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DEPARTMENT OF MATHEMATICS

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UNIT – V

- **LAPLACE TRANSFORMS**
- **INVERSE LAPLACE TRANSFORMS**

LAPLACE TRANSFORMS

DEFINITION:

Let f(t) be a function of the variable t which is defined for all positive values of t. Let s be a real constant. If the integral $\int_0^\infty e^{-st}$ f(t) dt exists and is equal to F(s) then F(s) is called the **Laplace Transform of f(t)** and it is denoted by **L{f(t)}.**

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\{f(t)\} = F(s)$$

IMPORTANT RESULTS:

RESULT 1:

Prove that L[e^{at}] = $\frac{1}{s-a}$, provided s – a > 0.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{e^{\infty}}{-(s-a)} + \frac{e^{0}}{-(s-a)} \qquad (\because e^{-\infty} = 0, e^0 = 1)$$

$$= \frac{1}{(s-a)}$$

$$\therefore L[e^{at}] = \frac{1}{(s-a)}$$

RESULT 2:

Prove that L[e^{-at}] = $\frac{1}{s+a}$, provided s – a > 0.

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[e^{-at}] = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\,dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{e^{\infty}}{-(s+a)} + \frac{e^{0}}{-(s+a)} \qquad (\because e^{-\infty} = 0, e^0 = 1)$$

$$= \frac{1}{(s+a)}$$

$$\therefore\ L[e^{-at}] = \frac{1}{(s+a)}$$

RESULT 3:

Prove that L[cos h at] $= \frac{\boldsymbol{s}}{(\boldsymbol{s^2 - a^2})}$

Proof:

$$L[\cos h\ at] = \left[L\frac{\boldsymbol{e^{at} + e^{-at}}}{(\boldsymbol{2})}\right]$$

$$= \frac{1}{2}\{L(e^{at}) + L(e^{-at})\}$$

$$= \frac{1}{2}\left[\frac{1}{(s+a)} + \frac{1}{(s+a)}\right] = \frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\ \frac{2s}{s^2-a^2}$$

$$= \frac{s}{(s^2-a^2)}$$

$$\therefore\ L[\cos h\ at] = \frac{s}{(s^2-a^2)}$$

RESULT 4:

Prove that L[sin h at] $= \frac{\boldsymbol{a}}{(\boldsymbol{s^2 - a^2})}$

Proof:

$$L[\sin h\ at] = \left[L\frac{\boldsymbol{e^{at} - e^{-at}}}{(\boldsymbol{2})}\right]$$

$$= \frac{1}{2}\{L(e^{at}) - L(e^{-at})\}$$

$$= \frac{1}{2}\left[\frac{1}{(s+a)} - \frac{1}{(s+a)}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2} \quad \frac{2a}{s^2-a^2}$$

$$= \frac{a}{(s^2-a^2)}$$

$$\therefore \text{ L[sin h at] } = \frac{a}{(s^2-a^2)}$$

RESULT 5:

Prove that L[cos at] $= \frac{s}{(s^2+a^2)}$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\{\cos at\} = \int_0^\infty e^{-st} \cos at\, dt$$

$$= \left[\frac{e^{-st}(-s\cos at + a\sin at)}{(s^2+a^2)}\right]_0^\infty$$

$$\left[\int e^{ax} \cos bx\, dx \quad = \frac{e^{ax}}{(a^2+b^2)}(a \cos bx + b \sin bx)\right]$$

$$= \frac{e^{-\infty}}{(s^2+a^2)} - \frac{e^0(-s)}{(s^2+a^2)} \quad = \quad 0 + \frac{s}{(s^2+a^2)} \qquad (\because e^{-\infty} = 0, e^0 = 1)$$

$$= \frac{s}{(s^2+a^2)}$$

$$\therefore \text{ L[cos at] } = \frac{s}{(s^2+a^2)}$$

RESULT 6:

Prove that L[sin at] $= \frac{a}{(s^2+a^2)}$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt$$

$$= \left[\frac{e^{-st}(-s\sin at - a\cos at)}{(s^2+a^2)}\right]_0^\infty$$

$$\left[\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{(a^2+b^2)} (a \sin bx - b \cos bx) \right]$$

$$= \frac{e^{-\infty}}{(s^2+a^2)} - \frac{e^{0}(-a)}{(s^2+a^2)} = 0 + \frac{a}{(s^2+a^2)} \qquad (\because e^{-\infty} = 0, e^{0} = 1)$$

$$= \frac{a}{(s^2+a^2)}$$

$$\therefore \; L[\sin at] = \frac{a}{(s^2+a^2)}$$

RESULT 7:

Prove that L[1] = $\frac{1}{s}$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L(1) = L(e^{0t})$$

$$L\{e^{0t}\} = \int_0^\infty e^{-st} e^{0t}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{e^{-\infty} - e^{0}}{-s}$$

$$= \frac{1}{s}$$

$$\therefore \; L[1] = \frac{1}{s}$$

LAPLACE TRANSFORMS OF DERIVATIVES:

THEOREM :

$$L[f'(t)] = sL[f(t)] - f(0)$$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt = \int_0^\infty e^{-st} d[f(t)]\, dt$$

