t) = f'(t)

w.k.t L{$f'(t)$} = s L{$f(t)$} – f(0).

= s L{$F(t)$} – F(0).

= s L{$f'(t)$} – f'(0).

= S[s L{$f(t)$} – f(0)] – f'(0)

L{$f''(t)$} = s^2L{$f(t)$} – sf(0)- f'(0)

5. <u>Initial Value Theorem :</u>

If L$\{f(t)\}$= F(s) then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof :

Let L$\{f(t)\}$= F(s)

w.k.t L$\{f'(t)\}$ = s L$\{f(t)\}$ – f(0).

= s F(s) – f(0).

Taking limit as s→ ∞ on both sides

$$\lim_{s\to\infty} L\{f'(t)\} = \lim_{s\to\infty} [\, s\,F(s) - f(0)]$$

$$\lim_{s\to\infty} [\, s\,F(s) - f(0)] = \lim_{s\to\infty} L\{f'(t)\}$$

$$= \lim_{s\to\infty} \int_0^\infty e^{-st} f'(t)dt$$

$$\lim_{s\to\infty} [\, s\,F(s) - f(0)] = 0$$

$$\lim_{s\to\infty} s\,F(s) = \lim_{s\to\infty} f(0)$$

$$= f(0)$$

$$\lim_{s\to\infty} s\,F(s) = \lim_{t\to 0} f(t)$$

Hence the proof.

6. <u>Final Value Theorem</u> :

If L$\{f(t)\}$= F(s) then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Proof :

Let L$\{f(t)\}$ = F(s)

w.k.t L$\{f'(t)\}$ = s L$\{f(t)\}$ – f(0).

= s F(s) – f(0).

Taking limit as s→ 0 on both sides

$\lim_{s\to 0}$ L$\{f'(t)\}$ = $\lim_{s\to 0}$ [s F(s) – f(0)]

$\lim_{s\to 0}$ [s F(s) – f(0)] = $\lim_{s\to 0}$ L$\{f'(t)\}$

= $\lim_{s\to 0} \int_0^\infty e^{-st} f'(t)dt$

$\lim_{s\to 0}$ [s F(s) – f(0)] = [f(t)]

$\lim_{s\to 0}$ [s F(s) – f(0)] = $\lim_{t\to\infty}$ f(t) - $\lim_{t\to 0}$ f(t)

= $\lim_{t\to\infty}$ f(t) - f(0)

$\lim_{s\to 0}$ s F(s) = $\lim_{t\to\infty}$ f(t) - f(0) + $\lim_{s\to 0}$ f(0)

= $\lim_{t\to\infty}$ f(t) - f(0) + f(0)

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Hence the proof.

7. If L$\{f(t)\}$= F(s) thenL$\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$.

Proof:

w.k.t L$\{f(t)\}$ = $\int_0^\infty e^{-st} f(t)dt$

L $\{f(at)\}$ = $\int_0^\infty e^{-st} [f(at)]dt$

Put at = y

t= y/a

dt = dy/a

L $\{f(at)\}$ = $\int_0^\infty e^{-s\left(\frac{y}{a}\right)} [f(y)] \frac{dy}{a}$

$$= \frac{1}{a}\int_0^\infty e^{\frac{-sy}{a}} [f(y)]dy$$

L$\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$.

Hence the proof.

8.If L$\{f(t)\}$= F(s) thenL $\{e^{-at}f(t)\}$ = F (s+a).

Proof:

w.k.t L$\{f(t)\}$ = $\int_0^\infty e^{-st} f(t)dt$

L $\{e^{-at}f(t)\}$ = $\int_0^\infty e^{-st}e^{-at} f(t)dt$

= $\int_0^\infty e^{-st-at} f(t)dt$

= $\int_0^\infty e^{-t(s+a)} f(t)dt$

L $\{e^{-at}f(t)\}$ = F (s+a).

Hence the proof.

9.If L$\{f(t)\}$= F(s) thenL $\{tf(t)\}$ = - $\frac{d}{ds}$F (s).

Proof:

w.k.t L$\{f(t)\}$ = $\int_0^\infty e^{-st} f(t)dt$

F(s) = $\int_0^\infty e^{-st} f(t)dt$

$\frac{d}{ds}$F(s) $= \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt$

$= \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)dt$

$= \int_0^\infty -te^{-st} f(t)dt$

$= -\int_0^\infty te^{-st} f(t)dt$

= - L $\{tf(t)\}$

$-\frac{d}{ds}$F (s) = L $\{tf(t)\}$

Hence the proof.

10..If L$\{f(t)\}$= F(s) and if f(t)/t has limit as t → 0, then

L $\{ \frac{f(t)}{t} \} = \int_s^\infty F(s)ds.$

Proof:

w.k.t L$\{f(t)\}$ $= \int_0^\infty e^{-st} f(t)dt$

F(s) $= \int_0^\infty e^{-st} f(t)dt$

$\int_s^\infty F(s)ds. = \int_s^\infty \int_0^\infty e^{-st} f(t)dtds$

$=\int_0^\infty \int_s^\infty e^{-st} f(t)dsdt$

On interchanging the order of integration

$=\int_0^\infty f(t) \left[\frac{e^{-st}}{t}\right]_s^\infty$ dt

$= \int_0^\infty \frac{f(t)}{t} e^{-st}dt$

L $\{ \frac{f(t)}{t} \} = \int_s^\infty F(s)ds.$

Hence the proof.