Differential Equations and Laplace Transforms

Differential Equations of first order and Higher Degree

A differential equation of first order in which the differential co-efficient dy/dx occur is of the form.

$$f(x, y, dy/dx) = 0$$

If we denote $p = dy/dx$, then equ ① can be written as $f(x, y, p) = 0$

This differential equ is also called differential equation of first order and first degree ($\because$ the degree of p is one)

Differential equations of 1st order (i.e there exists only dy/dx & not $\frac{d^2y}{dx^2}$ etc...) and of higher degree (i.e exist $(dy/dx)^2$, $(dy/dx)^3$. etc...) The general form of such an equation is given by $p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_{n-1} p + P_n = 0$ where $p = dy/dx$ & $P_1, P_2 \dots P_n$ are functions of x and y

To solve this equation, we have the following three methods.

* Equation solvable for p
* Equation solvable for x

* Equation solvable for y.

EQUATION SOLVABLE FOR p :-

Suppose the given differential equation (3) can be factorised as linear factors of the form.

$$(p-f_1)\ (p-f_2)\ \dots\ (p-f_n)=0.$$

where $p = dy/dx$ and $f_1, f_2, \dots, f_n$ are functions of x and y, then clearly

$$p-f_1=0 \quad p-f_2=0\ ; \dots\ ; p-f_n=0$$

(i.e) $p=f_1$; $p=f_2$; ... $p=f_n$

(i.e) $dy/dx = f_1(x,y)$: $dy/dx = f_2(x,y)$, ... $\frac{dy}{dx} = f_n(x,y)$

clearly these differential equation are of 1st degree and of 1st order which can be solved mostly by using the method of seperation of variables.

Let the solutions be of the form.

$$F_1(x,y,c_1)=0\ ,\ F_2(x,y,c_2)=0 \dots F_n(x,y,c_n)=0.$$

where $c_1, c_2 \dots c_n$ are constants. since 1st order differential equation admits only one arbitary constant we can take

$c_1 = c_2 = \dots c_n$

Hence the general soln of (1) is

$$F_1(x,y,c) \cdot F_2(x,y,c) \dots F_n(x,y,c)=0.$$

Solvable For $P[dy/dx]$

1) Solve $P^2 - 3P + 2 = 0$.

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Solution:-

Given equation $P^2 - 3P + 2 = 0$

$$(P-1)(P-2) = 0$$

$$(P-1) = 0 \text{ (or) } (P-2) = 0$$

$$P = 1 \text{ (or) } P = 2$$

$$\frac{dy}{dx} = 1 \text{ (or) } \frac{dy}{dx} = 2$$

$$dy = dx \text{ (or) } dy = 2dx$$

Integrating on both sides

$$\int dy = \int dx \text{ (or) } \int dy = 2\int dx$$

$$y = x + c_1 \text{ (or) } y = 2x + c_2$$

Hence the complete solution is

$$(-x + y - c_1)(y - 2x - c_2) = 0$$

2) Solve $p^3 - 7p - 6 = 0$.

Solution:-

Given equation

$$p^3 - 7p - 6 = 0$$

$$\begin{array}{r|rrrr} & 1 & 0 & -7 & -6 \\ -1 & 0 & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$(p+1)$ is a factor

$p^2 - p - 6 = 0$

$(p-3)(p+2) = 0$.

$(p+1)(p-3)(p+2) = 0$

$p = -1$ (or) $p = 3$ (or) $p = -2$

$\frac{dy}{dx} = -1$ (or) $\frac{dy}{dx} = 3$ (or) $\frac{dy}{dx} = -2$

$dy = -dx$ (or) $dy = 3dx$ (or) $dy = -2dx$

Integrating on both sides

$\int dy = -\int dx$ (or) $\int dy = 3\int dx$ (or) $\int dy = -2\int dx$

$y = -x + c_1$ (or) $y = 3x + c_2$ (or) $y = -2x + c_3$

~~Hence the~~

$y + x - c_1 = 0$ (or) $y - 3x - c_2 = 0$ (or) $y + 2x - c_3 = 0$

Hence the complete solution is

$$(y + x - c_1)(y - 3x - c_2)(y + 2x - c_3) = 0.$$

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3) Solve $2p^2 - (x+2y^2)p + xy^2 = 0$.

