De Moiver's Theorem.

If n is any integer, then.

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

If n is a fraction, then. $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Note : 1.

$$\frac{1}{\cos\theta \pm i\sin\theta} = (\cos\theta \pm i\sin\theta)^{-1}$$

$$= (\cos\theta \pm i\sin\theta)$$

Note : 2.

$$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta.$$

1) Prove that $(\sin x + i\cos x)^n = \cos n(\pi/2 - x) + i\sin n(\pi/2 - x)$

Proof:

$$(\sin x + i\cos x)^n = \left[\cos(\pi/2 - x) + i\sin(\pi/2 - x)\right]^n$$

$$= \left[\cos n(\pi/2 - x) + i\sin n(\pi/2 - x)\right]$$

[∵ by De Moiver's theorem]

2) Prove that $(1 + i\sqrt{3})^n + (1 - i\sqrt{3})^n = 2^{n+1}\cos\frac{n\pi}{3}$.

Solu:

$$(1 + i\sqrt{3}) = 2\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$$

$= 2\left[\cos(\pi/3) + i\sin(\pi/3)\right]$

$(1 - i\sqrt{3}) = 2\left(\cos \pi/3 - i\sin \pi/3\right).$

$(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = \left[2(\cos \pi/3 + i\sin \pi/3)\right]^n + \left[2(\cos \pi/3 - i\sin \pi/3)\right]^n.$

$= 2^n (\cos n\pi/3 + i\sin n\pi/3) + 2^n(\cos n\pi/3 - i\sin n\pi/3)$

$= 2^n\left[\cos n\pi/3 + \cancel{i\sin n\pi/3} + \cos n\pi/3 - \cancel{i\sin n\pi/3}\right]$

$= 2^n (2\cos n\pi/3)$

$= 2^{n+1}\cos\frac{n\pi}{3}.$

Expansion of $\cos n\theta$ & $\sin n\theta$ in powers of $\sin\theta$ & $\cos\theta$, n being a positive integer.

$$\boxed{\begin{aligned}\cos n\theta &= \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \dots\\ \sin n\theta &= {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \dots\end{aligned}}$$

Expansion of $\tan n\theta$.

We know that, $\tan n\theta = \dfrac{\sin n\theta}{\cos n\theta}.$

$$\boxed{\tan n\theta = \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \dots}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \dots}}$$

Dividing both numerator & denominator by $\cos^n\theta$.

$$\boxed{\tan n\theta = \frac{nc_1 \tan\theta - nc_3 \tan^3\theta + nc_5 \tan^5\theta - \dots}{1 - nc_2 \tan^2\theta + nc_4 \tan^4\theta - \dots}}$$

1) Express $\cos 5\theta$ interms of $\cos\theta$.

$\frac{6x^2+5x+4}{1+x+x^2}$

solu:

We know that,

$$\cos n\theta = \cos^n\theta - nc_2 \cos^{n-2}\theta \sin^2\theta + nc_4 \cos^{n-4}\theta \sin^4\theta - \dots$$

put $n=5$.

$$\cos 5\theta = \cos^5\theta - 5c_2 \cos^3\theta \sin^2\theta + 5c_4 \cos\theta \sin^4\theta.$$

$$= \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta(\sin^2\theta)^2. \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1+\cos^4\theta - 2\cos^2\theta)$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta + 5\cos^5\theta - 10\cos^3\theta.$$

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

2) Express $\sin 7\theta$ interms of $\sin\theta$.

solu:

We know that,

$$\sin n\theta = nc_1 \cos^{n-1}\theta \sin\theta - nc_3 \cos^{n-3}\theta \sin^3\theta + \dots$$

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put $n=7$.

$$\sin 7\theta = 7C_1 \sin\theta \cos^6\theta - 7C_3 \cos^4\theta \sin^3\theta + 7C_5 \cos^2\theta \sin^5\theta - 7C_7 \cos^0\theta \sin^7\theta.$$

$$= 7(\cos^2\theta)^3 \sin\theta - 35(\cos^2\theta)^2 \sin^3\theta + 21 \cos^2\theta \sin^5\theta - \sin^7\theta.$$

