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UNIT : 1

1. Theory of simplex methods

i) Simplex Method

ii) BIG-M Method

iii) Two Phase Simplex Method.

2. Duality

i) Primal Dual conversion

ii) Duality & solve.

3. Dual Simplex Method.

Examples:

1 To solve the following LPP.

Maximize $Z = 3x_1 + 2x_2 + 5x_3$

s.t.c.

$x_1 + 4x_2 \le 420 \longrightarrow (1)$

$3x_1 + 2x_3 \le 460 \longrightarrow (2)$

$x_1 + 2x_2 + x_3 \le 430 \longrightarrow (3)$

$x_1, x_2, x_3 \ge 0.$

solution:

check the initial Conditions.

i) The objective function is of Maximization type.

ii) The constraints are $\le$ type

iii) The constants are non-negative.

(2)

Introduce the slack variables s_1, s_2 & $s_3 \geq 0$ in the (1), (2) & (3) equations respectively.

$$x_1 + 4x_2 + s_1 = 420$$
$$3x_1 + 2x_3 + s_2 = 460$$
$$x_1 + 2x_2 + x_3 + s_3 = 430.$$

The New objective function becomes

$$\text{Max } Z^* = 3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$$

<u>Initial Iteration Table:</u>

	C_B	0	3	2	5	0	0	0	$X_B/E.C$
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	Ratio
0	s_1	420	1	4	0	1	0	0	—
0 L.R	s_2	460	3	0	2^*	0	1	0	$\frac{460}{2} = 230$ (Mini)
0	s_3	430	1	2	1	0	0	1	$\frac{430}{1} = 430$
	$Z_j - C_j$	0	−3	−2	−5	0	0	0	—

Most (−ve) E.C

s_2 leave the basis, x_3 enter the basis. 2^* is a leading element.

Now Leading element convert into Unity & all other elements in the Entering column is Zero.

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First iteration Table

	C_j	0	3	2	5	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	Ratio
0	s_1	420	1	4	0	1	0	0	$\frac{420}{4}=105$
(Key row) 5	x_3	230	$3/2$	0	1	0	$1/2$	0	—
0	s_3	200	$-1/2$	2^*	0	0	$-1/2$	1	$\frac{200}{2}=100$ ✓
	z_j-c_j	1150	$9/2$	-2	0	0	$5/2$	0	—

Rough work:

$-1\times KR \Rightarrow$ -1×230 $\quad -1\times 3/2$ $\quad -1\times 0$ $\quad -1\times 1$ $\quad -1\times 0$ $\quad -1\times 1/2$ $\quad -1\times 0$

	-230	$-3/2$	0	-1	0	$-1/2$	0
Add $x_3 \Rightarrow$	430	1	2	1	0	0	1
	200	$-1/2$	2	0	0	$-1/2$	1

$5\times KR \Rightarrow$ 5×230 $\quad 5\times 3/2$ $\quad 5\times 0$ $\quad 5\times 1$ $\quad 5\times 0$ $\quad 5\times 1/2$ $\quad 5\times 0$

	1150	$15/2$	0	5	0	$5/2$	0
Add z_j-c_j	0	-3	-2	-5	0	0	0
	1150	$9/2$	-2	0	0	$5/2$	0

s_3 leave the basis, x_2 enter the basis, 2^* is a leading element.

(4) **Second Iteration Table**

	C_B	0	3	2	5	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
0	S_1	20	2	0	0	1	1	-2
5	x_3	230	$3/2$	0	1	0	$1/2$	0
(Key row) 2	x_2	100	$-1/4$	1	0	0	$-1/4$	$1/2$
	$z_j c_j$	1350	4	0	0	0	2	1

Roughwork

$-4 \times KR \Rightarrow$	-4×100	$-4\times -1/4$	-4×1	-4×0	-4×0	$-4\times -1/4$	$-4\times 1/2$
	-400	1	-4	0	0	1	-2
Add $S_1 \Rightarrow$	420	1	4	0	1	0	0
	20	2	0	0	1	1	-2

$2 \times KR \Rightarrow$	2×100	$2\times -1/4$	2×1	2×0	2×0	$2\times -1/4$	$2\times 1/2$
	200	$-1/2$	2	0	0	$-1/2$	1
Add $z_j - c_j$	1150	$9/2$	-2	0	0	$5/2$	0
	1350	4	0	0	0	2	1

Conclusion. All $z_j - c_j \geq 0$.

The optimum solution will be obtained.

$$x_1 = 0 \; ; \; x_2 = 100 \; ; \; x_3 = 230$$

$$\text{Max } Z^* = 3(0) + 2(100) + 5(230)$$
$$= 0 + 200 + 1150$$
$$= 1350.$$

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BIG-M Method / Penality Method

charnes method of Penality

Solve the following LPP.

Maximize $Z = 3x_1 + 2x_2$

s.t.c.

$2x_1 + x_2 \leq 2 \longrightarrow (1)$

$3x_1 + 4x_2 \geq 12 \longrightarrow (2)$

$x_1, x_2 \geq 0.$

solution:

Introduce the slack Variable $S_1 > 0$ (add) and the surplus variable $S_2 > 0$ (subtract) in the (1) & (2) equations respectively.

$2x_1 + x_2 + s_1 = 2$

$3x_1 + 4x_2 - s_2 = 12$

The surplus Variable carrying negative coefficients, so we add one more artifical Variable $A_1 > 0$ in the (2) eqns.

$3x_1 + 4x_2 - S_2 + A_1 = 12$

We assign a cost $-M$ to the artifical Variable to the New Objective function.

(6)

$$\text{Max } Z^* = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA_1$$

s.t.c.

$$2x_1 + x_2 + S_1 = 2$$

$$3x_1 + 4x_2 - S_2 + A_1 = 12$$

$$x_1, x_2, S_1, S_2, A_1 > 0.$$

Initial Iteration Table

	CB	0	3	2	0	0	-M	
C_B	Y_B	X_B	X_1	X_2	S_1	S_2	A_1	Ratio
0 (l.R)	S_1	2	2	1^*	1	0	0	$2/1 = 2$
$-M$	A_1	12	3	4	0	-1	1	$12/4 = 3$
	$Z_j - C_j$	$-12M$	$-3M-3$	$\boxed{-4M-2}$	0	M	0	—

Most -ve EC

S_1 leave the basis, X_2 enter the basis
1^* is leading element.

Now Leading element convert into one and all other Variables in the entering column is zero.

First iteration Table:

	CB	0	3	2	0	0	-M
C_B	Y_B	X_B	X_1	X_2	S_1	S_2	A_1
(Key Row) 2	X_2	2	2	1	1	0	0
$-M$	A_1	4	-5	0	-4	-1	1
	$Z_j - C_j$	$-4M+4$	$5M+1$	0	$4M+2$	M	0

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Rough work:

$-4\times K_R \Rightarrow$	-4×2	-4×2	-4×1	-4×1	-4×0	-4×0
	-8	-8	-4	-4	0	0
Add $A_1 \Rightarrow$	12	3	4	0	-1	1
	4	-5	0	-4	-1	1

All $z_j - c_j \geq 0$ in the coefficient of M. An artificial Variable A_1 appears in the optimum basis at a positive level.

So, The given LPP doesnot have any feasible solution.

