Exact Equation :

The first order differential equation

$$\frac{dy}{dx} = f(x,y) \longrightarrow ①$$

may also be expresesed in the differential form

$$M(x,y)\,dx + N(x,y)\,dy = 0 \longrightarrow ②$$

The differential form

$M(x,y)\,dx + N(x,y)\,dy$ is said to be extract if there exists a function $F(x,y)$ which that,

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

the total differential of $F(x,y)$ satisfies the relation $dF(x,y) = M(x,y)\,dx + N(x,y)\,dy$

If $M(x,y)\,dx + N(x,y)\,dy$ is an extract differential form, than $M(x,y)\,dx + N(x,y)\,dy = 0$ is called an extract equation.

To solve the exact equation $Mdx + Ndy = 0$

1. Integrate M with respect to x keeping y constant.

2. those terms is N not containg x respect to y.

3. The sum of those two integrals equation to c is the Solution.

1. Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

Soln :

$$\frac{dy}{dx} = -\left[\frac{y\cos x + \sin y + y}{\sin x + x\cos y + x}\right]$$

$$[\sin x + x\cos y + x]\,dy = -[y\cos x + \sin y + y]\,dx$$

$$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$$

$$M = y\cos x + \sin y + y$$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$$

$$N = \sin x + x\cos y + x$$

$$\frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This equation is exact

$$\int M\,dx = \int (y\cos x + \sin y + y)\,dx$$

$$\int M\,dx = y\sin x + \sin y\, x + yx$$

Since the solution $y\sin x + x\sin y + yx = c$.

2. Solve $(1+e^{x/y})\,dx + (1 - x/y)\,e^{x/y}\,dy = 0$

Soln :

$$M = 1 + e^{x/y}$$

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$$\frac{\partial m}{\partial y} = e^{x/y}\left[-x/y^2\right]$$

$$\frac{\partial m}{\partial y} = \frac{-xe^{x/y}}{y^2}$$

$$N = (1 - x/y)\, e^{x/y}$$

$$\frac{\partial N}{\partial x} = (1 - x/y)\, e^{x/y}(1/y) + e^{x/y}(-1/y)$$

$$= \left(\frac{y-x}{y}\right) e^{x/y}(1/y) - \frac{e^{x/y}}{y}$$

$$= \frac{ye^{x/y}}{y^2} - \frac{xe^{x/y}}{y^2} - \frac{e^{x/y}}{y}$$

$$= \frac{-xe^{x/y}}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This equation is extract

$$\int m\,dx = \int (1+e^{x/y})\,dx$$

$$= x + \frac{e^{x/y}}{1/y} = x + e^{x/y}$$

Hence the complete solution is

$$x + ye^{x/y} \pm c$$

3. Solve $\left[\frac{2}{\sqrt{1-x^2}} + y\cos(xy)\right]dx + \left[x\cos xy - y^{-1/3}\right]dy = 0$

Soln :

$$M = \frac{2}{\sqrt{1-x^2}} + y\cos(xy)$$

$$\frac{\partial m}{\partial y} = y\left[-\sin(xy)(x)\right] + \cos xy$$

$$\frac{\partial m}{\partial y} = -xy\sin(xy) + \cos xy$$

$$N = x\cos xy - y^{-1/3}$$

$$\frac{\partial N}{\partial x} = x\left[(-\sin xy)y\right] + \cos xy$$

$$\frac{\partial N}{\partial x} = -xy\sin xy + \cos xy$$

$$\frac{\partial m}{\partial y} = \frac{\partial N}{\partial x}$$

This equation is exact

$$\int m\,dx = \int\left[\frac{2}{\sqrt{1-x^2}} + y\cos(xy)\right]dx$$

$$\int m\,dx = \int\left[\frac{2}{\sqrt{1-x^2}} + y\cos(xy)\right]dx$$

$$= 2\sin^{-1}(x) + y\,\frac{\sin(xy)}{y}$$

$$\int m\,dx = 2\sin^{-1}(x) + \sin xy$$

$$\int N\,dy = \int -y^{-1/3}\,dy = -y\,\frac{-1/3+1}{-1/3+1}$$

$$= \frac{-y^{2/3}}{2/3}$$

$$\int N\,dy = -3/2\, y^{2/3}$$

$$\int M\,dx + \int N\,dy = 0$$

Hence the complete solution is

$$2\sin^{-1}(x) + \sin xy - 3/2\, y^{2/3} = c$$

4. Solve $(D^2+2D+2)y = \sinh x$

soln :

$$(D^2+2D+2)y = \frac{e^{x} - e^{-x}}{2}$$

$$(D^2+2D+2)y = \frac{e^{x}}{2} - \frac{e^{-x}}{2}$$

The auxillary equation is

$$(m^2+2m+2) = 0$$

$$= \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4-4(2)}}{2}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

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$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$$m = -1 \pm i$$

$$y_c = e^{\alpha x} [A \cos \beta x + B \sin \beta x]$$

$$y_c = e^{-x} [A \cos x + B \sin x]$$

$$y_{p_1} = \frac{1}{2} \frac{e^x}{D^2 + 2D + 2}$$

$$= \frac{1}{2} \frac{e^x}{1+2+2}$$

$$y_{p_1} = \frac{e^x}{10}$$

$$y_{p_2} = -\frac{1}{2} \frac{e^{-x}}{D^2+2D+2}$$

$$= -\frac{1}{2} \left[\frac{e^{-x}}{1-2+2} \right]$$

$$y_{p_2} = \frac{-e^{-x}}{2}$$

$$y = y_c + y_{p_1} + y_{p_2}$$

$$= e^{-x} [A \cos x + B \sin x] + \frac{e^x}{10} - \frac{e^{-x}}{2}$$

5. Solve $(D^3 - 3D^2 + 4D - 2)y = e^x$

Soln :

The given equation

$$D^3 - 3D^2 - 4D - 2$$

The Auxillary equation is

$$m^3 - 3m^2 - 4m - 2 = 0$$

$$\begin{array}{c|cccc} 1 & 1 & -3 & 4 & -2 \\ & 0 & 1 & -2 & 2 \\ \hline & 1 & -2 & 2 & \underline{0} \end{array}$$

$(m-1)$ is a factor

$$m^2 - 2m + 2 = 0$$

$$= \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2} = \frac{2(1 \pm i)}{2}$$

$$m = 1 \pm i, \quad m = 1$$

$$y_c = Ae^x + e^x [B \cos x + c \sin x]$$

$$y_p = \frac{e^x}{D^3 - 3D^2 + 4D - 2}$$

$$= \frac{e^x}{1 - 3 + 4 - 2}$$

$$= \frac{xe^x}{3D^2 - 6D + 4}$$

$$= \frac{xe^x}{3 - 6 + 4}$$

$$y_p = xe^x$$

$$y = y_c + y_p$$

$$y = Ae^x + e^x \left[B\cos x + c\sin x \right] + xe^x$$