Unit - II

Linear Differential Equations with constant co-efficiences

The general linear differential equation of order his of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_n y = f(x)$$

where $a_1, a_2 \ldots\ldots a_n$ are real constant.

This equation can also be written in operator as

$$(D^n y + a_1 D^{n-1} y + \ldots + a_n y) y = f(x) \longrightarrow (1)$$

The solution of (1) consists of two parts viz

i) complementary function ii) particular Integral

i.e ....$y = y_c + y_p$ where y_c is a complementary function y_p is particular integral.

To complete complementary function:

To find complementary function we have to form the auxillary equation with is abtained by putting $D = m$ and $f(x) = 0$

Therefore the auxillary equation of

i) $D\cdot(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n) = 0 \longrightarrow (2)$

Equation (2) is an ordinary algebrail equation in m of degree 'n' by Solving this equation we get n roots e or values,

for m_1, Say $m_1, m_2 \dots m_n$

Case (i)

If all the roots $m_1, m_2 \dots m_n$ are real and different then the complementary function

$$y_c = Ae^{m_1 x} + Be^{m_2 x} + Ce^{m_3 x} + \dots\dots$$

case (ii)

If any two roots are equal say $m_1 = m_2 = m$ then the complementary function is given by

$$y = (Ax + B)e^{mx}$$

Case (iii):

If any three roots are equal say $m_1 = m_2 = m_3 = m$ then the complementary function is $y = (Ax^2 + Bx + C)e^{mx}$

case (iv):

If the roots are imaginary say

$m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the

complementry function is $y_c = e^{\alpha x}$

$$(A \cos \beta x + B \sin \beta x)$$

To find particular integral (P.I)

Type I :

If $f(x) = e^{ax}$, then the particular

integral is given by

$$P.I = \frac{1}{\phi(D)} e^{ax}$$

$$= \frac{1}{\phi(a)} e^{ax} \quad \text{provided } \phi(a) \neq 0$$

If $\phi(a) = 0$ then $P.I = \frac{1}{\phi(D)} e^{ax}$

$$= x \frac{1}{\phi'(D)} e^{ax}$$

where $\phi'(D)$ means derivatio of $\phi(D)$

withe respect (D)

$$= x \frac{1}{\phi'(a)} e^{ax} \quad \text{provided } \phi'(a) \text{ to}$$

If $\phi'(a) = 0$, then $P.I = x^2 \frac{1}{\quad} e^{ax}$

$$= x^2 \frac{1}{\phi''(a)} e^{ax} \quad \text{pracuided } \phi''(a) \neq 0$$

If $\phi''(a) = 0$ then $p \cdot I = x^3 \frac{1}{\phi'''(D)} e^{ax}$

$$= x^3 \frac{1}{\phi'''(a)} e^{ax} \text{ pracuided}$$

$\phi'''(a) \neq 0$ and so one

Finding of particular Integral using this method is very clear from the folloing examply.

Type II

If $f(x) = \sin ax$ (or), $\cos ax$, Then particular integral is given by

$$P.I = \frac{1}{\phi(D)} \sin ax \ (or) \ \cos ax$$

In $\phi(D)$ replacing D^2 by a^2, pracuidod $\phi(D) \neq 0$

If $\phi(D) = 0$, when we replacing D^2 by a^2 then

$$P.I = x \cdot \frac{1}{\phi'(D)} \sin ax \ (or) \ \cos ax$$

Again replace D^2 by a^2 in $\phi'(D)$ pracuided ϕ'

$I(D) \neq 0$, then

$$P.I = x^2 \frac{1}{\phi''(D)} \sin ax \text{ (or) } \cos ax$$

and this process may be respected if $\phi''(D) = 0$ and $\to 0$ on

Type III :

If $f(x) = a_0 x^n + a_1 x^{n-1} + \dots a_n$

where $a_0 x^n + a_1 x^{n-1} + \dots + a_n$ is a pure algebric function then

$$P.I = \frac{1}{\phi(D)} a_0 x^n + a_1 x^{n-1} + \dots + a_n$$

Expand $[\phi(D)]^{-1}$ by using Binamial the then operate arem is ascending powers of D and an $a_0 x^n + a_1 x^{n-1} + \dots + a_n$

Type IV :

If $f(x) = e^{ax} x$ where x is same function of x, then $P.I = \frac{1}{\phi(D)} e^{ax} \cdot x$

$$= \frac{e^{ax}}{\phi(D+a)} x$$

Type V

To find particular integral when $f(x) = x^n \sin ax$ (or) $x^n \cos ax$.

Soln :

$$P.I = \frac{1}{F(D)} x^n \sin ax \text{ (or) } x^n \cos ax$$

Now

$$\frac{1}{F(D)} x^n (\cos ax + i \sin ax)$$

$$= \frac{1}{F(D)} x^n . e^{iax}$$

$$= e^{iax} \frac{1}{F(D+ia)} x^n$$

$\therefore \frac{1}{F(D)} x^n \sin ax =$ imaginary part of $e^{iax} \frac{1}{F(D+ia)} . x^n$

$\frac{1}{F(D)} x^n \cos ax =$ Real part of

$$e^{iax} \frac{1}{F(D+ia)} x^n .$$