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2) Use BIG-M method to solve the following LPP.

$$\text{Minimize } Z = 4x_1 + 3x_2$$

s.t.c.

$$2x_1 + x_2 \geq 10 \rightarrow (1)$$

$$-3x_1 + 2x_2 \leq 6 \rightarrow (2)$$

$$x_1 + x_2 \geq 6 \rightarrow (3)$$

$$x_1, x_2 \geq 0.$$

Solution:

Given Minimize $Z = 4x_1 + 3x_2$

Convert into Maximization,

$$\text{Max } Z = -4x_1 - 3x_2$$

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Introduce the surplus variable $S_1 > 0$ (-subtract), an artifical variable $A_1 > 0$ (add), the slack variable $S_2 > 0$ (add), and the surplus variable $S_3 > 0$ (subtract), an artifical variable $A_2 > 0$ (add) in the ①, ② & ③ equations respectively.

$$2x_1 + x_2 - S_1 + A_1 = 10$$

$$-3x_1 + 2x_2 + S_2 = 6$$

$$x_1 + x_2 - S_3 + A_2 = 6$$

We assign a cost $-M$ to the artifical variables in the New objective function.

The New objective function becomes

$$\text{Max } Z = -4x_1 - 3x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2$$

Initial Iteration Table

	C_B	0	-4	-3	0	0	0	-M	-M	
C_B	Y_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	A_2	Ratio
-M	A_1	10	2^*	1	-1	0	0	1	0	$10/2 = 5$
0	S_2	6	-3	2	0	1	0	0	0	—
-M	A_2	6	1	1	0	0	-1	0	1	$6/1 = 6$
	$Z_j - C_j$	-16M	-3M+4	-2M+3	M	0	M	0	0	—

A_1 leave the basis

x_1 enter the basis.

2^* is a leading element.

(9)

First Iteration Table:

	C_B	0	-4	-3	0	0	0	$-M$	
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	A_2	Rati
(Key Row) -4	x_1	5	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{5}{1/2}=$
0	S_2	21	0	$\frac{7}{2}$	$-\frac{3}{2}$	1	0	0	$\frac{21\times 2}{7}=$
$-M$	A_2	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1	1	$\frac{1}{1/2}=$
	z_j-c_j	$-M-20$	0	$-\frac{M}{2}+2$	$-\frac{M}{2}+4$	0	M	0	—

Roughwork:

$3\times KR \Rightarrow$ 3×5 3×1 $3\times\frac{1}{2}$ $3\times-\frac{1}{2}$ 3×0 3×0 3×0

	15	3	$\frac{3}{2}$	$-\frac{3}{2}$	0	0	0
Add $S_2 \Rightarrow$	6	-3	2	0	1	0	0
	21	0	$\frac{7}{2}$	$-\frac{3}{2}$	1	0	0

$-1\times KR \Rightarrow$ -1×5 -1×1 $-1\times\frac{1}{2}$ $-1\times-\frac{1}{2}$ -1×0 -1×0 -1×0

	-5	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0
Add A_2	6	1	1	0	0	-1	1
	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1	1

A_2 leave the basis,
x_2 enter the basis.
$\frac{1}{2}$ is a Leading element.

Second iteration Table:

	C_B	0	-4	-3	0	0	0
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
-4	x_1	4	1	0	-1	0	1
0	s_2	14	0	0	-5	1	7
(Key Row) -3	x_2	2	0	1	1	0	-2
	$z_j - c_j$	-22	0	0	1	0	2

Roughwork:

$-\frac{1}{2} \times KR \Rightarrow$ $-\frac{1}{2}\times 2$ $\quad -\frac{1}{2}\times 0$ $\quad -\frac{1}{2}\times 1$ $\quad -\frac{1}{2}\times 1$ $\quad -\frac{1}{2}\times 0$ $\quad -\frac{1}{2}\times -2$

	-1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	1
Add $x_1 \Rightarrow$	5	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
	4	1	0	-1	0	1

$-\frac{7}{2} \times KR \Rightarrow$ $-\frac{7}{2}\times 2$ $\quad -\frac{7}{2}\times 0$ $\quad -\frac{7}{2}\times 1$ $\quad -\frac{7}{2}\times 1$ $\quad -\frac{7}{2}\times 0$ $\quad -\frac{7}{2}\times -2$

	-7	0	$-\frac{7}{2}$	$-\frac{7}{2}$	0	7
Add $s_2 \Rightarrow$	21	0	$\frac{7}{2}$	$-\frac{3}{2}$	1	0
	14	0	0	-5	1	7

$\cancel{-\frac{1}{2} \times KR} \Rightarrow$ $\cancel{-\frac{1}{2}\times 2}$ $\quad \cancel{-\frac{1}{2}\times 0}$ $\quad \cancel{-\frac{1}{2}\times 1}$ $\quad \cancel{-\frac{1}{2}\times 1}$ $\quad \cancel{-\frac{1}{2}\times 0}$ $\quad \cancel{-\frac{1}{2}\times -2}$

$\cancel{-1} \quad \cancel{0} \quad \cancel{-\frac{1}{2}} \quad \cancel{-\frac{1}{2}} \quad \cancel{0} \quad \cancel{1}$

~~Add~~

All $z_j - c_j \geq 0$

$$\text{Max } z^* = -22$$

$$\text{Min } z = -(-22) = 22$$

$$x_1 = 4 \,;\; x_2 = 2$$

$$\text{Min } z = 22$$

TWO-PHASE SIMPLEX METHOD

Use Two phase simplex method to solve the following LPP.

$$\text{Max } z = 5x_1 + 3x_2$$

s.t.c.

$$2x_1 + x_2 \leq 1 \longrightarrow (1)$$

$$x_1 + 4x_2 \geq 6 \longrightarrow (2)$$

$$x_1, x_2 \geq 0.$$

Solution:

Introduce the slack Variable $s_1 > 0$ (add), and the surplus Variable $s_2 > 0$ (subtract), and an artifical Variable $A_1 > 0$ (Add) in the (1) & (2) equations respectively

$$2x_1 + x_2 + s_1 = 1$$

$$x_1 + 4x_2 - s_2 + A_1 = 6.$$

PHASE: (1)

We assign a cost -1 to the artifical Variable and a cost to all other Variables including Original Variables in the New objective function

The New objective function becomes

$$\text{Max } z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 - A_1$$

$$\text{ie) Max } z^* = -A_1$$

Initial iteration Table:

	C_B	0	0	0	0	0	-1	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	A_1	Ratio
0	S_1	1	2	$\boxed{1}$	1	0	0	$1/1 = 1$ ✓
-1	A_1	6	1	4	0	-1	1	$6/4 = 1.5$
	$z_j - c_j$	-6	-1	-4	0	1	0	—

S_1 leave the basis,

x_2 enter the basis

1^* is a Leading element.

First Iteration Table:

	C_B	0	0	0	0	0	-1
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	A_1
(KeyRow) 0	x_2	1	2	1	1	0	0
-1	A_1	2	-7	0	-4	-1	1
	$z_j - c_j$	-2	7	0	4	1	0

Roughwork

$-4 \times KR \Rightarrow$	-4×1	-4×2	-4×1	-4×1	-4×0	-4×0
	-4	-8	-4	-4	0	0
Add $A_1 \Rightarrow$	6	1	4	0	-1	1
	2	-7	0	-4	-1	1

13 All $z_j - c_j \geq 0$.

Max $z^* < 0$ and an artifical Variable A_1 appears in the optimum basis at a positive level.

The given LPP has no feasible solution.

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2 solve: Minimize $Z = -2x_1 - x_2$

s.t.c.

$$x_1 + x_2 \geq 2 \longrightarrow (1)$$

$$x_1 + x_2 \leq 4 \longrightarrow (2)$$

$$x_1, x_2 \geq 0$$

solution:

Given Minimize $Z = -2x_1 - x_2$

$\Rightarrow$ Maximize $Z = 2x_1 + x_2$

Introduce the sUrplus Variable $s_1 > 0$ (subtract) & add an artifical variable $A_1 > 0$ (add), the Slack Variables $s_2 > 0$(add) in the (1), (2) equations respeetively.

$$x_1 + x_2 - s_1 + A_1 = 2$$

$$x_1 + x_2 + S_2 = 4$$

PHASE: (1)

We assign a cost -1 to the artifical Variable and a cost 0 to all Other Variables including Original Variables in the New Objective function.

