Unit-5.

Exponential Function.

Exponential function of a complex variable:

1) $e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \infty$

We can define the exponential function of the complex variable $z = x + iy$, as.

$$e^{z} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots \infty$$

2) $e^{iy} = \cos y + i \sin y.$

3) $e^{-iy} = \cos y - i \sin y.$

Circular function of a complex variable:

The circular function of a complex variable z by the equation,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z} = \left(\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right) \frac{1}{i}.$$

Note:

1) $(e^{i\theta})^n = e^{in\theta}$

2) $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$.

3) $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$.

4) $\tan x = \dfrac{\sin x}{\cos x} = \left(\dfrac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}\right)\dfrac{1}{i}$.

5) $\sec x = \dfrac{1}{\cos x} = \dfrac{2}{e^{ix} + e^{-ix}}$

6) $\operatorname{cosec} x = \dfrac{1}{\sin x} = \dfrac{2i}{e^{ix} - e^{-ix}}$.

7) $\cot x = \dfrac{1}{\tan x} = i\left(\dfrac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}\right)$.

Hyperbolic function:

1) $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$.

2) $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$.

3) $\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$.

4) $\operatorname{sech} x = \dfrac{2}{e^{x} + e^{-x}}$.

5) $\operatorname{cosech} x = \dfrac{2}{e^{x} - e^{-x}}$

6) $\coth x = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$.

$\dfrac{e^{x} + e^{-x}}{2} = \dfrac{2}{e^{x} + e}$

$\tanh x = \dfrac{e^{x} - e}{e^{x}}$

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Relation b/w Hyperbolic function & circular function.

1) $\sin ix = i \sinh x$.

2) $\cos ix = \cosh x$.

3) $\tan ix = i \tanh x$.

4) $\sinh ix = i \sin x$.

5) $\cosh ix = \cos x$.

6) $\tanh ix = i \tan x$.

Formulae of Hyperbolic function.

1) $\cosh^2 x - \sinh^2 x = 1$

2) $\operatorname{sech}^2 x + \tanh^2 x = 1$.

3) $\coth^2 x - \operatorname{cosech}^2 x = 1$.

4) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$.

5) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$.

6) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 + \tanh x \tanh y}$.

7) $\sinh 2x = 2 \sinh x \cdot \cosh x$.

8) $\cosh 2x = 1 + 2\sinh^2 x$.

9) $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$.

10) $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$.

11) $\cosh 3x = 4\cosh^3 x - 3\cosh x$.

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12) $\tanh 3x = \dfrac{3\tanh x \pm \tanh^3 x}{1+3\tanh^2 x}.$

13) $\sinh x + \sinh y = 2\sinh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}.$

14) $\sinh x - \sinh y = 2\cosh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}.$

15) $\cosh x + \cosh y = 2\cosh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}.$

16) $\cosh x - \cosh y = 2\sinh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}.$

Inverse hyperbolic function:

Prove That $\sinh^{-1} x = \log(x+\sqrt{x^2+1})$.

Solution:

Let, $\sinh^{-1} x = y \rightarrow (1).$

$$x = \sinh y$$

$$= \frac{e^y - e^{-y}}{2}$$

$$= \frac{e^y - \frac{1}{e^y}}{2}.$$

$$= \frac{\frac{e^y \cdot e^y - 1}{e^y}}{2}$$

$$x = \frac{e^{2y} - 1}{2e^y} \Rightarrow 2e^y x = e^{2y} - 1$$

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$e^{2y} - 1 - 2e^y x = 0,$

$(e^y)^2 - 2xe^y - 1 = 0.$

$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$

$= \frac{2x \pm \sqrt{4x^2+4}}{2}$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$= \frac{2x \pm \sqrt{4(x^2+1)}}{2}$$

$$= \frac{\cancel{2}x \pm \cancel{2}\sqrt{x^2+1}}{\cancel{2}}.$$

$e^y = x \pm \sqrt{x^2+1}$

$\therefore e^y > 0$, '+ve' sign must be taken.

$e^y = x + \sqrt{x^2+1}$

Taking log on both sides,

$\log e^y = \log(x + \sqrt{x^2+1}).$

$y = \log(x + \sqrt{x^2+1})$

$\therefore \sinh^{-1} x = \log(x + \sqrt{x^2+1}).$

2) Prove That $\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}.$

solution :

Let $\tanh^{-1} x = y. \rightarrow (1).$

$x = \tanh y.$

$$= \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^y - 1/e^y}{e^y + 1/e^y}$$

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$$= \frac{\frac{e^y . e^y - 1}{e^y}}{\frac{e^y . e^y + 1}{e^y}} \quad x = \frac{\frac{e^{2y} - 1}{\cancel{e^y}}}{\frac{e^{2y}+1}{\cancel{e^y}}}$$

$$x = \frac{e^{2y}-1}{e^{2y}+1}$$

$$x(e^{2y}+1) = e^{2y}-1$$

$$xe^{2y}+x = e^{2y}-1$$

$$1+x = e^{2y} - xe^{2y}$$

$$1+x = e^{2y}(1-x)$$

$$e^{2y} = \frac{1+x}{1-x}.$$

Taking log on both sides,

$$\log e^{2y} = \log \frac{1+x}{1-x}.$$

$$2y = \log \frac{1+x}{1-x}.$$

$$y = 1/2 \log \frac{1+x}{1-x}.$$

$$\therefore \tanh^{-1} x = 1/2 \log \frac{1+x}{1-x}.$$