By using Integration by parts,

Take $u = e^{-st}$ $\quad \int f'(t) dt = dv$

$du = -se^{-st}\, dt$ $\quad f(t) = v$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)\, d[e^{-st}]\, dt$$

$= 0 - e^0 \; f(0) - \int_0^\infty f(t)\,[e^{-st}]\,[-s]\,dt$

$= - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$

$\therefore \quad L[f'(t)] = sL[f(t)] - f(0)$

Similarly,

$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$

In general

$L[f^n(t)] = s^n L[f(t)] - s^{n-1}[f(0)] - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots\ldots - f^{n-1}(0)$

LINEARITY PROPERTY:

If c_1 and c_2 are constants and $f_1(t)$ and $f_2(t)$ are given functions, then

$L[c_1 f_1(t) + c_2 f_2(t)] = c_1 L[f_1(t)] + c_2 L[f_2(t)]$

Proof:

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$L[c_1 f_1(t) + c_2 f_2(t)] = \int_0^\infty e^{-st}\{c_1 f_1(t) + c_2 f_2(t)\}\,dt$

$= \int_0^\infty \{c_1 e^{-st} f_1(t) + c_2 e^{-st} f_2(t)\}\,dt$

$= c_1 \int_0^\infty e^{-st} f_1(t)\,dt + c_2 \int_0^\infty e^{-st} f_2(t)\,dt$

$= c_1 L[f_1(t)] + c_2 L[f_2(t)]$

$\therefore L[c_1 f_1(t) + c_2 f_2(t)] = c_1 L[f_1(t)] + c_2 L[f_2(t)]$

WORKED EXAMPLES:

1. Find $L(e^{2t} + 3e^{-5t})$

Solution:

$L[e^{at}] = \frac{1}{(s-a)}$

$L(e^{2t} + 3e^{-5t}) = L(e^{2t}) + L(3e^{-5t})$

$= \frac{1}{(s-2)} + 3\,\frac{1}{(s+5)}$

$\therefore L(e^{2t} + 3e^{-5t}) = \frac{1}{(s-2)} + \frac{3}{(s+5)}$

2. Find L(sin h 6t + 3e^{-5t} + cos 5t)

Solution:

$$L[e^{-at}] = \frac{1}{(s+a)}, \quad L[\sin h\ at] = \frac{a}{(s^2-a^2)}, \quad L[\cos\ at] = \frac{s}{(s^2+a^2)}$$

$L(\sin h\ 6t + 3e^{-5t} + \cos 5t) = L(\sin h\ 6t) + L(3e^{-5t}) + L(\cos 5t)$

$= L(\sin h\ 6t) + 3\,L(e^{-5t}) + L(\cos 5t)$

$= \frac{6}{(s^2-6^2)} + 3\,\frac{1}{(s+5)} + \frac{5}{(s^2+5^2)}$

$= \frac{6}{(s^2-36)} + 3\,\frac{1}{(s+5)} + \frac{5}{(s^2+25)}$

$\therefore L(\sin h\ 6t + 3e^{-5t} + \cos 5t) = \frac{6}{(s^2-36)} + 3\,\frac{1}{(s+5)} + \frac{5}{(s^2+25)}$

3. Find L($\boldsymbol{sin^2}$2t)

Solution:

$$sin^2\ x = \frac{1-\cos 2x}{2}$$

$L(sin^2 2t) = L\left[\frac{1-\cos 2x}{2}\right]$

$= L\left(\frac{1}{2}\right) - L\left[\frac{\cos 4t}{2}\right]$

$= \frac{1}{2}\,L(1) - \frac{1}{2}\,L(\cos 4t) = \frac{1}{2s} - \frac{1}{2}\,\frac{s}{(s^2+4^2)}$

$\therefore\ L(sin^2 2t) = \frac{1}{2s} - \frac{1}{2}\,\frac{s}{(s^2+4^2)}$

4. Find L($\boldsymbol{cos^3}$3t)

Solution:

$$cos^3\ x = \frac{\cos 3x + 3\ cos x}{4}$$

$L(cos^3 3t) = L\left[\frac{\cos 9t + 3\ cos 3t}{4}\right] = \frac{1}{4}\left[L(\cos 9t) + L(3\cos 3t)\right]$

$= \frac{1}{4}\left[\frac{s}{(s^2+9^2)} + \frac{3s}{(s^2+3^2)}\right] = \frac{s}{4}\left[\frac{1}{(s^2+81)} + \frac{3}{(s^2+9)}\right]$

$\therefore\ L(cos^3 3t) = \frac{s}{4}\left[\frac{1}{(s^2+81)} + \frac{3}{(s^2+9)}\right]$

5. Find L[sin (ωt+ α)] , α is a constant

Solution:

$\sin(\omega t + \alpha) = \sin \omega t \cos \alpha + \cos \omega t \sin \alpha$

$L[\sin(\omega t + \alpha)] = L[\sin \omega t \cos \alpha + \cos \omega t \sin \alpha]$

$= L[\sin \omega t \cos \alpha] + L[\cos \omega t \sin \alpha]$

$= \cos \alpha\, L[\sin \omega t] + \sin \alpha\, L[\cos \omega t]$ [$\because \sin \alpha$, $\cos \alpha$ are constant]