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The New objective function becomes,

$$\text{Max } z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 - A_1$$

$$\text{(i.e.) Max } z^* = -A_1$$

Initial Iteration Table:

	C_B	0	0	0	0	0	-1	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	A_1	Ratio
-1	A_1	2	1^*	1	-1	0	1	$2/1 = 2$
0	S_2	4	1	1	1	0	0	$4/1 = 4$
	$z_j c_j$	-2	-1	-1	1	0	-1	—

A_1 leave the basis,

x_1 enter the basis

1^* is a Leading element.

First Iteration Table:

	C_B	0	0	0	0	0	-1
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	A_1
(Key Row) 0	x_1	2	1	1	-1	0	1
0	S_2	2	0	0	2	0	-1
	$z_j c_j$	0	0	0	0	0	1

Roughwork

$-1 \times KR \Rightarrow$	-1×2	-1×1	-1×1	-1×-1	-1×0	-1×1
	-2	-1	-1	1	0	-1
Add $S_2 \Rightarrow$	4	1	1	1	0	0
	2	0	0	2	0	-1

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Max $z^* = 0$ and No artifical Variable appears in the optimum basis.

$\therefore$ We Proceed Phase - 2

PHASE: 2

The New objective function becomes,

$$\text{Max } z^* = 2x_1 + x_2 + 0s_1 + 0s_2$$

Initial Iteration Table:

	C_B	0	2	1	0	0	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	Ratio
2	x_1	2	1	1	-1	0	-
0	s_2	2	0	0	1^*	1	$2/1 = 2$
	$z_j c_j$	4	0	1	-2	0	-

s_2 leave the basis

s_1 enter the basis

1^* is a Leading element.

First iteration Table

		C_B	0	2	1	0	0
	C_B	Y_B	X_B	x_1	x_2	s_1	s_2
	2	x_1	4	1	1	0	1
(Key Row)	0	s_1	2	0	0	1	1
		$z_j c_j$	8	0	1	0	2

All $z_j - c_j \geq 0$

the optimum solution is

$x_1 = 4; \quad x_2 = 0$

$\text{Max } z = \cdot 8$

$\text{Min } z = -8$

DUALITY

To construct the dual Problem, we adopt the following rules:

* The Maximization Problem in the Primal becomes the Minimization Problem in the dual vice-versa
* The Maximization Problem has $\leq$ constraints while the Minimization Problem has $\geq$ constraints.
* The Transpose of the matrix of the Primal Problem gives the matrix of the dual and vice-versa
* The constants in the objective function of the Primal appears in the constraints of the dual.
* The constants in the constraints of the Primal appear in the objective function of the dual.
* The variables in both Problems are non-negative.

EXAMPLES

(1) Write the dual of the following LPP.

$$\text{Maximize } Z = x_1 + 2x_2 + x_3$$

s.t.c.

$$2x_1 + x_2 - x_3 \leq 2$$

$$2x_1 + x_2 - 5x_3 \leq 6$$

$$4x_1 + x_2 + x_2 \leq 6$$

$$x_1, x_2, x_3 \geq 0.$$

Solution: The objective function is Maximizatio

So, the constraints are $\leq$ type.

Let y_1, y_2, y_3 are the dual Variables.

The dual Problem is

$$\text{Minimize } Z = 2y_1 + 6y_2 + 6y_3$$

s.t.c.

$$2y_1 + 2y_2 + 4y_3 \geq 1$$

$$y_1 + y_2 + y_3 \geq 2$$

$$-y_1 - 5y_2 + y_3 \geq 1$$

$$y_1, y_2, y_3 \geq 0.$$

(18) (2) Write the dual of the following LPP

$$\text{Minimize } Z = 4x_1 + 6x_2 + 18x_3$$

s.t.c.

$$x_1 + 3x_2 \geq 3$$

$$x_2 + 2x_3 \geq 5$$

$$x_1, x_2, x_3 \geq 0$$

solution:

The objective function is Minimization, so all the constraints are $\geq$ type.

Let y_1, y_2 are dual Variables

The dual Problem is

$$\text{Maximize } Z = 3y_1 + 5y_2$$

s.t.c.

$$y_1 \leq 4$$

$$3y_1 + y_2 \leq 6$$

$$2y_2 \leq 18$$

$$y_1, y_2 \geq 0.$$

③ Write the dual of the following LPP

$$\text{Min } Z = 3x_1 - 2x_2 + 4x_3$$

s.t.c.

$$3x_1 + 5x_2 + 4x_3 \geq 7 \rightarrow ①$$

$$6x_1 + x_2 + 3x_3 \geq 4 \rightarrow ②$$

$$7x_1 - 2x_2 - x_3 \leq 10 \rightarrow ③$$

$$x_1 - 2x_2 + 5x_3 \geq 3 \rightarrow ④$$

$$4x_1 + 7x_2 - 2x_3 \geq 2 \rightarrow ⑤$$

$$x_1, x_2, x_3 \geq 0.$$

Solution:

The objective function of the Primal Problem is minimization.

The eqn ③ can be converted into $\geq$ type.

so multiply bothsides by -1

ie) $-7x_1 + 2x_2 + x_3 \geq -10$.

The given LPP can be rewritten as

$$\text{Minimize } Z = 3x_1 - 2x_2 + 4x_3$$

s.t.c.

$$3x_1 + 5x_2 + 4x_3 \geq 7$$

$$6x_1 + x_2 + 3x_3 \geq 4$$

$$-7x_1 + 2x_2 - x_3 \geq -10$$

$$x_1 - 2x_2 + 5x_3 \geq 3$$

$$4x_1 + 7x_2 - 2x_3 \geq 2$$

$$x_1, x_2, x_3 \geq 0.$$

(20) Let y_1, y_2, y_3, y_4, y_5 are the dual Variables.

Maximize $Z = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2$

s.t.c.

$$3y_1 + 6y_2 + 7y_3 + y_4 + 4y_5 \leq 3$$

$$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$$

$$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0.$$

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(4) Write the dual of the following LPP.

Maximize $Z = 3x_1 + 10x_2 + 2x_3$

s.t.c.

$$2x_1 + 3x_2 + 2x_3 \leq 7 \rightarrow (1)$$

$$3x_1 - 2x_2 + 4x_3 = 3 \rightarrow (2)$$

$$x_1, x_2, x_3 \geq 0.$$

solution:

The objective function of the Primal is of maximization.

The eqn (2) can be rewritten as

$$3x_1 - 2x_2 + 4x_3 \leq 3$$

$$-3x_1 + 2x_2 - 4x_3 \leq -3$$

The given Primal LPP can be rewritten as

(21) Maximize $Z = 3x_1 + 10x_2 + 2x_3$

s.t.c.

$2x_1 + 3x_2 + 2x_3 \leq 7$

$3x_1 - 2x_2 + 4x_3 \leq 3$

$-3x_1 + 2x_2 - 4x_3 \leq -3$

$x_1, x_2, x_3 \geq 0.$

Let y_1, y_2, y_3 are the dual Variables

The dual Problem is

Minimize $Z = 7y_1 + 3y_2 - 3y_3$

s.t.c.

$2y_1 + 3y_2 - 3y_3 \geq 3$

$3y_1 - 2y_2 + 2y_3 \geq 10$

$2y_1 + 4y_2 - 4y_3 \geq 2$

$y_1, y_2, y_3 \geq 0.$

(5) Write the dual of the following LPP

Minimize $Z = 4x_1 + 5x_2 - 3x_3$

s.t.c.