Solution:-

Given equation $2p^2 - (x+2y^2)p + xy^2 = 0$.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = p$$

$$\frac{(x+2y^2) \pm \sqrt{(x+2y^2)^2 - 4(2)(xy^2)}}{2(2)} = p$$

$$\frac{(x+2y^2) \pm \sqrt{x^2 + 4xy^2 + 4y^4 - 8xy^2}}{4} = p$$

$$\frac{(x+2y^2) \pm \sqrt{x^2 - 4xy^2 + 4y^4}}{4} = p$$

$$\frac{(x+2y^2) \pm \sqrt{(x-2y^2)^{\cancel{2}}}}{4} = p.$$

$$\frac{(x+2y^2) \pm (x-2y^2)}{4} = p$$

$$\frac{(x+2y^2) + (x-2y^2)}{4} = p, \quad \frac{(x+2y^2) - (x-2y^2)}{4} = p$$

$$\frac{x+2y^2+x-2y^2}{4}=P, \quad \frac{x+2y^2-x+2y^2}{4}=P$$

$$\frac{2x}{4}=P, \quad 4y^2/_4=P$$

$$x/_2=P, \quad y^2=P$$

$$\frac{dy}{dx}=x/_2 \; ; \quad \frac{dy}{dx}=y^2$$

$$dy=1/_2\, x\,dx \; ; \quad \frac{dy}{y^2}=dx$$

$$dy=1/_2\, x\,dx, \quad y^{-2}dy=dx$$

Integrating on both sides.

$$\int dy = 1/_2\int x\,dx, \quad \int y^{-2}dy=\int dx$$

$$y=1/_2\, x^2/_2+c, \quad \frac{y^{-1}}{-1}=x+c_2$$

$$y=\frac{x^2}{4}+c_1, \quad \frac{-1}{y}=x+c_2$$

$$4y=x^2+4c_1 \; ; \quad -1/_y-x-c_2=0.$$

$$\frac{-1-xy-yc_2}{y}=0$$

$$-1-xy-yc_2=0$$

$$-(1+xy+yc_2)=0 \Rightarrow (1+xy+yc_2)=0.$$

Hence the complete solution is

$$(1+xy+yc_2)(4y-x^2-4c_1)=0.$$

4. Solve $y(1-p^2) = 2px$

Soln:-

Given equation $y(1-p^2) = 2px$

$$y - yp^2 - 2px = 0$$

$$-y + yp^2 + 2px = 0$$

$$yp^2 + 2px - y = 0.$$

$$p = \frac{-2x \pm \sqrt{4x^2 + 4(y)y}}{2y}$$

$$= \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y}$$

$$= \frac{2\left[-x \pm \sqrt{x^2+y^2}\right]}{2y}$$

$$p = \frac{-x \pm \sqrt{x^2+y^2}}{y}$$

$$\frac{dy}{dx} = \frac{-x \pm \sqrt{x^2+y^2}}{y}$$

$$y\,dy = \left[-x \pm \sqrt{x^2+y^2}\right] dx$$

$$y\,dy = -x\,dx \pm \sqrt{x^2+y^2}\,dx$$

$$y\,dy + x\,dx = \pm\sqrt{x^2+y^2}\,dx \rightarrow (1).$$

put $x^2+y^2 = t \rightarrow (2)$

$$2x + 2y\,dy/dx = dt/dx$$

$$\frac{2x\,dx + 2y\,dy}{dx} = dt/dx$$

$$2x\,dx + 2y\,dy = dt$$

$$2[x\,dx + y\,dy] = dt$$

$$x\,dx + y\,dy = \frac{dt}{2} \rightarrow (3).$$

Sub equ (2) & (3) in equ (1) we get

$$\frac{dt}{2} = \pm\sqrt{t}\,dx$$

$$\frac{dt}{\sqrt{t}} = \pm 2dx.$$

$$t^{-1/2}\,dt = 2dx$$

Integrate on both sides

$$\int t^{-1/2}\,dt = 2\int dx$$

$$\frac{t^{-1/2+1}}{-1/2+1} = 2x+c$$

$$\frac{t^{1/2}}{1/2} = 2x+c$$

$$2t^{1/2} = 2x + c$$

$$t^{1/2} = x + c$$

squaring on both sides.

