Differential Equations and Laplace Transforms

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SOLVABLE FOR X

Let the given differential eqn be of the form $F(x, y, p) = 0$ where $p = dy/dx$

From equation (1) if it is possible to find x in terms of y & p explicity (without any difficulty) then eqn (1) can be written as $x = F(y, p)$

Now differentiate (2) w.r. to y we get

$$\frac{dx}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial y}$$

$$1/p = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial y}$$

As already explained equ (3) is an ordinary 1st order differential equn in y & p. let the soln of (3) be of the For

$$\phi(y, p, c) = 0.$$

Now the elimination of p from (2) & (4) gives the solution of the given differential equation.

Note - I

If it is not possible to eliminate 'p' from (2) & (4) Then equ- (2) together with equ (4) gives the solution. In this case we can express x and y in terms of p as a parameter.

Solvable for x

1) Solve $x = 1 - \frac{p}{\sqrt{p^2+1}}$

Solution:-

Given equation $x = 1 - \frac{p}{\sqrt{p^2+1}}$

Differentiate with respect to y

$$\frac{dx}{dy} = -\left[\frac{\sqrt{p^2+1}\, dp/dy - p\left(\frac{1}{2}(p^2+1)^{-1/2}(+2p)\right)\frac{dp}{dy}}{\sqrt{(p^2+1)}^{2}}\right]$$

$$\frac{1}{p} = -\frac{\left[\sqrt{p^2+1}\, dp/dy - \frac{p^2}{\sqrt{p^2+1}}\, dp/dy\right]}{(p^2+1)}$$

$$\frac{1}{p} = -dp/dy\left[\sqrt{p^2+1} - \frac{p^2}{\sqrt{p^2+1}}\right]$$

$$\frac{1}{p} = -dp/dy\left[\frac{p^2+1-p^2}{\sqrt{p^2+1}\,(p^2+1)}\right]$$

$$\frac{1}{p} = -dp/dy\left[\frac{1}{\sqrt{p^2+1}\,(p^2+1)}\right]$$

$$dy = -dp\left[\frac{p}{(p^2+1)^{3/2}}\right]$$

$$p^2+1 = t^2$$

$$2p\frac{dp}{dy} = 2t\frac{dt}{dy}$$

$$p\,dp = t\,dt$$

$$dy = -\frac{t\,dt}{t^3}$$

$$dy = -\frac{dt}{t^2}$$

$$dy = -t^{-2}dt$$

Integrate on both sides.

$$y = +t^{-1} + c$$

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$$y = \frac{1}{t} + c_1$$

$$y = \frac{1}{\sqrt{p^2+1}} + c$$

$$y - c = \frac{1}{\sqrt{p^2+1}}$$

squaring on both sides

$$(y-c)^2 = \frac{1}{p^2+1} \rightarrow (1).$$

From the equation

$$x - 1 = \frac{-p}{\sqrt{p^2+1}}$$

squaring on both sides

$$(x-1)^2 = \frac{p^2}{(p^2+1)}$$

$$(x-1)^2 = \frac{p^2+1-1}{p^2+1} \leftarrow \text{split}$$

$$(x-1)^2 = 1 - \frac{1}{p^2+1} \rightarrow (2).$$

sub equ (1) in (2).

$$(x-1)^2 = 1 - (y-c)^2$$

Hence the complete solution

$$(x-1)^2 + (y-c)^2 = 1.$$

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2) Solve $x = y + a\log p$

Soln:

Given equation $x = y + a\log p$

Differentiate with respect to y

$$\frac{dx}{dy} = 1 + a\frac{1}{p}\frac{dp}{dy}$$

$$\frac{1}{p} = 1 + \frac{a}{p}\frac{dp}{dy}$$

$$\frac{1}{p} - 1 = \frac{a}{p}\frac{dp}{dy}$$

$$\frac{1-p}{p} = \frac{a}{p}\frac{dp}{dy}$$

$$1-p = a\frac{dp}{dy}$$

$$dy = \frac{a\,dp}{1-p}$$

Integrate on both sides.