$= \cos \alpha \frac{\omega}{(s^2+\omega^2)} + \sin \alpha \frac{s}{(s^2+\omega^2)}$

$\therefore L[\sin(\omega t + \alpha)] = \cos \alpha \frac{\omega}{(s^2+\omega^2)} + \sin \alpha \frac{s}{(s^2+\omega^2)}$

RESULT 8:

Prove that $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ or $L[t^n] = \frac{n!}{s^{n+1}}$

Proof:

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$L\{t^n\} = \int_0^\infty e^{-st} t^n\, dt$

Put $st = x$ when $t = 0, x = 0$

$s\,dt = dx$ when $t = \infty, x = \infty$

$L\{t^n\} = \int_0^\infty e^{-st} \left(\frac{x}{s}\right)^n \frac{1}{s}\, dt$

$= \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x}\, dx$

$\therefore L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ $\left[\because \int_0^\infty x^n e^{-x}\, dx = \Gamma(n+1)\right]$

When n is appositive integer, then $\Gamma(n+1) = n!$

$\therefore L[t^n] = \frac{n!}{s^{n+1}}$

Corollary:

(i) $L(1) = [L[t^o] = \frac{0!}{s^{n+1}} = \frac{1}{s}$

$\therefore L(1) = \frac{1}{s}$

(ii) L(t) $= \frac{1}{s^2}$

(iii) L$(t^2) = \frac{2}{s^3}$

(iv) L$(t^3) = \frac{6}{s^4}$

(v) L$(\sqrt{t})$ = L$(t^{\frac{1}{2}}) = \frac{\Gamma(\frac{1}{2}+1)}{s^{\frac{1}{2}+1}}$ [Here n is not an integer]

$= \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{\frac{3}{2}}} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$ $[\because \Gamma\frac{1}{2} = \sqrt{\pi}]$

L$(\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$

(vi) L$(t^{\frac{-1}{2}}) = \frac{\Gamma(\frac{-1}{2}+1)}{s^{\frac{-1}{2}+1}}$ [Here n is not an integer]

$= \frac{\Gamma\frac{1}{2}}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{s^{\frac{1}{2}}}$

L$(t^{\frac{-1}{2}}) = \sqrt{\frac{\pi}{s}}$

WORKED EXAMPLES:

1.Find L$\left(a + bt + \frac{c}{\sqrt{t}}\right)$

Solution:

L$\left(a + bt + \frac{c}{\sqrt{t}}\right)$ = L(a) + L(bt) + L $\left(\frac{c}{\sqrt{t}}\right)$

= a L(1) + b L(t) + c L$\left(t^{\frac{-1}{2}}\right)$

$= \frac{a}{s} + \frac{b}{s^2} + \sqrt{\frac{\pi}{s}}$

$\therefore$ L$\left(a + bt + \frac{c}{\sqrt{t}}\right) = \frac{a}{s} + \frac{b}{s^2} + \sqrt{\frac{\pi}{s}}$

2. Find L (5 – 3t – 2e^{-t})

Solution:

L (5 – 3t – 2e^{-t}) = L (5) – L(3t) – L(2e^{-t})

= 5 L (1) – 3L(t) – 2L(e^{-t})

$= \frac{5}{s} - \frac{3}{s^2} - \frac{2}{s+1}$

$= \frac{3s^2+2s-3}{s^2(s+1)}$

∴ L (5 – 3t – 2e^{-t}) = $\frac{3s^2+2s-3}{s^2(s+1)}$

3. Find L [$(t+1)^2$]

Solution:

L [$(t+1)^2$ = L[t^2+2t +1]

= L(t^2) + 2 L(t) + L(1)

$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$

$= \frac{2+2s+s^2}{s^3}$

∴ L [$(t+1)^2$ = $\frac{2+2s+s^2}{s^3}$

3. Find L [5e^{8t} + cos h 3t + sin 5t]

Solution:

L [5e^{8t} + cos h 3t + sin 5t] = L [5e^{8t}] + L[cos h 3t] + L[sin 5t]

$= \frac{5}{s-8} + \frac{s}{s^2-9} + \frac{5}{s^2+25}$

∴ L [5e^{8t} + cos h 3t + sin 5t] = $\frac{5}{s-8} + \frac{s}{s^2-9} + \frac{5}{s^2+25}$

4. Find L[cos h at + sin 2t +t^3]

Solution:

L[cos h at + sin 2t +t^3] = L[cos h at]+ L[sin 2t] +L[t^3]

$= \frac{s}{s^2-a^2} + \frac{2}{s^2+4} + \frac{3!}{s^4}$

∴ L[cos h at + sin 2t +t^3] = $\frac{s}{s^2-a^2} + \frac{2}{s^2+4} + \frac{3!}{s^4}$

FIRST SHIFTING THEOREM (FIRST TRANSLATION)

If L{f(t)} = F(s), then L[$e^{at}f(t)$] = F(s – a)

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$L[e^{at}f(t)] = \int_0^\infty e^{-st}[e^{at}f(t)]\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt, \quad s - a > 0$$

$$= F(s - a)$$

$\therefore L[e^{at}f(t)] = F(s - a)$ where F(s) = L[f(t)]