$(a^3 - 3a^2b + 3ab^2 - b^3)$

$$= 7\sin\theta(1-\sin^2\theta)^3 - 35\sin^3\theta(1-\sin^2\theta)^2 + 21\sin^5\theta(1-\sin^2\theta) - \sin^7\theta.$$

$$= 7\sin\theta(1-3\sin^2\theta + 3\sin^4\theta - \sin^6\theta) - 35\sin^3\theta(1+\sin^4\theta - 2\sin^2\theta) + 21\sin^5\theta - 21\sin^7\theta - \sin^7\theta.$$

$$= 7\sin\theta - 21\sin^3\theta + 21\sin^5\theta - 7\sin^7\theta - 35\sin^3\theta - 35\sin^7\theta + 70\sin^5\theta + 21\sin^5\theta - 21\sin^7\theta - \sin^7\theta.$$

$$= -64\sin^7\theta + 112\sin^5\theta - 56\sin^3\theta + 7\sin\theta.$$

$$= 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta.$$

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1) Show that $\frac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$

Solu:

We know that;

$\sin n\theta = nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta + \dots$

put $n=6$.

$\sin 6\theta = 6C_1 \cos^5\theta \sin\theta - 6C_3 \cos^3\theta \sin^3\theta + 6C_5 \cos\theta \sin^5$

$\frac{6 \cdot 5 \cdot 4}{3!}$

$= 6\cos^5\theta \sin\theta - 20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta$

$\div \sin\theta \Rightarrow \frac{\sin 6\theta}{\sin\theta} = 6\cos^5\theta - 20\cos^3\theta \sin^2\theta + 6\cos\theta \sin^4$

$= 6\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1-\cos^2\theta)$

$= 6\cos^5\theta - 20\cos^3\theta + 20\cos^5\theta + 6\cos\theta(1 + \cos^4\theta - 2\cos^2$

$= 26\cos^5\theta - 20\cos^3\theta + 6\cos\theta + 6\cos^5\theta - 12\cos^3$

$\frac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$.

Expansion of $\sin^n\theta$ & $\cos^n\theta$ interms of sines & cosines of multiples of θ, n being a +ve integer

Let $x = \cos\theta + i\sin\theta \rightarrow ①$.

$\frac{1}{x} = \cos\theta - i\sin\theta \rightarrow ②$

$x^n = \cos n\theta + i\sin n\theta \rightarrow ③$

$\frac{1}{x^n} = \cos n\theta - i\sin n\theta \rightarrow ④$.

① + ② $\rightarrow x + \frac{1}{x} = 2\cos\theta$.

① − ② $\rightarrow x - \frac{1}{x} = 2i\sin\theta$.

③ + ④ $\rightarrow x^n + \frac{1}{x^n} = 2\cos n\theta$.

③ − ④ $\rightarrow x^n - \frac{1}{x^n} = 2i\sin n\theta$.

To get expansion of $\cos^n\theta$:

$$(2\cos\theta)^n = \left(x+\frac{1}{x}\right)^n \quad (\because \text{from } ⓐ)$$

To get expansion of $\sin^n\theta$:

$$(2i\sin\theta)^n = \left(x-\frac{1}{x}\right)^n.$$

To get expansion of $\sin^m\theta\cos^n\theta$:

$$(2i\sin\theta)^m(2\cos\theta)^n = \left(x-\frac{1}{x}\right)^m\left(x+\frac{1}{x}\right)^n.$$

1) Prove that $\sin^5\theta = \frac{1}{16}\left[\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right]$

Solution :

Let $x = \cos\theta + i\sin\theta \rightarrow ①$

$\frac{1}{x} = \cos\theta - i\sin\theta \rightarrow ②$

$x^n = \cos n\theta + i\sin n\theta \rightarrow ③$

$\frac{1}{x^n} = \cos n\theta - i\sin n\theta \rightarrow 4$

① + ② $\rightarrow x + \frac{1}{x} = 2\cos\theta$.