$x_1 + x_2 + x_3 = 22 \longrightarrow (1)$

$3x_1 + 5x_2 - 2x_3 \leq 65 \longrightarrow (2)$

$x_1 + 7x_2 + 4x_3 \geq 120 \longrightarrow (3)$

$x_1, x_2 \geq 0$ & x_3 is unrestricted.

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Solution: The objective function of the given LPP is Minimization.

The eqn ③ can be converted into $\geq$ type.

i.e) $-3x_1 - 5x_2 + 2x_3 \geq -65$

The given Primal LPP can be rewritten as

Minimize $Z = 4x_1 + 5x_2 - 3x_3$

s.t.c.

$$x_1 + x_2 + x_3 = 22 \rightarrow ①$$

$$-3x_1 - 5x_2 + 2x_3 \geq -65 \rightarrow ②$$

$$x_1 + 7x_2 + 4x_3 \geq 120 \rightarrow ③$$

$x_1, x_2 \geq 0$ & x_3 is Unrestricted.

Let y_1, y_2, y_3 are dual variables.

In the Primal Problem the first Constraint are equality sign, the Corresponding first dual variable y_1 is unrestricted.

In the Primal Problem the third Variable x_3 is Unrestricted, the Corresponding the third dual constraint is in equality sign.

The dual Problem is

$$\text{Maximize } Z = 22y_1 - 65y_2 + 120y_3$$

s.t.c.

$$y_1 - 3y_2 + y_3 \leq 4$$

$$y_1 - 5y_2 + 7y_3 \leq 5$$

$$y_1 + 2y_2 + 4y_3 = -3$$

y_1 is unrestricted.

$$y_2, y_3 \geq 0.$$

DUALITY AND SOLVE

(1) Use duality to solve the following LPP.

$$\text{Minimize } Z = 2x_1 + 2x_2$$

s.t.c.

$$2x_1 + 4x_2 \geq 1$$

$$x_1 + 2x_2 \geq 1$$

$$2x_1 + x_2 \geq 1$$

$$x_1, x_2 \geq 0.$$

Solution: The objective function of the Primal Problem is minimization, so all the constraints are $\geq$ type.

Let y_1, y_2, y_3 are the dual variables.

Then the dual Problem is

$$\text{Maximize } Z = y_1 + y_2 + y_3$$

s.t.c.

$$2y_1 + y_2 + 2y_3 \le 2 \rightarrow (1)$$

$$4y_1 + 2y_2 + y_3 \le 2 \rightarrow (2)$$

$$y_1, y_2, y_3 \ge 0.$$

Apply simplex method to solve the dual problem.

Introduce the slack variables $s_1, s_2 \ge 0$ (add) in the (1), (5) eqns respectively.

$$2y_1 + y_2 + 2y_3 + s_1 = 2$$

$$4y_1 + 2y_2 + y_3 + s_2 = 2$$

The New objective function becomes

$$\text{Max } z^* = y_1 + y_2 + y_3 + 0s_1 + 0s_2$$

<u>Initial iteration table</u>

	C_B	0	1	1	1	0	0	
C_B	Y_B	X_B	X_1	X_2	Y_3	S_1	S_2	Ratio
0	S_1	2	2	1	2	1	0	$2/1 = 2$
0	S_2	2	4	2^*	1	0	1	$2/2 = 1$
	$z_j - c_j$	0	-1	-1	-1	0	0	—

S_2 leave the basis

y_2 enter the basis

2^* is a Leading element.

First iteration Table

	C_B	0	1	1	1	0	0	
C_B	Y_B	X_B	y_1	y_2	y_3	s_1	s_2	Ratio
0	s_1	1	0	0	$3/2$	1	$-1/2$	$\frac{1}{3/2} = 2/3$
(Key Row) 1	y_2	1	2	1	$1/2$	0	$1/2$	$\frac{1}{1/2} = 2/1 =$
	$z_j - c_j$	1	1	0	$-1/2$	0	$1/2$	—

Roughwork:

$-1 \times KR \Rightarrow$	-1×1	-1×2	-1×1	$-1\times 1/2$	-1×0	$-1\times 1/2$
	-1	-2	-1	$-1/2$	0	$-1/2$
Add $s_1 \Rightarrow$	2	2	1	2	1	0
	1	0	0	$3/2$	1	$-1/2$

$1 \times KR \Rightarrow$	1×1	1×2	1×1	$1\times 1/2$	1×0	$1\times 1/2$
	1	2	1	$1/2$	0	$1/2$
Add $z_j c_j \Rightarrow$	0	-1	-1	-1	0	0
	1	1	0	$-1/2$	0	$1/2$

s_1 leave the basis

y_3 enter the basis

$3/2^*$ is a leading element.

Second Iteration Table

	C_B	0	1	1	1	0	0
C_B	Y_B	X_B	y_1	y_2	y_3	s_1	s_2
(keyRow) 1	y_3	$2/3$	0	0	1	$2/3$	$-1/3$
1	y_2	$2/3$	2	1	0	$-1/3$	$2/3$
	$z_j c_j$	$4/3$	1	0	0	$1/3$	$1/3$

Roughwork

$-\frac{1}{2} \times KR \Rightarrow$	$-\frac{1}{2} \times \frac{2}{3}$	$-\frac{1}{2} \times 0$	$-\frac{1}{2} \times 0$	$-\frac{1}{2} \times 1$	$-\frac{1}{2} \times \frac{2}{3}$	$-\frac{1}{2} \times -\frac{1}{3}$
	$-1/3$	0	0	$-1/2$	$-1/3$	$1/6$
Add $y_2 \Rightarrow$	1	2	1	$1/2$	0	$1/2$
	$2/3$	2	1	0	$-1/3$	$2/3$

$\frac{1}{2} \times KR \Rightarrow$	$\frac{1}{2} \times \frac{2}{3}$	$\frac{1}{2} \times 0$	$\frac{1}{2} \times 0$	$\frac{1}{2} \times 1$	$+\frac{1}{2} \times \frac{2}{3}$	$\frac{1}{2} \times -\frac{1}{3}$
	$1/3$	0	0	$1/2$	$+1/3$	$-1/6$
Add $z_j c_j \Rightarrow$	1	1	0	$-1/2$	0	$1/2$
	$4/3$	1	0	0	$+1/3$	$1/3$

All $z_j - c_j \geq 0$.

The optimum solution of the dual Problem is

$y_1 = 0 \quad y_2 = 2/3 \quad y_3 = 2/3$

$\text{Max } Z^* = 4/3.$

(27) The solution of the Primal Problem is.

$$x_1 = 1/3 \; ; \; x_2 = 1/3.$$

$$\text{Min } Z^* = 4/3$$

Since the primal variable x_1, x_2 are associated with the corresponding slack variables.

(i.e) $s_1 = x_1 = 1/3 \; ; \; s_2 = x_2 = 1/3.$

DUAL SIMPLEX METHOD

(1) Use dual simplex method to solve the following LPP.

$$\text{Max } Z = 6x_1 + 4x_2 + 4x_3$$

s.t.c.

$$3x_1 + x_2 + 2x_3 \geq 2$$

$$2x_1 + x_2 - x_3 \geq 1$$

$$-x_1 + x_2 + 2x_3 \geq 1$$

$$x_1, x_2, x_3 \geq 0.$$

Solution:

$$\text{Max } Z = 6x_1 + 4x_2 + 4x_3$$

s.t.c.

$$-3x_1 - x_2 - 2x_3 \leq -2$$

$$-2x_1 - x_2 + x_3 \leq -1$$

$$x_1 - x_2 - 2x_3 \leq -1$$

$$x_1, x_2, x_3 \geq 0.$$

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The standard form an LPP is

$$\text{Max } z^* = 6x_1 + 4x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$$

s.t.c.

$$-3x_1 - x_2 - 2x_3 + s_1 = -2$$

$$-2x_1 - x_2 + x_3 + s_2 = -1$$

$$x_1 - x_2 - 2x_3 + s_3 = -1$$

Initial Iteration Table

	C_B	0	6	4	4	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	-2	-3	-1	-2	1	0	0
0	s_2	-1	-2	-1	1	0	1	0
0	s_3	-1	1	-1	-2	0	0	1
	$z_j - c_j$	0	-6	-4	-4	0	0	0

All $z_j - c_j < 0$, The method fails.