Corollary: $L[e^{-at}f(t)] = F(s + a)$ where F(s) = L[f(t)]

THE UNIT STEP FUNCTION (OR) HEAVISIDE'S UNIT FUNCTION:

This function is denoted by H(t) and is defined as

$$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

We have also $H(t-a) = \begin{cases} 1 & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$ where a > 0

SECOND SHIFTING THEOREM (SECOND TRANSLATION)

If L[f(t)] = F(s) and $G(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$ **then L[G(t)] =** e^{-as} **F(s)**

Solution:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{G(t)\} = \int_0^\infty e^{-st} G(t)\,dt$$

$$= \int_0^a e^{-st} G(t)\,dt + \int_a^\infty e^{-st} G(t)\,dt$$

$$= \int_0^a e^{-st}\, 0\,dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t - a = u$ when t = a, u = 0

$dt = du$ when $t = \infty$, $u = \infty$

$$L\{G(t)\} = \int_0^\infty e^{-s(u+a)} f(u)\, du$$

$$= e^{-sa} \int_0^\infty e^{-su} f(u)\, du$$

In $\int_0^\infty e^{-su} f(u)\, du$, u is dummy variable. Hence we can replace it by the variable t.

$$L\{G(t)\} = e^{-sa} \int_0^\infty e^{-su} f(t)\, dt$$

$$= e^{-sa} L[f(t)]$$

$$= e^{-sa} F(s)$$

$$\therefore L[G(t)] = e^{-as} F(s)$$

WORKED EXAMPLES:

1. Find L [$e^{-3t} sin^2 t$]

Solution:

We know that L[$e^{-at} f(t)$] = F(s + a), where F(s) = L[f(t)]

$$f(t) = sin^2 t$$

$$L[f(t)] = L[sin^2 t]$$

$$L[sin^2 t] = L\left[\frac{1 - cos2t}{2}\right]$$

$$= \frac{1}{2} [L(1) – L(\cos 2t)]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$\therefore L [e^{-3t} sin^2 t] = \frac{1}{2}\left[\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right]$$

2. Find L[(cos h 2t + $\frac{1}{2}$ sin h 2t)]

Solution:

We know that L[$e^{-at} f(t)$] = F(s + a), where F(s) = L[f(t)]

$$f(t) = \cos h\, 2t + \frac{1}{2} \sin h\, 2t$$

$$L[f(t)] = L[\cos h\, 2t + \frac{1}{2} \sin h\, 2t]$$

$$= L[\cos h\, 2t] + \frac{1}{2} L[\sin h\, 2t]$$

$$= \frac{s}{s^2 - 4} + \frac{1}{2}\, \frac{2}{s^2 - 4}$$

$$\therefore L[(\cos h\, 2t + \frac{1}{2} \sin h\, 2t)] = \frac{s}{s^2 - 4} + \frac{1}{2}\, \frac{2}{s^2 - 4}$$

RESULT: CHANGE OF SCALE PROPERTY

If L[f(t)] = F(s), then L[f(at)] = $\frac{1}{a}$ F$\left(\frac{s}{a}\right)$

Proof:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\, dt$$

Put at = x when t = 0, x = 0

a dt = dx when t = ∞, x = ∞

$$L\{f(at)\} = \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x)\, dx$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t} f(t)\, dt \qquad [\because \text{ x is dummy variable}]$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$\therefore \quad L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

WORKED EXAMPLES:

1. Find L[f(t)], where f(t) = $\begin{cases} \mathbf{0, \text{ when } 0 < t < 2} \\ \mathbf{3, \text{ when } t > 2} \end{cases}$

Solution:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^2 e^{-st} f(t)\, dt + \int_2^\infty e^{-st} f(t)\, dt$$

$$= \int_0^2 e^{-st}\, 0\, dt + \int_2^\infty e^{-st}\, 3\, dt \qquad [\because f(t) = 0 \text{ in } (0, 2), f(t) = 3 \text{ in } (2, \infty)]$$

$$= 3\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$= 3\left[\frac{e^{-\infty} - e^{-2s}}{-s}\right]$$

$$= \frac{3e^{-2s}}{s}$$

$$\therefore \quad L\{f(t)\} = \frac{3e^{-2s}}{s}$$

2. Find L[f(t)], where f(t) = $\begin{cases} e^t, \text{ when } 0 < t < 2 \\ 3, \text{ when } t > 2 \end{cases}$

Solution:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} f(t)\, dt + \int_1^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} f(t)\, dt + \int_1^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} \boldsymbol{e^t}\, dt + \int_1^\infty e^{-st}\, 0\, dt$$

$$= \int_0^1 e^{-(s-1)t}\, dt \quad = \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^1$$

$$= \frac{e^{-(s-1)}}{-(s-1)} + \frac{e^0}{(s-1)}$$

$$= \frac{1 - e^{-(s-1)}}{(s-1)}$$

$\therefore \quad L\{f(t)\} = \dfrac{1 - e^{-(s-1)}}{(s-1)}$

THEOREM:

If L[f(t)] = F(s), then L{tf(t)} = $\boldsymbol{\frac{-d}{ds}}$ **F(s)**

Proof:

L[f(t)] = F(s)

F(s) = L[f(t)]

Taking derivatives on both sides with respect to s, we get

$$\frac{-d}{ds} F(s) = \frac{-d}{ds} L[f(t)]$$

$$= \frac{-d}{ds}\left\{\int_0^\infty e^{-st} f(t)\, dt\right\} \quad = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\right] dt$$

$$= \int_0^\infty -t\, e^{-st} f(t)\, dt$$

$$= -\int_0^\infty e^{-st}\, [t\, f(t)]\, dt$$

$$\frac{d}{ds} F(s) = -L[t\, f(t)]$$

$$\therefore L[t\, f(t)] = \frac{-d}{ds} F(s)$$

(or) $\quad \therefore L[t\, f(t)] = -F'(s)$

Corollary:

We have $L[tf(t)] = \frac{-d}{ds} F(s)$

$L[t^2 f(t)] = L[t.tf(t)]$

$= \frac{-d}{ds} L[t.f(t)]$

$= \frac{-d}{ds} \frac{-d}{ds} L[f(t)]$

$\therefore \quad L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$

In general, we have

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

WORKED EXAMPLES:

1. Find L[t sin 2t]

Solution:

We know that $L[t f(t)] = \frac{-d}{ds} F(s)$ where $F(s) = L[f(t)]$

Here f(t) = sin 2t.

$$F(s) = L[f(t)] = L[\sin 2t] = \frac{2}{s^2+4}$$

$$F(s) = \frac{2}{s^2+4}$$

$$L[t \sin 2t] = \frac{-d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= - \left[\frac{0-2\,.\,2s}{(s^2+4)^2}\right]$$

$$= \frac{4s}{(s^2+4)^2}$$

$$\therefore L[t \sin 2t] = \frac{4s}{(s^2+4)^2}$$

2. Find L $[\boldsymbol{t^2 e^{-3t}}]$

Solution:

We know that $L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$ where $F(s) = L[f(t)]$. Here $f(t) = e^{-3t}$

$$L[t^2 e^{-3t}] = (-1)^2 \frac{d^2}{ds^2} L[e^{-3t}] = \frac{d^2}{ds^2} \frac{1}{s+3}$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{s+3}\right)\right]$$

$$= \frac{d}{ds}\left[\left(\frac{0-1}{(s+3)^2}\right)\right]$$

$$= \frac{0+2(s+3)}{(s+3)^4}$$

$$= \frac{2}{(s+3)^3}$$

$$\therefore \quad L[t^2 e^{-3t}] = \frac{2}{(s+3)^3}$$

3. Find L[t cos^3t]

Solution:

$$L[t\ cos^3 t] = L\left[t\left(\frac{\cos 3t + 3\cos t}{4}\right)\right]$$

$$= \frac{1}{4}\{L[t \cos 3t] + 3L[t \cos t]\}$$

$$= \frac{1}{4}\left\{\frac{-d}{ds} L[\cos 3t] - 3\frac{d}{ds} L[\cos t]\right\}$$

$$= \frac{1}{4}\left\{\frac{-d}{ds}\left(\frac{s}{s^2+9}\right) - 3\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right\}$$

$$= \frac{-1}{4}\left\{\frac{(s^2+9)-s.2s}{(s^2+9)^2}\right\} - \frac{3}{4}\left\{\frac{(s^2+1)-s.2s}{(s^2+1)^2}\right\}$$

$$= \frac{-1}{4}\left\{\frac{-s^2+9}{(s^2+9)^2}\right\} - \frac{3}{4}\left\{\frac{-s^2+1}{(s^2+1)^2}\right\}$$

$$= \frac{1}{4}\left\{\frac{s^2+9}{(s^2+9)^2}\right\} + \frac{3}{4}\left\{\frac{s^2+1}{(s^2+1)^2}\right\}$$

$$\therefore L[t\ cos^3 t] = \frac{1}{4}\left\{\frac{s^2+9}{(s^2+9)^2}\right\} + \frac{3}{4}\left\{\frac{s^2+1}{(s^2+1)^2}\right\}$$

4. Find L[$t^2 e^t\ sin\ t$]

Solution:

We know that $L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$ where F(s) = L[f(t)].

$$L[t^2 e^t\ sin\ t] = (-1)^2 \frac{d^2}{ds^2} L[e^t\ sin\ t] \qquad \textbf{(1)}$$

Now, $L[e^t\ sin\ t] = \frac{1}{(s-1)^2+1}$ —————— (2)