① − ② $\rightarrow x - \frac{1}{x} = 2i\sin\theta$

③ + ④ $\rightarrow x^n + \frac{1}{x^n} = 2\cos n\theta$.

③ − ④ $\rightarrow x^n - \frac{1}{x^n} = 2i\sin n\theta$.

$(2i\sin\theta)^n = \left(x-\frac{1}{x}\right)^n$

put $n=5$.

$(2i\sin\theta)^5 = \left(x-\frac{1}{x}\right)^5$

Using Binomial theorem:

$$\left(x-\frac{1}{x}\right)^n = x^n - nC_1\, x^{n-1}\frac{1}{x} + nC_2\, x^{n-2}\frac{1}{x^2} - nC_3\, x^{n-3}\frac{1}{x^3}\cdots$$

$$(2i\sin\theta)^5 = x^5 - 5C_1\, x^4 \frac{1}{x} + 5C_2\, x^3 \frac{1}{x^2} - 5C_3\, x^2\frac{1}{x^3} + 5C_4\, x\frac{1}{x^4} - 5C_5\frac{1}{x^5}$$

$$2^5 i^5 \sin^5\theta = x^5 - 5x^3 + 10x - 10\frac{1}{x} + 5\frac{1}{x^3} - \frac{1}{x^5}.$$

$$2^5 i \sin^5\theta = \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right).$$

$$2^5 i \sin^5\theta = \cancel{2i}\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$

$$\cancel{2^5}\cancel{i} \sin^5\theta = \cancel{2i}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

$$\sin^5\theta = \frac{1}{16}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$$

2) Prove that $\cos^6\theta = \frac{1}{32}[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta]$

solution:

Let $x = \cos\theta + i\sin\theta \rightarrow ①$.

$\frac{1}{x} = \cos\theta - i\sin\theta \rightarrow ②$.

$x^n = \cos n\theta + i\sin n\theta \rightarrow ③$.

$\frac{1}{x^n} = \cos n\theta - i\sin n\theta \rightarrow ④$.

① + ② $\Rightarrow x + \frac{1}{x} = 2\cos\theta$

① − ② $\Rightarrow x - \frac{1}{x} = 2i\sin\theta$.

③ + ④ $\Rightarrow x^n + \frac{1}{x^n} = 2\cos n\theta$.

③ − ④ $\Rightarrow x^n - \frac{1}{x^n} = 2i\sin n\theta$.

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$(2\cos\theta)^n = (x + 1/x)^n.$

By Using Binomial theorem:

$(x+1/x)^n = x^n + nC_1 x^{n-1} 1/x + nC_2 x^{n-2} 1/x^2 + \dots$

$(2\cos\theta)^6 = (x + 1/x)^6.$

$2^6 \cos^6\theta = x^6 + 6C_1 x^5 1/x + 6C_2 x^4 1/x^2 + 6C_3 x^3 1/x^3 +$
$6C_4 x^2 1/x^4 + 6C_5 x 1/x^5 + 6C_6 1/x^6$

$= x^6 + 6x^4 + 15x^2 + 20 + 15 1/x^2 + 6 1/x^4 + 1/x^6.$

$= (x^6 + 1/x^6) + 6(x^4 + 1/x^4) + 15(x^2 + 1/x^2) + 20.$

$= (2\cos 6\theta) + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20.$

$\cancel{2^6}\cos^6\theta = \cancel{2}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

2) Prove that $\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$.