So, The given LPP can not solve by dual simplex method.

29 2 Using dual simplex Method to solve the following LPP.

$$\text{Minimize } Z = 2x_1 + x_2$$

s.t.c.

$$3x_1 + x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \geq 3$$
$$x_1, x_2 \geq 0.$$

solution:

$$\text{Max } Z^* = -2x_1 - x_2$$

s.t.c.

$$-3x_1 - x_2 \leq -3$$
$$-4x_1 - 3x_2 \leq -6$$
$$-x_1 - 2x_2 \leq -3$$
$$x_1, x_2 \geq 0.$$

The standard form of an LPP is

$$\text{Max } Z^* = -2x_1 - x_2 + 0s_1 + 0s_2 + 0s_3$$

s.t.c.

$$-3x_1 - x_2 + 0s_1 = -3$$
$$-4x_1 - 3x_2 + 0s_2 = -6$$
$$-x_1 - 2x_2 + 0s_3 = -3$$

Initial Iteration Table

C_B	C_B / Y_B	0 / X_B	-2 / x_1	-1 / x_2	0 / s_1	0 / s_2	0 / s_3
0	s_1	-3	-3	-1	1	0	0
0	s_2	-6	-4	-3	0	1	0
0	s_3	-3	-1	-2	0	0	1
	$z_j - c_j$	0	2	1	0	0	0

All $z_j - c_j \geq 0$ & all $X_{B_i} < 0$

choose most negative of X_{B_i}

i.e) $X_{B_2} = -6$ is most negative

Ratio $\quad \text{Max}\left\{ \frac{z_r - c_r}{y_{kr}} \Big/ y_{kr} < 0 \right\}$

$$\text{Max}\left\{ \frac{2}{-4}, \frac{1}{-3} \right\} = \{-0.5, -0.3\} = -0.3$$

s_2 leave the basis, x_2 enter the basis

(-3) is a leading element.

First Iteration Table

C_B	C_B / Y_B	0 / X_B	-2 / x_1	-1 / x_2	0 / s_1	0 / s_2	0 / s_3
0	s_1	-1	$-5/3$	0	1	$-1/3$	0
(Keyrow) 1	x_2	2	$4/3$	1	0	$-1/3$	0
0	s_3	1	$5/3$	0	0	$-2/3$	1
	$z_j - c_j$	-2	$2/3$	0	0	$1/3$	0

All $z_j - c_j \geq 0$ & one $X_{B_i} < 0$

Roughwork

$1\times KR \Rightarrow$	1×2	$1\times 4/3$	1×1	1×0	$1\times -1/3$	1×0
	2	$4/3$	1	0	$-1/3$	0
Add s_1	-3	-3	-1	1	0	0
	-1	$-5/3$	0	1	$-1/3$	0

$+2\times KR \Rightarrow$	$+2\times 2$	$2\times 4/3$	2×1	2×0	$2\times -1/3$	2×0
	4	$8/3$	2	0	$-2/3$	0
Add s_3	-3	-1	-2	0	0	1
	1	$5/3$	0	0	$-2/3$	1

$-1\times KR \Rightarrow$	-1×2	$-1\times 4/3$	-1×1	-1×0	$-1\times -1/3$	$-1\times$
	-2	$-4/3$	-1	0	$1/3$	0
Add $z_j c_j$	0	2	1	0	0	0
	-2	$2/3$	0	0	$1/3$	0

Ratio $\text{Max}\left\{ \frac{z_r - c_r}{y_{kr}} \Big/ y_{kr} < 0 \right\}$

$$\text{Max}\left\{ \frac{\frac{2}{3}}{-5/3}, \frac{1/3}{-1/3} \right\} = \text{Max}\left\{ -2/5, -1 \right\}$$
$$= -2/5$$

s_1 leave the basis,

x_1 enter the basis

$-5/3$ is a Leading element.

Second Iteration Table

	C_B	0	-2	-1	0	0	0
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
Key Row -2	x_1	$3/5$	1	0	$-3/5$	$1/5$	0
-1	x_2	$6/5$	0	1	$4/5$	$-3/5$	0
0	s_3	0	0	0	1	-1	1
	$z_j c_j$	$-12/5$	0	0	$2/5$	$1/5$	0

Roughwork

$-\frac{4}{3}\times KR \Rightarrow$	$-\frac{4}{3}\times\frac{3}{5}$	$-\frac{4}{3}\times 1$	$-\frac{4}{3}\times 0$	$-\frac{4}{3}\times 1$	$-\frac{4}{3}\times -\frac{1}{3}$	$-\frac{4}{3}\times 0$
	$-\frac{4}{5}$	$-\frac{4}{3}$	0	$-\frac{4}{3}$	$\frac{4}{9}$	0
Add x_2	2	$\frac{4}{3}$	1	0	$-\frac{1}{3}$	0
	$\frac{6}{5}$	0	1	$-\frac{4}{3}$	$-\frac{3}{5}$	0

$-\frac{5}{3}\times KR \Rightarrow$	$-\frac{5}{3}\times\frac{3}{5}$	$-\frac{5}{3}\times 1$	$-\frac{5}{3}\times 0$	$-\frac{5}{3}\times 1$	$-\frac{5}{3}\times -\frac{1}{3}$	$-\frac{5}{3}\times 0$
	-1	$-\frac{5}{3}$	0	$-\frac{5}{3}$	$\frac{5}{9}$	0
Add s_3	1	$\frac{5}{3}$	0	$-\frac{2}{3}$	$-\frac{2}{3}$	1
	0	0	0	1	-1	1

$-\frac{2}{3}\times KR \Rightarrow$	$-\frac{2}{3}\times\frac{3}{5}$	$-\frac{2}{3}\times 1$	$-\frac{2}{3}\times 0$	$-\frac{2}{3}\times 1$	$-\frac{2}{3}\times -\frac{1}{3}$	$-\frac{2}{3}\times 0$
	$-\frac{2}{5}$	$-\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{2}{9}$	0
Add z_j	-2	$\frac{2}{3}$	0	0	$\frac{1}{3}$	0
	$-\frac{12}{5}$	0	0	$\frac{2}{5}$	$\frac{1}{5}$	0

Conclusion:

All $z_j - c_j \geq 0$ & all $x_{B_i} \geq 0$

Then the optimum solution of the given LPP is

$$x_1 = \frac{3}{5}\ ;\ x_2 = \frac{6}{5}$$

$$\text{Max } z^* = -\frac{12}{5}$$

$$\text{Min } z = -\left(-\frac{12}{5}\right) = \frac{12}{5}$$

GAME THEORY

GAMES WITHOUT SADDLE POINTS.