Substituting (2) in (1), we get

$$L[t^2 e^t \sin t\,] = \frac{d^2}{ds^2}\left[\frac{1}{(s-1)^2+1}\right]$$

$$= \frac{-d}{ds}\left[\frac{0-2\,(s-1)}{((s-1)^2+1)^2}\right]$$

$$= \frac{-d}{ds}\left[\frac{-2\,(s-1)}{(s^2-2s\,+2)^2}\right]$$

$$= \frac{\left(s^2-2s\,+2\right)^2(-2)+2\,(s-1)2(s^2-2s\,+2)(2s-2)}{(s^2-2s\,+2)^4}$$

$$= \frac{2\left(s^2-2s\,+2\right)\left[-\left(s^2-2s\,+2\right)+4(s-1)^2\right]}{(s^2-2s\,+2)^4}$$

$$= \frac{2\left(s^2-2s\,+2\right)\left[-s^2+2s-2+4s^2+4-8s\right]}{(s^2-2s\,+2)^4}$$

$$= \frac{2\left(s^2-2s\,+2\right)\left[3s^2-6s+2\right]}{(s^2-2s\,+2)^4}$$

$$= \frac{2\left[3s^2-6s+2\right]}{(s^2-2s\,+2)^3}$$

$$\therefore\ L[t^2 e^t \sin t\,] = \frac{2\left[3s^2-6s+2\right]}{(s^2-2s\,+2)^3}$$

THEOREM:

If L[f(t)] = F(s) and if $\boldsymbol{\frac{f(t)}{t}}$ has a limit as t → 0, then L $\left[\boldsymbol{\frac{f(t)}{t}}\right] = \int_s^\infty \boldsymbol{F(s)}$ ds

Proof:

Given F(s) = L{f(t)} = $\int_0^\infty e^{-st} f(t)\,dt$

$$\int_s^\infty F(s)\,ds = \int_s^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds$$

$$= \int_s^\infty ds \int_0^\infty e^{-st} f(t)\,dt$$

[∵ s and t are independent variables and hence the order of integration in the double integral can be interchanged]

$$\int_s^\infty F(s)\,ds = \int_0^\infty dt \int_s^\infty e^{-st} f(t)$$

$$= \int_0^\infty f(t)dt \int_s^\infty e^{-st}\,ds$$

$$= \int_0^\infty f(t)dt \left[\frac{e^{-st}}{-t}\right]_s^\infty$$

$$= \int_0^\infty \frac{e^{-st}}{-t} f(t)dt$$

$$= \int_0^\infty e^{-st} \left[\frac{f(t)}{t}\right] dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

$$\therefore \; L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds \text{ ,F(s) = L[f(t)]}$$

WORKED EXAMPLES:

1. Find L $\left[\frac{\sin at}{t}\right]$. Hence show that $\int_0^\infty \frac{\sin t}{t}$ dt = $\frac{\pi}{2}$

Solution:

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty L[sinat]\, ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2} ds$$

$$= a \int_s^\infty \frac{1}{s^2+a^2} ds$$

$$= a \left[\frac{1}{a}\, tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty$$

$$= \tan^{-1}\infty - tan^{-1}\left(\frac{s}{a}\right)$$

$$= \frac{\pi}{2} - tan^{-1}\left(\frac{s}{a}\right)$$

$$= cot^{-1}\left(\frac{s}{a}\right)$$

$$\therefore \quad L\left[\frac{\sin at}{t}\right] = tan^{-1}\left(\frac{a}{s}\right)$$

Deduction:

We have $L\left[\frac{\sin at}{t}\right] = tan^{-1}\left(\frac{a}{s}\right)$

Putting a = 1, and s → 0, we get

$$\int_0^\infty \frac{\sin t}{t} dt = \tan^{-1}(\infty)$$

$$= \frac{\pi}{2}$$

$$\therefore \int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

2. Find L $\left[\frac{1- e^t}{t}\right]$

Solution:

$$L\left[\frac{1- e^t}{t}\right] = \int_s^\infty L[1 - e^t]\, ds$$

$$= \int_s^\infty L[1] - L[e^t]\, ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{1}{s-1}\right] ds$$

$$= [\log s - \log(s-1)]_s^\infty$$

$$= \log\left[\frac{s}{s-1}\right]_s^\infty$$

$$= \log\left[\frac{1}{1-\frac{1}{s}}\right]_s^\infty$$

$$= \log 1 - \log \frac{1}{1-\frac{1}{s}}$$

$$= 0 - \log \frac{s}{s-1}$$

$$= \log\left[\frac{s}{s-1}\right]^{-1}$$

$$= \log\left[\frac{s-1}{s}\right]$$

$$\therefore\ L\left[\frac{1- e^t}{t}\right] = \log\left[\frac{s-1}{s}\right]$$

3. Find L $\left[\frac{e^{at} - \cos 6t}{t}\right]$

Solution:

$$L\left[\frac{e^{at} - \cos 6t}{t}\right] = \int_s^\infty L[e^{at} - \cos 6t]\, ds$$

$$= \int_s^\infty \{L[e^{at}] - L[\cos 6t]\}\, ds$$

$$= \int_s^\infty \left[\frac{1}{s-a} - \frac{s}{s^2+36}\right] ds$$

$$= \left[\log(s-a) - \frac{1}{2}\log(s^2+36)\right]_s^\infty$$

$$= \left[\log(s-a) - \log(s^2+36)^{\frac{1}{2}}\right]_s^\infty$$

$$= \left[\log \frac{(s-a)}{\sqrt{s^2+36}}\right]_s^\infty$$

$$= \left[\log \frac{1-\frac{a}{s}}{\sqrt{1+\frac{36}{s^2}}}\right]_s^\infty$$

$$= \log 1 - \log \left[\frac{1-\frac{a}{s}}{\sqrt{1+\frac{36}{s^2}}}\right]$$

$$= -\log \left[\frac{(s-a)}{\sqrt{s^2+36}}\right]$$

$$= \log \left[\frac{(s-a)}{\sqrt{s^2+36}}\right]^{-1}$$

$$= \log \left[\frac{\sqrt{s^2+36}}{(s-a)}\right]$$

$$\therefore \quad L\left[\frac{e^{at}-\cos 6t}{t}\right] = \log \left[\frac{\sqrt{s^2+36}}{(s-a)}\right]$$