$$t = (x+c)^2$$

$$(x^2 + y^2 = x^2 + c^2 + 2cx$$

$$\cancel{x^2} + y^2 - \cancel{x^2} - c^2 - 2cx = 0.$$

Hence the complete solution

$$(y^2 - c^2 - 2cx) = 0.$$

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5. Solve $p^2 - 2py = 3y^2$

Soln:-

Given equation $p^2 - 2py = 3y^2$

$$p^2 - 2py - 3y^2 = 0$$

$$p = \frac{+2y \pm \sqrt{4y^2 + 4(3y^2)}}{2}$$

$$= \frac{2y \pm \sqrt{4y^2 + 12y^2}}{2}$$

$$= \frac{2y \pm \sqrt{16y^2}}{2}$$

$$= \frac{2y \pm 4y}{2}$$

$$P = \frac{2y+4y}{2}, \quad P = \frac{2y-4y}{2}$$

$$P = 6y/2, \quad P = -2y/2$$

$$P = 3y, \quad P = -y.$$

$$\frac{dy}{dx} = 3y, \quad \frac{dy}{dx} = -y$$

$$\frac{dy}{y} = 3dx, \quad \frac{dy}{y} = -dx$$

Integrate on both sides

$$\int \frac{dy}{y} = 3\int dx, \quad \int \frac{dy}{y} = -\int dx$$

$$\log y = 3x + \log c_1; \quad \log y = -x + \log c_2$$

$$\log y - \log c_1 = 3x, \quad \log y - \log c_2 = -x$$

$$\log\left[y/c_1\right] = 3x, \quad \log\left[y/c_2\right] = -x$$

$$y/c_1 = e^{3x}, \quad y/c_2 = e^{-x}$$

$$y = c_1 e^{3x}, \quad y = c_2 e^{-x}$$

$(y - c_1 e^{3x}) = 0, \quad (y - c_2 e^{-x}) = 0$

The complete solution is $(y - c_1 e^{3x})(y - c_2 e^{-x})$

6) Solve $4p^2 - 8p + 3 = 0$

Solution:-

Given equation is $4p^2 - 8p + 3 = 0$

$$4p^2 - 6p - 2p + 3 = 0$$

$$4p^2 - 2p - 6p + 3 = 0$$

$$2p(2p-1) - 3(2p-1) = 0$$

$$(2p-1)(2p-3) = 0.$$

$(2p-1) = 0$	$2p - 3 = 0$
$2p = 1$	$2p = 3$
$p = 1/2$	$p = 3/2$
$\dfrac{dy}{dx} = 1/2$	$\dfrac{dy}{dx} = 3/2$
$2dy = dx$	$2dy = 3dx$

Integrate on both sides

$2\int dy = \int dx$	$2\int dy = 3\int dx$
$2y = x + c_1$	$2y = 3x + c_2$
$2y - x - c_1 = 0$	$2y - 3x - c_2 = 0$