MIXED STRATEGIES

SOLUTION OF 2x2 GAMES WITHOUT SADDLE POINT

For any 2x2 two person zero-sum game without any saddle point having the Pay off matrix for Player A is

$$\text{Player A} \quad \begin{array}{c} \\ \\ A_1 \\ A_2 \end{array} \begin{array}{c} \text{Player B} \\ B_1 \quad B_2 \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$$

The optimum mixed strategie for Player A is $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$

where $p_1 = \dfrac{d-c}{(a+d)-(b+c)}$, $p_2 = 1 - p_1$.

the optimum mixed Strategie for player B is $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$

where $q_1 = \dfrac{d-b}{(a+d)-(b+c)}$, $q_2 = 1 - q_1$

The value of the game

$$V = \frac{ad-bc}{(a+d)-(b+c)}$$

EXAMPLE: 1

Solve the following 2×2 game.

$$\text{Player A}\quad \begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 5 & 1 \\ A_2 & 3 & 4 \end{array}$$

Solution

Player B

$$\text{Player A}\quad \begin{array}{ccc|c} & B_1 & B_2 & \text{Row-Minimum} \\ A_1 & 5 & 1 & 1 \\ A_2 & 3 & 4 & 3 \\ \text{Column Maximum} & 5 & 4 & \end{array}$$

Maximum of Row-minimum $= 3 = \underline{v}$

Minimum of Column-Maximum $= 4 = \overline{v}$

$$\underline{v} \neq \overline{v}$$

$\therefore$ The given game doesnot have any Saddle point.

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{4-3}{(5+4)-(1+3)} = \frac{1}{9-4} = \frac{1}{5}$$

$$p_2 = 1 - p_1 = 1 - \tfrac{1}{5} = \tfrac{4}{5}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{4-1}{(5+4)-(1+3)} = \frac{3}{9-4} = \tfrac{3}{5}$$

$$q_2 = 1 - q_1 = 1 - \tfrac{3}{5} = \tfrac{2}{5}.$$

$$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{(5)(4)-(1)(3)}{(5+4)-(1+3)} = \frac{20-3}{9-4} = 17/5$$

The optimum strategie for player A, $S_A = \begin{pmatrix} A_1 & A_2 \\ 1/5 & 4/5 \end{pmatrix}$

The optimum strategie for player B, $S_B = \begin{pmatrix} B_1 & B_2 \\ 3/5 & 2/5 \end{pmatrix}$

Value of the game $v = 17/5$

EXAMPLE: 2

Solve the following 2×2 game without Saddle point.

$$\text{Player A} \begin{array}{c} \text{Player B} \\ \begin{pmatrix} 6 & -3 \\ -3 & 0 \end{pmatrix} \end{array}$$

Solution:

$$\begin{array}{ccc} & \text{Player B} & \text{Row-minimum} \\ \text{Player A} & \begin{pmatrix} 6 & -3 \\ -3 & 0 \end{pmatrix} & \begin{array}{c} -3 \\ -3 \end{array} \\ \text{Column maximum} & \begin{array}{cc} 6 & 0 \end{array} & \end{array}$$

Maximum of Row-minimum $= \underline{V} = -3$

Minimum of column-maximum $= \overline{V} = 6$

$$\underline{V} \neq \overline{V}$$

The given game doesnot have any saddle point.

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{0-(-3)}{(6+0)-(-3-3)} = \frac{3}{6+6} = \frac{3}{12} = \frac{1}{4}$$

$$p_2 = 1 - p_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{0-(-3)}{(6+0)-(-3-3)} = \frac{3}{6+6} = \frac{3}{12} = \frac{1}{4}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{(6)(0)-(-3)(-3)}{(6+0)-(-3-3)} = \frac{6-9}{6+6} = \frac{-3}{12} = -\frac{1}{4}$$

The optimum strategie for Player A is

$$S_A = \begin{pmatrix} A_1 & A_2 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

The optimum strategie for player B is

$$S_B = \begin{pmatrix} B_1 & B_2 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix},$$ Value of the game

$$V = -\frac{1}{4}$$

GRAPICAL METHOD FOR $n \times 2$ and $2 \times m$ Games.

EXAMPLE: 1 Solve the following 2×4 game graphically.

Player B

$$\text{Player A} \begin{pmatrix} 1 & 0 & 4 & -1 \\ -1 & 1 & -2 & 5 \end{pmatrix}$$

Solution:

$$\begin{array}{c|c|c} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} & \text{Row-minimum} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 1 & 0 & 4 & -1 \\ -1 & [redacted] & -2 & 5 \end{bmatrix} & \begin{array}{c} -1 \\ -2 \end{array} \end{array}$$

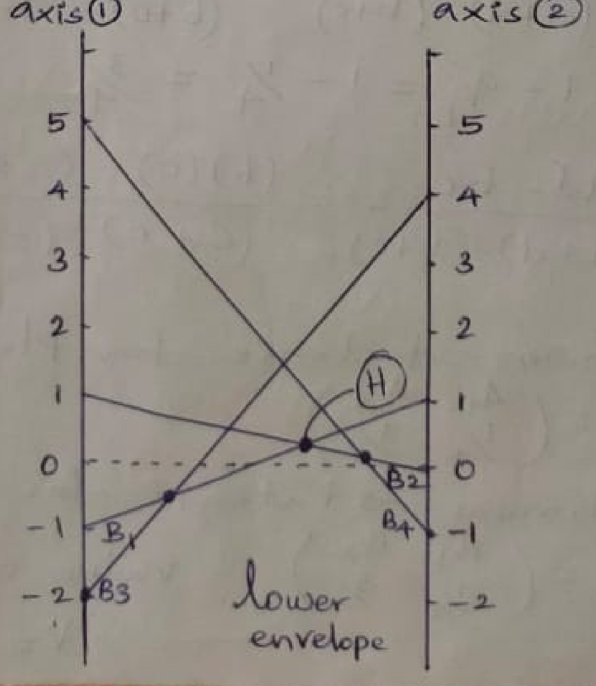
Column Maximum $\quad 1 \quad$ [redacted] $\quad 4 \quad 5$

Maximum of Row-minimum $= \underline{v} = -1$

Minimum of column-maximum $= \overline{v} =$ [redacted]

$\underline{v} \neq \overline{v}$ ie) $-1 \neq$ [redacted]

The given game doesnot have any saddle point.



The point (H) represents the higeast point of the lower envelope.

The Lines B_1 and B_2 passes through (H).

The given 2×4 game is reduced to 2×2 game.

$$\begin{array}{c|cc|c} & B_1 & B_2 & \text{Row-minimum} \\ \hline A_1 & 1 & 0 & 0 \\ A_2 & -1 & 1 & -1 \\ \hline \text{Column Maximum} & 1 & 1 & \end{array}$$

Maximum of Row-minimum $= \underline{v} = -1$

Minimum of column-maximum $= \bar{v} = 1$

$\underline{v} \neq \bar{v}$ ie) $-1 \neq 1$.

The given game doesnot have any saddle point.

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{1-(-1)}{(1+1)-(0-1)} = \frac{1+1}{2+1} = 2/3$$

$$p_2 = 1 - p_1 = 1 - 2/3 = 1/3$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{1-0}{(1+1)-(0-1)} = \frac{1}{2+1} = 1/3$$

$$q_2 = 1 - \frac{1}{3} = 2/3.$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{(1)(1)-(0)(-1)}{(1+1)-(0-1)} = \frac{1}{2+1} = 1/3$$

The optimum strategie for Player A is

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ 2/3 & 1/3 \end{pmatrix}$$

The optimum strategie for player B is

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ q_1 & q_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 1/3 & 2/3 & 0 & 0 \end{pmatrix}$$

The value of the game $V = \frac{1}{3}$.