Solution:

$$\sin n\theta = nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta + nC_5 \cos^{n-5}\theta \sin^5\theta \ldots$$

put $n=7$

$$\sin 7\theta = 7C_1 \cos^6\theta \sin\theta - 7C_3 \cos^4\theta \sin^3\theta + 7C_5 \cos^2\theta \sin^5\theta - 7C_7 \sin^7\theta$$

$$= 7\cos^6\theta \sin\theta - 35\sin^3\theta(\cos^2\theta)^2 + 21\sin^5\theta\cos^2\theta - \sin^7\theta.$$

$$= 7\sin\theta(1-\sin^2\theta)^3 - 35\sin^3\theta(1-\sin^2\theta)^2 + 21\sin^5\theta(1-\sin^2\theta) - \sin^7\theta.$$

$$= 7\sin\theta(1-\sin^6\theta - 3\sin^2\theta + 3\sin^4\theta) - 35\sin^3\theta(1+\sin^4\theta - 2\sin^2\theta) + 21\sin^5\theta - 21\sin^7\theta - \sin^7\theta.$$

$$= 7\sin\theta - 7\sin^7\theta - 21\sin^3\theta + 21\sin^5\theta - 35\sin^3\theta - 35\sin^7\theta + 70\sin^5\theta + 21\sin^5\theta + 21\sin^7\theta - \sin^7\theta$$

$$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta.$$

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta.$$

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Solve approximately $\sin(\pi/6 + \theta) = 0.51$.

Solution:

We know that,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

$$\sin(\pi/6 + \theta) = \sin \pi/6 \cos\theta + \cos \pi/6 \sin\theta.$$

$$= 1/2 \cos\theta + \sqrt{3}/2 \sin\theta.$$

$$= 1/2 \left[1 - \theta^2/2! + \theta^4/4 \cdots\right] + \frac{\sqrt{3}}{2}\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right]$$

$$(\sin(\pi/6 + \theta)) = 1/2 + \sqrt{3}/2\,\theta \to ⑩$$ [$\because$ omitting ~~the~~ θ^2 and highest power of θ]

Given $\sin\left(\pi/6 + \theta\right) = 0.51.$

From ① & ②

$$\Rightarrow \frac{1}{2} + \frac{\sqrt{3}}{2}\theta = 0.51$$

$$\frac{\sqrt{3}}{2}\theta = 0.51 - \frac{1}{2}.$$

$$= 0.51 - 0.5$$

$$= 0.01.$$

$$\frac{\sqrt{3}}{2}\theta = \frac{1}{100}.$$

$$\theta = \frac{1}{\cancel{100}} \times \frac{\cancel{2}}{\sqrt{3}}.$$

$$= \frac{1}{50} \times \frac{1}{\sqrt{3}} . \frac{\sqrt{3}}{\sqrt{3}}.$$

$$= \frac{\sqrt{3}}{50 \times 3} = \frac{\sqrt{3}}{150} \text{ radians.}$$

$$= \frac{\sqrt{3}}{150} \times \frac{180}{\pi} \text{ degrees.}$$

$$\theta = 39^\circ 78'.$$

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3) $\frac{\tan\theta}{\theta} = \frac{2524}{2523}$ then find the value of θ.

Solution:

We know that,

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + \dots$$

$$\frac{\tan\theta}{\theta} = 1 + \theta^2/3 + \frac{2}{15}\theta^4 + \dots \rightarrow (1).$$

Given, $\frac{\tan\theta}{\theta} = \frac{2524}{2523}$.

$$\frac{\tan\theta}{\theta} = 1 + \frac{1}{2523}. \rightarrow (2).$$

Equating (1) & (2)

$$1 + \theta^2/3 + \frac{2}{15}\theta^4 + \dots = 1 + \frac{1}{2523}.$$

$$\theta^2/3 = 1/2523.$$

$$\theta^2 = \cancel{3}/\cancel{2523}.$$

$$\theta^2 = \frac{1}{891}.$$

$$\theta = \sqrt{1/891}.$$

$$\theta = 1/29 \text{ radians}.$$

$$= \frac{1}{29} \times \frac{180}{\pi}.$$

$$= 1^\circ 58'.$$