$\therefore$ The complete solution is

$$(2x - x - c_1)(2y - 3x - c_2) = 0$$

7) Solve $x^2p^2 + 3xyp + 2y^2 = 0$

Solution:-

Given equation $x^2p^2 + 3xyp + 2y^2 = 0$

$$a = x^2,\ b = 3xy \quad c = 2y^2$$

$$p = \frac{-3xy \pm \sqrt{9x^2y^2 - 4(x^2)(2y^2)}}{2x^2}$$

$$= \frac{-3xy \pm \sqrt{9x^2y^2 - 8x^2y^2}}{2x^2}$$

$$= \frac{-3xy \pm \sqrt{x^2y^2}}{2x^2}$$

$$= \frac{-3xy \pm \sqrt{(xy)^2}}{2x^2}$$

$$p = \frac{-3xy \pm xy}{2x^2}$$

$$p = \frac{-3xy + xy}{2x^2}$$

$$= \frac{-2xy}{2x^2}$$

$$\frac{dy}{dx} = -y/x$$

$$\frac{dy}{y} = -dx/x$$

;

$$p = \frac{-3xy - xy}{2x^2}$$

$$= \frac{-4xy}{2x^2}$$

$$\frac{dy}{dx} = -\frac{2y}{x}$$

$$\frac{dy}{y} = -2\left(\frac{dx}{x}\right)$$

Integrate on both sides

$$\int \frac{dy}{y} = -\int \frac{dx}{x} \quad ; \quad \int \frac{dy}{y} = -2\int \frac{dx}{x}$$

$\log y = -\log x + \log c_1$; $\log y = -2\log x + \log c_2$

$\log y + \log x = \log c_1$ | $\log y + 2\log x = \log c_2$

$\log(xy) = \log c_1$ | $\log y + \log x^2 = \log c_2$

$\log(x^2 y) = \log c_2$

Taking exponential powers on both sides.,

$xy = c_1$ | $x^2 y = c_2$

$xy - c_1 = 0$ | $x^2 y - c_2 = 0.$

The complete solution is $(xy - c_1)(x^2 y - c_2) = 0$

8) Solve $p^2 - p(e^x + e^{-x}) + 1 = 0.$

Solution:-

let the given equation is $p^2 - p(e^x + e^{-x}) + 1 = 0$

$$p = \frac{(e^x + e^{-x}) \pm \sqrt{(e^x + e^{-x})^2 - 4}}{2}$$

$$= \frac{(e^x + e^{-x}) \pm \sqrt{e^{2x} + e^{-2x} + 2 - 4}}{2}$$

$$= \frac{(e^x + e^{-x}) \pm \sqrt{e^{2x} + e^{-2x} - 2}}{2}$$

$$= \frac{(e^x + e^{-x}) \pm \sqrt{(e^x - e^{-x})^2}}{2}$$

$$p = \frac{(e^x + e^{-x}) \pm (e^x - e^{-x})}{2}$$

$$= \frac{(e^x+e^{-x}) \pm \sqrt{(e^x-e^{-x})^2}}{2}$$

$$p = \frac{(e^x+e^{-x}) \pm (e^x-e^{-x})}{2}$$

$$p = \frac{e^x+e^{-x}+e^x-e^{-x}}{2} \quad ; \quad p = \frac{e^x+e^{-x}-e^x+e^{-x}}{2}$$

$$p = \frac{2e^x}{2} \qquad\qquad = \frac{2e^{-x}}{2}$$

$$p = e^x$$

$$\frac{dy}{dx} = e^x \qquad\qquad \frac{dy}{dx} = e^{-x}$$

$$dy = e^x dx \qquad\qquad dy = e^{-x} dx$$

Integrating on both sides,

$$\int dy = \int e^x dx \qquad\qquad \int dy = \int e^{-x} dx$$

$$y = e^x + c_1 \qquad\qquad y = -e^{-x} + c_2$$

$$y - e^x - c_1 = 0 \qquad\qquad y + e^{-x} - c_2 = 0$$

The complete solution is $(y-e^x-c_1)(y+e^{-x}-c_2)=0$

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9) Solve $y\left[\frac{dy}{dx}\right]^2 + (x-y)\frac{dy}{dx} - x = 0.$