$$\int dy = a\int \frac{dp}{1-p}$$

$$y = a\log(1-p) + c$$

$$x = y + a\log p$$

$$x = c - a\log(1-p) + a\log p$$

$$= c + a\log p - a\log(1-p)$$

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Hence the complete solution

$$x = c + a \log\left[\frac{p}{1-p}\right]$$

(or)

$$x = c - a \log\left[\frac{p}{1-p}\right]_{//}$$

5. Solve $x = p^2 + y$.

Soln:

Given equation

$$x = p^2 + y \rightarrow (1)$$

Differentiate with respect to y

$$\frac{dx}{dy} = 2p\frac{dp}{dy} + 1$$

$$1/p = 2p\, dp/dy + 1$$

$$1/p - 1 = 2p\, dp/dy$$

$$\frac{1-p}{p} = 2p\, dp/dy$$

$$dy = \frac{2p^2}{1-p}\, dp$$

$$\int dy = 2\int \frac{p^2}{1-p}\, dp$$

$$\int dy = 2\int \frac{p^2+1-1}{1-p}\, dp$$

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$$\int dy = 2\left[\int \frac{p^2-1}{1-p}\,dp + \int \frac{1}{1-p}\,dp\right]$$

$$= 2\left[\int \frac{(p+1)(p-1)}{1-p}\,dp + \int \frac{dp}{1-p}\right]$$

$$= 2\left[-\int (p+1)\,dp - \int \frac{dp}{p-1}\right]$$

$$= -2\left[\int (p+1)\,dp + \int \frac{dp}{p-1}\right]$$

$$\int dy = -2\left[\frac{p^2}{2} + p + \log(p-1) + c\right].$$

$$y = c - 2\left[\frac{p^2}{2} + p + \log(p-1)\right] \rightarrow (2)$$

Here eliminate of p is very difficult

from equation (1) & (2)

$$x = p^2 + y$$

$$y = c - 2\left[\frac{p^2}{2} + p + \log(p-1)\right].$$

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4) solve $p^3 - 4xyp + 8y^2 = 0$

Soln:

Given equation

$$p^3 - 4xyp + 8y^2 = 0 \rightarrow (1)$$

$$-4xyp = -p^3 - 8y^2$$

$$4xyp = p^3 + 8y^2$$

$$x = \frac{p^3}{4yp} + \frac{8y^2}{4yp}$$

$$x = \frac{p^2}{4y} + \frac{2y}{p}$$

Differentiate with respect to y

$$\frac{dx}{dy} = \left[\frac{4y(2p\, dp/dy) - p^2(4)}{(4y)^2}\right] + \left[\frac{p(2) - 2y\frac{dp}{dy}}{p^2}\right]$$

$$1/p = \frac{4y\left[2p\, dp/dy\right]}{16y^2} - \frac{4p^2}{16y^2} + \frac{2p}{p^2} - \frac{2y\frac{dp}{dy}}{p^2}$$

$$1/p = \frac{p\frac{dp}{dy}}{2y} - \frac{p^2}{4y^2} + 2/p - \frac{2y\frac{dp}{dy}}{p^2}$$

$$1/p = \frac{p^2}{4y^2} - 2/p = \frac{p\frac{dp}{dy}}{2y} - \frac{2y\frac{dp}{dy}}{p^2}$$

$$\frac{p^2}{4y^2} - 1/p = \frac{dp}{dy}\left[\frac{p}{2y} - \frac{2y}{p^2}\right]$$

$$\frac{p^3 - 4y^2}{4y^2p} = \frac{dp}{dy}\left[\frac{p^3 - 4y^2}{2yp^2}\right]$$

$$\frac{p^3-4y^2}{4y^2p} - \left(\frac{p^3-4y^2}{2yp^2}\right)\frac{dp}{dy} = 0$$

$$(p^3-4y^2)\left[\frac{1}{4y^2p} - \frac{1}{2yp^2}\frac{dp}{dy}\right] = 0$$

case (i)

$$p^3 - 4y^2 = 0$$

$$p^3 = 4y^2$$

$$p = (4y^2)^{1/3} \longrightarrow (2).$$

equation (1) $\Rightarrow$

$$4y^2 - 4xy(4y^2)^{1/3} + 8y^2 = 0$$

$$-4xy(4y^2)^{1/3} + 12y^2 = 0$$

$$-4xy(4y^2)^{1/3} = -12y^2$$

$$x(4y^2)^{1/3} = 3y$$

$$x^3 4y^2 = 2+y^3$$

$$2+y = 4x^3$$

which gives the singular solution.