4. Find L $\left[\frac{\cos 4t \sin 2t}{t}\right]$

Solution:

$$L\left[\frac{\cos 4t \sin 2t}{t}\right] = \int_s^\infty L[\cos 4t \sin 2t]\, ds$$

$$= \int_s^\infty L\left[\frac{\sin 6t - \sin 2t}{2}\right] ds$$

$$= \frac{1}{2}\int_s^\infty L[\sin 6t] - L[\sin 2t]\, ds$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{6}{s^2+36} - \frac{2}{s^2+4}\right] ds$$

$$= \frac{1}{2}\left[6\frac{1}{6}\tan^{-1}\left(\frac{s}{6}\right) - 2\frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\tan^{-1}(\infty) - \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{6}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{6}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{2}\right) - \tan^{-1}\left(\frac{s}{6}\right)\right]$$

$$\therefore L\left[\frac{\cos 4t \sin 2t}{t}\right] = \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{2}\right) - \tan^{-1}\left(\frac{s}{6}\right)\right]$$

INVERSE LAPLACE TRANSFORM

DEFINITION:

If the Laplace Transform of a function f(t) is F(s) that is L[f(t)] = F(s) then f(t) is called an **Inverse Laplace Transform of F(s)** and it is denoted by

$$f(t) = L^{-1}[F(s)]$$

Here L^{-1} is called the **Inverse Laplace Transform Operator.**

Thus if $\qquad L[e^{at}] = \frac{1}{s-a}$

Then $\qquad L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

Some Standard Inverse Laplace Transform as follows:

1.	$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$	$L[e^{at}] = \frac{1}{s-a}$
2.	$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$	$L[e^{-at}] = \frac{1}{s+a}$
3.	$L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$	$L[\sin at] = \frac{a}{s^2+a^2}$
4.	$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$	$L[\cos at] = \frac{s}{s^2+a^2}$
5.	$L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sin h\ at$	$L[\sin h\ at] = \frac{a}{s^2-a^2}$
6.	$L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cos h\ at$	$L[\cos h\ at] = \frac{s}{s^2-a^2}$
7.	$L^{-1}\left[\frac{1}{s}\right] = 1$	$L[1] = \frac{1}{s}$
8.	$L^{-1}\left[\frac{1}{s^2}\right] = t$	$L\ [t] = \frac{1}{s^2}$
9.	$L^{-1}\left[\frac{n!}{s^2}\right] = t^n$	$L[t^n] = \frac{n!}{s^2}$
10	$L^{-1}\left[\frac{1}{(s-a)^2}\right] = te^{at}$	$L[te^{at}] = \frac{1}{(s-a)^2}$

LINEAR PROPERTY:

If F_1(s) and F_2(s) are Laplace Transforms of f_1(t) and f_2(t) respectively, then

$$L^{-1}[\, c_1F_1(s) + c_2F_2(s)\,] = c_1\, L^{-1}[F_1(s)] + c_2\, L^{-1}[F_2(s)]$$

Where c_1 and c_2 are constants.

Proof:

We know that $L[c_1f_1(t) + c_2f_2(t)] = c_1\, L\,[f_1(t)] + c_2\, L\,[f_2(t)]$

$$L[c_1f_1(t) + c_2f_2(t)] = c_1\, F_1(s) + c_2\, F_2(s)$$

$\{\because L[f_1(t)] = F_1(s)$ and $L[f_1(t)] = F_2(s)\}$

$$c_1f_1(t) + c_2f_2(t) = L^{-1}[c_1\, F_1(s) + c_2\, F_2(s)]$$

$$= c_1\, L^{-1}[F_1(s)] + c_2\, L^{-1}[F_2(s)]$$

$$\therefore\ L^{-1}[\, c_1F_1(s) + c_2F_2(s)\,] = c_1\, L^{-1}[F_1(s)] + c_2\, L^{-1}[F_2(s)]$$

WORKED EXAMPLES:

1. Find $L^{-1}\left[\frac{1}{s-3} + \frac{1}{s} + \frac{s}{s^2-4}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s-3} + \frac{1}{s} + \frac{s}{s^2-4}\right] = L^{-1}\left[\frac{1}{s-3}\right] + L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{s}{s^2-4}\right]$$

$$= e^{3t} + 1 + \cos h\ 2t$$

$$= e^{3t} + \cos h\ 2t + 1$$

$$\therefore\ L^{-1}\left[\frac{1}{s-3} + \frac{1}{s} + \frac{s}{s^2-4}\right] = e^{3t} + \cos h\ 2t + 1$$