EXAMPLE: 2

Solve the following 5x2 game graphically

Player B

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} -4 & 3 \\ -7 & 1 \\ -2 & -4 \\ -5 & -2 \\ -1 & -6 \end{bmatrix} \end{array}$$

Solution:

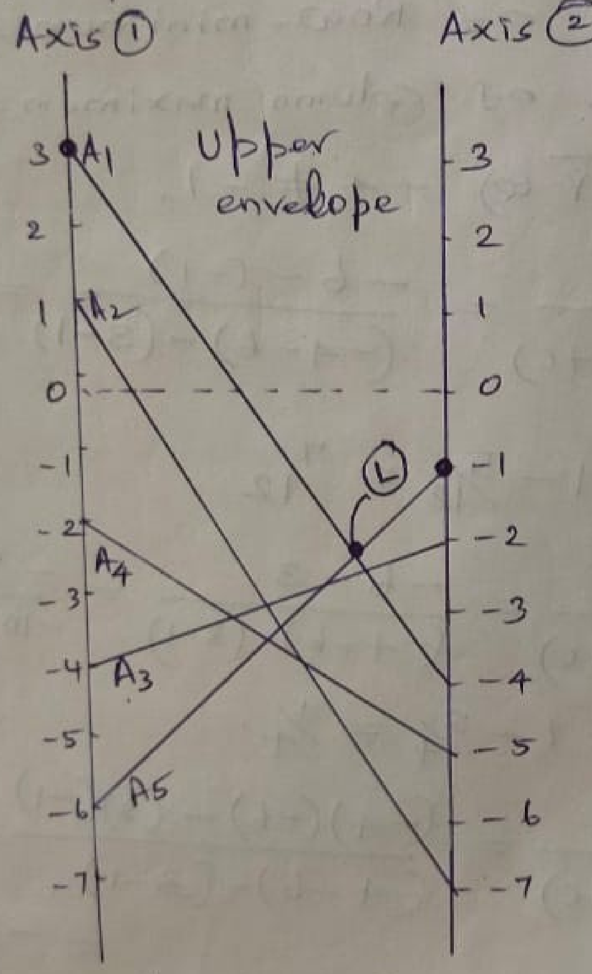
$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ \text{Column Maximum} \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} -4 & 3 \\ -7 & 1 \\ -2 & -4 \\ -5 & -2 \\ -1 & -6 \end{bmatrix} \\ \begin{array}{cc} -1 & 3 \end{array} \end{array} \begin{array}{c} \text{Row-minimum} \\ -4 \\ -7 \\ -4 \\ -5 \\ -6 \\ \\ \end{array}$$

Maximum of Row-minimum $= \underline{V} = -4$

Minimum of Column-maximum $= \overline{V} = -1$

$$\underline{V} = \overline{V} = -4 \neq -1.$$

The given game doesnot have any Saddle point.



The point L represents the Lowest point of the Upper envelope.

The lines A_1 and A_5 passes through L.

The given 5×2 game is reduced to 2×2 game.

$$\begin{array}{c|cc|c} & B_1 & B_2 & \text{row minimum} \\ A_1 & -4 & 3 & -4 \\ A_5 & -1 & -6 & -6 \\ \text{column maximum} & -1 & +3 & \end{array}$$

Maximum of Row-minimum $= \underline{v} = -4$

Minimum of column maximum $= \overline{v} = -1$

$\underline{v} \neq \overline{v}$ (i.e) $-4 \neq -1$.

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{-6-(-1)}{(-4-6)-(3-1)} = \frac{-6+1}{-10-2} = \frac{-5}{-12} = \frac{5}{12}$$

$$p_2 = 1 - p_1 = 1 - 5/12 = 7/12$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{-6-3}{(-4-6)-(3-1)} = \frac{-9}{-10-2} = \frac{-9}{-12} = \frac{3}{4}$$

$$q_2 = 1 - q_1 = 1 - 3/4 = 1/4.$$

$$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{(-4)(-6)-(3)(-1)}{(-4-6)-(3-1)} = \frac{24+3}{-10-2} = \frac{27}{-12} = -\frac{9}{4}.$$

The optimum strategie for player A is

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 5/12 & 0 & 0 & 0 & 7/12 \end{pmatrix}$$

The optimum strategie for player B is

$$S_B = \begin{pmatrix} B_1 & B_2 \\ 3/4 & 1/4 \end{pmatrix}$$

The value of the game

$$v = -\frac{9}{4}.$$

Dominance Property

Rules:

1. If all the elements of a row, say k^{th} are less than are equal to the corresponding elements of any other row, say r^{th}. then k^{th} row is dominated by r^{th} row.
2. If all the elements of a column, say k^{th}, are greater than are equal to the corresponding elements of any other column, say r^{th}. then k^{th} column is dominated by r^{th} column.
3. Omit dominated rows (or) columns.

EXAMPLE: 1

Solve the following game using dominance Property.

Player B

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 6 & 1 & 6 \end{bmatrix} \end{array}$$

Solution:

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \text{Row-minimum} \\ A_1 & 1 & 7 & 2 & 1 \\ A_2 & 6 & 2 & 7 & 2 \\ A_3 & 6 & 1 & 6 & 1 \\ \text{column maximum} & 6 & 7 & 7 & \end{array}$$

Maximum of row-minimum $= \underline{V} = 2$

Minimum of column Maximum $= \overline{V} = 6$

$\overline{V} \neq \underline{V}$ ie) $b \neq 2$

The given game doesnot have any saddle point.

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \\ A_1 & 1 & 7 & 2 & R_1 \\ A_2 & 6 & 2 & 7 & R_2 \\ A_3 & 6 & 1 & 6 & R_3 \\ & C_1 & C_2 & C_3 & \end{array}$$

R_3 is less than or equal to R_2.

$\therefore R_3$ is dominated by R_2

so, we omit R_3.

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \\ A_1 & 1 & 7 & 2 & R_1 \\ A_2 & 6 & 2 & 7 & R_2 \\ & C_1 & C_2 & C_3 & \end{array}$$

B_3 is greater than (or) equal to B_1

$\therefore C_3$ is dominated by C_1

so, we omit C_3.

$$\begin{array}{c|cc|} & B_1 & B_2 \\ A_1 & 1 & 7 \\ A_2 & 6 & 2 \end{array}$$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{2-6}{(1+2)-(7+6)} = \frac{-4}{3-13} = \frac{-4}{-10} = 2/5$$

$$p_2 = 1 - p_1 = 1 - 2/5 = 3/5$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{2-7}{(1+2)-(7+6)} = \frac{-5}{3-13} = \frac{-5}{-10} = 1/2$$

$$q_2 = 1 - q_1 = 1 - 1/2 = 1/2$$

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{(1)(2)-(7)(6)}{(1+2)-(7+6)} = \frac{2-42}{3-13}$$

$$= \frac{-40}{-10} = 4$$

The optimum strategie for Player A is

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & 0 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & A_3 \\ 2/5 & 3/5 & 0 \end{pmatrix}$$

the optimum strategie for player B is

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & 0 \end{pmatrix} = \begin{pmatrix} B_1 & B_2 & B_3 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

The value of the game $v = 4$.

Example: 2 Use dominance Property to simplify the rectangular game with the following Pay off and solve it.