case (ii)

$$\frac{1}{4y^2p} - \frac{1}{2yp^2}\frac{dp}{dy} = 0.$$

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$$\frac{1}{4yp} = \frac{1}{2yp^2}\frac{dp}{dy}$$

$$\frac{1}{2y} = 1/p \; dp/dy$$

$$1/2 \; \frac{dy}{y} = dp/p$$

$$\int \frac{dy}{y} = 2\int \frac{dp}{p}$$

$$\log y = 2\log p + \log c$$

$$\log y = \log p^2 + \log c$$

$$\log y = \log p^2 c$$

$$y = p^2 c$$

$$p^2 = y/c \longrightarrow ③$$

$$p = y^2/c^2$$

equation ① $\Rightarrow$

$$p^3 - 4xyp + 8y^2 = 0.$$

$$p^3 - 4xyp = -8y^2$$

$$p(p^2 - 4xy) = -8y^2$$

Squaring on both sides.

$$p^2(p^2-4xy)^2 = 64y^4$$

$$p^2(p^4 + 16x^2y^2 - 8p^2xy) = 64y^4$$

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$$(y/c)\left[y^2/c^2 + 16x^2y^2 - 8(y/c)xy\right] = 64y^4.$$

$$(y/c)\left[\frac{y^2 + 16x^2y^2c^2 - 8xy^2c}{c^2}\right] = 64y^4$$

$$\frac{y^3 + 16x^2y^3c^2 - 8xy^3c}{c^3} = 64y^4$$

$$y^3 + 16c^2x^2y^3 - 8xy^3c = 64c^3y^4$$

$$y^3 + 16c^2x^2y^3 - 8xy^3c - 64c^3y^4 = 0$$

$$y^3(1 + 16c^2x^2 - 8xc - 64c^3y) = 0$$

$$[(1-4cx)^2] - 64c^3y = 0.$$

Hence the complete solution

$$[(1-4cx)^2] - 64c^3y = 0.$$

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5) Solve $x^2 = 1 + p^2$

Soln:

Given equation

$$x^2 = 1 + p^2$$

$$x = \pm\sqrt{1+p^2} \longrightarrow (1).\ d_x$$

$$\frac{dx}{dy} = \frac{1}{\cancel{2}}\frac{1}{\sqrt{1+p^2}}\cancel{2}p\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{p}{\sqrt{1+p^2}}\frac{dp}{dy}$$

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$$dy = \frac{p^2}{\sqrt{1+p^2}}\, dp$$

$$\int dy = \int \frac{p^2}{\sqrt{1+p^2}}\, dp.$$

$$\int dy = \int \frac{p^2+1-1}{\sqrt{1+p^2}}\, dp$$

$$\int dy = \int \frac{p^2+1}{\sqrt{1+p^2}}\, dp - \int \frac{dp}{\sqrt{1+p^2}}$$

$$\int dy = \int (p^2+1)(p^2+1)^{-1/2}\, dp - \int \frac{dp}{\sqrt{1+p^2}}$$

$$\int dy = \int (p^2+1)^{1/2}\, dp - \int \frac{dp}{\sqrt{1+p^2}}$$

$$\int dy = \int \sqrt{p^2+1}\, dp - \int \frac{dp}{\sqrt{1+p^2}}$$

$$y = \frac{p}{2}\sqrt{p^2+1} + \frac{1}{2}\sinh^{-1}(p) - \sinh^{-1}(p)$$

$$y = \frac{p}{2}\sqrt{p^2+1} - \frac{1}{2}\sinh^{-1}(p) + c$$

$$y = \frac{1}{2}\left[p\sqrt{p^2+1} - \sinh^{-1}(p)\right] + c \rightarrow (2)$$

Here elimination of p from (1) & (2) is very difficult ∴ the general ~~equa~~ solution $x = \pm\sqrt{1+p^2}$

$$y = \frac{1}{2}\left[p\sqrt{p^2+1} - \sinh^{-1}(p)\right] + c.$$