2. Find $L^{-1}\left[\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}\right]$$

$$= L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s+4}\right] + L^{-1}\left[\frac{1}{s^2+4}\right] + L^{-1}\left[\frac{s}{s^2-9}\right]$$

$$= t + e^{-4t} + \frac{\sin 2t}{2} + \cos h\ 3t$$

$$\therefore\ L^{-1}\left[\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}\right] = t + e^{-4t} + \frac{\sin 2t}{2} + \cos h\ 3t$$

FIRST SHIFTING PROPERTY:

RESULT 1:

We know that if L[f(t)] = F(s), then

$$L[e^{-at} f(t)] = F(s + a)$$

$$L^{-1}[F(s + a)] = e^{-at} f(t)$$

$$= e^{-at} L^{-1}[F(s)]$$

WORKED EXAMPLES:

1. Find $\boldsymbol{L^{-1}\left[\frac{1}{(s+1)^2}\right]}$

Solution:

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-t} . t$$

$$\therefore \quad L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t} . t$$

2. Find $\boldsymbol{L^{-1}\left[\frac{s}{(s+2)^2+1}\right]}$

Solution:

$$L^{-1}\left[\frac{s}{(s+2)^2+1}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2+1}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{s}{s^2+1}\right] -2\, e^{-2t} L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-2t} \cos t - 2e^{-2t} \sin t$$

$$= e^{-2t} (\cos t - 2 \sin t)$$

$$\therefore L^{-1}\left[\frac{s}{(s+2)^2+1}\right] = e^{-2t} (\cos t - 2 \sin t)$$

3. Find $\boldsymbol{L^{-1}\left[\frac{1}{(s-4)^5}+\frac{5}{(s-2)^2+5^2}+\frac{s+3}{(s+3)^2-6^2}\right]}$

Solution:

$$L^{-1}\left[\frac{1}{(s-4)^5}+\frac{5}{(s-2)^2+5^2}+\frac{s+3}{(s+3)^2-6^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s-4)^5}\right]+L^{-1}\left[\frac{5}{(s-2)^2+5^2}\right]+L^{-1}\left[\frac{s+3}{(s+3)^2+6^2}\right]$$

$$= e^{4t}\,L^{-1}\left[\frac{1}{s^5}\right]+e^{-2t}\,L^{-1}\left[\frac{5}{s^2+5^2}\right]+e^{-3t}\,L^{-1}\left[\frac{s}{s^2+6^2}\right]$$

$$= e^{-2t}\,\frac{t^4}{4!} + e^{2t}\sin 5t + e^{-3t}\cos 6t$$

CHANGE OF SCALE PROPERTY:

If L[f(t)] = F(s), then $\boldsymbol{L^{-1}[F(as)] = \frac{1}{a}\,f\left(\frac{t}{a}\right)}$**, a > 0.**

Proof:

$$F(s) = L[f(t)]$$

$$= \int_0^{\infty} e^{-st}\, f(t)\, dt$$

$$F(as) = \int_0^{\infty} e^{-ast}\, f(t)\, dt$$

Put $at = t_1$ when $t = 0$, $t_1 = 0$

$dt = \frac{dt_1}{a}$ when $t = \infty$ $t_1 = \infty$

$$= \int_0^{\infty} e^{-st_1}\, f\left(\frac{t_1}{a}\right)\frac{dt_1}{a}$$

$$= \frac{1}{a}\int_0^{\infty} e^{-st_1}\, f\left(\frac{t_1}{a}\right) dt_1$$

$$= \frac{1}{a}\int_0^{\infty} e^{-st}\, f\left(\frac{t}{a}\right) dt$$

$$= \frac{1}{a}\, L\left[f\left(\frac{t}{a}\right)\right]$$

$$\therefore\ L^{-1}[F(as)] = \frac{1}{a}\, f\left(\frac{t}{a}\right)$$

WORKED EXAMPLE:

If $\boldsymbol{L^{-1}\left[\frac{s^2-1}{(s^2+1)^2}\right]}$ **= t cos t, then find** $\boldsymbol{L^{-1}\left[\frac{9s^2-1}{(9s^2+1)^2}\right]}$

Solution:

$$L^{-1}\left[\frac{s^2-1}{(s^2+1)^2}\right] = \text{t cos t}$$

Writing as for s,

$$L^{-1}\left[\frac{s^2-1}{(s^2+1)^2}\right] = \frac{1}{a}\,\frac{t}{a}\cos\left(\frac{t}{a}\right)$$

Putting a = 3,

$$L^{-1}\left[\frac{9s^2-1}{(9s^2+1)^2}\right] = \frac{1}{3}\,\frac{t}{3}\cos\left(\frac{t}{3}\right)$$

$\therefore$ $$L^{-1}\left[\frac{9s^2-1}{(9s^2+1)^2}\right] = \frac{t}{9}\cos\left(\frac{t}{3}\right)$$

RESULT:

We know that if L[f(t)] = F(s), then L[t f(t)] = $\frac{-d}{ds}$ F(s)

$$\therefore\ L^{-1}[F'(s)] = -t\ L^{-1}[F(s)]$$