Player B

$$\text{Player A} \quad \begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 18 & 4 & 6 & 4 \\ A_2 & 6 & 2 & 13 & 7 \\ A_3 & 11 & 5 & 17 & 3 \\ A_4 & 7 & 6 & 12 & 2 \end{array}$$

Solution:

$$\begin{array}{c|cccc|c} & B_1 & B_2 & B_3 & B_4 & \text{Row-minimum} \\ \hline A_1 & 18 & 4 & 6 & 4 & 4 \\ A_2 & 6 & 2 & 13 & 7 & 2 \\ A_3 & 11 & 5 & 17 & 3 & 3 \\ A_4 & 7 & 6 & 12 & 2 & 2 \\ \hline \text{Column Maximum} & 18 & 6 & 17 & 7 & \end{array}$$

Maximum of row minimum $= \underline{V} = 4$

Minimum of column Maximum $= \overline{V} = 6$

$\underline{V} \neq \overline{V} \Rightarrow 4 \neq 6.$

The given game doesnot have any saddle point.

$$\begin{array}{cc} \begin{bmatrix} 18 & 4 & 6 & 4 \\ 6 & 2 & 13 & 7 \\ 11 & 5 & 17 & 3 \\ 7 & 6 & 12 & 2 \end{bmatrix} & \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \\ \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} & \end{array}$$

C_1 is greater than C_2

C_1 is dominated by C_2

so we omit C_1

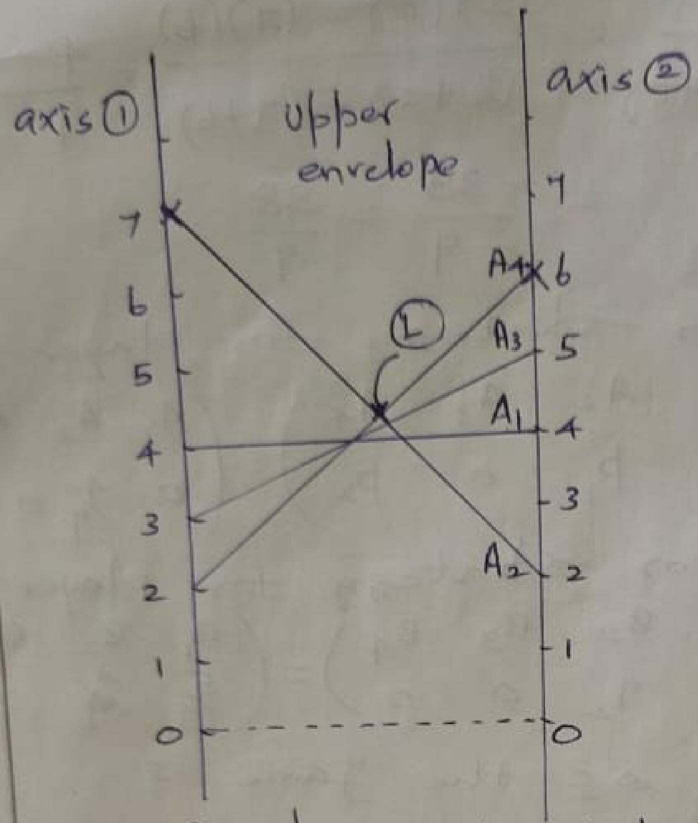
C_3 is greater than C_4

C_3 is dominated by C_4

so, we omit C_3

$$\text{Player A} \quad \begin{array}{c|c} & \begin{array}{cc} B_2 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 4 & 4 \\ 2 & 7 \\ 5 & 3 \\ 6 & 2 \end{bmatrix} \end{array}$$

we have to apply graphical method



The point L represents the lowest point of the upper envelope.

The line A_2 and A_4 passes through L.

$\therefore$ The given 4×2 game is reduced to 2×2 rectangular game.

$$\therefore \begin{array}{c} \\ A_2 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{pmatrix} 2 & 7 \\ 6 & 2 \end{pmatrix} \end{array}$$

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{2-6}{(2+2)-(7+6)} = \frac{-4}{4-13} = \frac{-4}{-9} = \frac{4}{9}$$

$$p_2 = 1 - 4/9 = 5/9$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{2-7}{(2+2)-(7+6)} = \frac{-5}{4-13} = \frac{-5}{-9} = 5/9$$

$$q_2 = 1 - 5/9 = 4/9$$

$$V = \frac{ad - bc}{(a+d)-(b+c)} = \frac{(2)(2)-(7)(6)}{(2+2)-(7+6)} = \frac{4-42}{4-13}$$

$$= \frac{-38}{-9} = \frac{38}{9}$$

The optimum ~~statagy~~ for player A is

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & p_1 & 0 & p_2 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 4/9 & 0 & 5/9 \end{pmatrix}$$

The optimum strategy for player B is

$$S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ q_1 & q_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 5/9 & 4/9 & 0 & 0 \end{pmatrix}$$

The value of the game is

$$V = 38/9$$

Solve the Game Using LPP

- solve the following game by using simplex method.

Player B

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix} \end{array}$$

solution:

Player B

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ \text{column maximum} \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix} \\ \begin{array}{ccc} 6 & 5 & 3 \end{array} \end{array} \begin{array}{c} \text{Row-minimum} \\ -1 \\ -3 \\ -2 \\ \\ \end{array}$$

Maximum of Row minimum $= \underline{v} = -1$

minimum of column Maximum $= \overline{v} = 3$

$\underline{v} \neq \overline{v}$ ie) $3 \neq -1$.

The given game doesnot have any saddle point.

Some of the entries in the Payoff matrix are negative

so we add a suitable Constant say $c = 4$ to the each element.

ie)
$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 5 & 3 & 7 \\ A_2 & 7 & 9 & 1 \\ A_3 & 10 & 6 & 2 \end{array}$$

The optimum strategies for two Players are

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix} \; \& \; S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$$

where $p_1 + p_2 + p_3 = 1$, $q_1 + q_2 + q_3 = 1$.

The LPP for player A is

Maximize v = Minimize $\frac{1}{v} = x_1 + x_2 + x_3$

s.t.c.

$$5x_1 + 7x_2 + 10x_3 \geq 1$$
$$3x_1 + 9x_2 + 6x_3 \geq 1$$
$$7x_1 + x_2 + 2x_3 \geq 1$$
$$x_1, x_2, x_3 \geq 0.$$

The LPP of player B is

Minimize $V =$ Maximize $\frac{1}{V} = y_1 + y_2 + y_3$

s.t.c.

$5y_1 + 3y_2 + 7y_3 \leq 1$

$7y_1 + 9y_2 + y_3 \leq 1$

$10y_1 + 6y_2 + 2y_3 \leq 1$

$y_1, y_2, y_3 \geq 0.$

We have to simplex method to solve the player B.

Intoduce the slack variables

s_1, s_2 & $s_3 \geq 0$

$5y_1 + 3y_2 + 7y_3 + s_1 = 1$

$7y_1 + 9y_2 + y_3 + s_2 = 1$

$10y_1 + 6y_2 + 2y_3 + s_3 = 1.$

The New objective function is

$\text{Max } \frac{1}{V} = y_1 + y_2 + y_3 + 0s_1 + 0s_2 + 0s_3.$

Initial Iteration Table

	C_B	0	1	1	1	0	0	0	
C_B	Y_B	X_B	y_1	y_2	$\boxed{y_3}$	s_1	s_2	s_3	Ratio
0	s_1	1	5	3	$\boxed{7}$	1	0	0	$1/7 = 0.14$
0	s_2	1	7	9	1	0	1	0	$1/1 = 1$
0	s_3	1	10	6	2	0	0	1	$1/2 = 0.5$
	z_j	0	−1	−1	−1	0	0	0	

s_1 leave the basis, y_3 enter the basis

7 is a leading element

Use duality,

The optimum strategie for player A is

$p_1^0 = y_4 = 2/15$; $p_2^0 = y_5 = 1/15$; $p_3^0 = y_6 = 0$.

$p_1 = p_1^0 \times v = 2/15 \times 5 = 2/3$ $p_2 = p_2^0 \times v = 1/15 \times 5 = 1/3$

$p_3 = 0$.

The optimum strategie for player A

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ 2/3 & 1/3 & 0 \end{bmatrix}$$

The optimum strategie for player B

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

The value of the game $v=1$.