Soln: The given equation is

$$y(dy/dx)^2 + (x-y)\,dy/dx - x = 0$$

$$\frac{dy}{dx} = \frac{-(x-y) \pm \sqrt{(x-y)^2 - 4xy}}{2y}$$

$$= \frac{-(x-y) \pm \sqrt{x^2+y^2-2xy+4xy}}{2y}$$

$$= \frac{-(x-y) \pm \sqrt{x^2+y^2+2xy}}{2y}$$

$$= \frac{-(x-y) \pm \sqrt{(x+y)^2}}{2y}$$

$$= \frac{-(x-y) \pm (x+y)}{2y}$$

$$\frac{dy}{dx} = \frac{-(x-y)+(x+y)}{2y}$$

$$= \frac{-x+y+x+y}{2y}$$

$$= \frac{2y}{2y}$$

$$dy = dx$$

$$\frac{dy}{dx} = \frac{-(x-y)-(x+y)}{2y}$$

$$= \frac{-x+y-x-y}{2y}$$

$$= -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -x/y$$

$$y\,dy = -x\,dx$$

Integrating on both sides

$$\int dy = \int dx$$

$$y = x + c_1$$

$$y - x - c_1 = 0$$

$$\int y\,dy = -\int x\,dx$$

$$y^2/2 = -x^2/2 + c_2/2$$

$$y^2 = -x^2 + c_2$$

$\therefore$ The complete solution is $(y-x-c_1)(y^2+x^2-c_2)=0$

10) Solve $p^2 + 2yp\cot x = y^2$

Soln:-

The given equation is $p^2 + 2yp\cot x - y^2 = 0$.

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2(\cot^2 x + 1)}}{2}$$

$$= \frac{-2y\cot x \pm 2y\operatorname{cosec} x}{2}$$

$$= -y\cot x \pm y\operatorname{cosec} x$$

$$p = -y\cot x \pm y\operatorname{cosec} x \rightarrow (1)$$

$$\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$$

$$\frac{dy}{y} = [-\cot x + \operatorname{cosec} x]\,dx$$

$$= -\cot x\,dx + \operatorname{cosec} x\,dx$$

Integrating on both sides

$$\int \frac{dy}{y} = -\int \cot x\,dx + \int \operatorname{cosec} x\,dx$$

$$\log y = -\log\sin x - \log(\operatorname{cosec} x + \cot x) + \log c_1$$

$$\log y + \log\sin x + \log(\operatorname{cosec} x + \cot x) = \log c_1$$

$$\log[\sin x(y)(\operatorname{cosec} x) + \cot x)] = \log c_1$$

$$y \sin x \,(\operatorname{cosec} x + \cot x) = c_1$$

$$y \sin x \left(\frac{\cos x}{\sin x} + 1/\sin x\right) = c_1$$

$$y \sin x \left(\frac{1+\cos x}{\sin x}\right) = c_1$$

$$y(1+\cos x) = c_1$$

$$y(1+\cos x) - c_1 = 0.$$

② $\rightarrow$ $p = y(-\cot x - \operatorname{cosec} x)$

$$\frac{dy}{dx} = y(-\cot x - \operatorname{cosec} x)$$

$$\frac{dy}{y} = dx\,(-\cot x - \operatorname{cosec} x)$$

Taking integration

$$\int \frac{dy}{y} = -\int \cot x\,dx - \int \operatorname{cosec} x\,dx$$

$$\log y = -\log \sin x + \log(\operatorname{cosec} x + \cot x) + \log c_2$$

$$\log y + \log \sin x - \log(\operatorname{cosec} x + \cot x) = \log c_2$$

$$\log\left[\frac{y \sin x}{\operatorname{cosec} x + \cot x}\right] = \log c_2$$

$$\frac{y \sin x}{\left[\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right]} = c_2$$

$$\frac{y \sin^2 x}{1+\cos x} = c_2$$

$$\frac{y(1-\cos^2 x)}{1+\cos x} = c_2$$

$$\frac{y\cancel{(1+\cos x)}(1-\cos x)}{\cancel{(1+\cos x)}} = c_2$$

$$y(1-\cos x) - c_2 = 0.$$

∴ The complete solution is $[y(1+\cos x)-c_1]$

$$[y(1-\cos x) - c_2] = 0.$$

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