Unit V

Laplace Transforms & Inverse Laplace Transforms:-

Definition:

If a function $f(t)$ is defined for all positive values of the variable t and if $\int_0^\infty e^{-st} f(t)\,dt$ exists and is equal to $F(S)$, then $F(S)$ is called the Laplace Transform of $f(t)$ and is denoted by $L\{f(t)\}$. (i.e) $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

Properties of Laplace Transforms:

(i) $L\{f(t)+\phi(t)\} = L\{f(t)\} + L\{\phi(t)\}$

Proof:

We know that

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{f(t)+\phi(t)\} = \int_0^\infty e^{-st}\left[f(t)+\phi(t)\right]dt$$

$$= \int_0^\infty e^{-st} f(t)\,dt + \int_0^\infty e^{-st}\phi(t)\,dt$$

$$L\{f(t)+\phi(t)\} = L\{f(t)\} + L\{\phi(t)\}$$

Hence Proved.

(ii) $L\{c\,f(t)\} = c\,L\{f(t)\}$, where c is a constant.

Proof: $L\{c\,f(t)\} = \int_0^\infty e^{-st}(c\,f(t))\,dt = c\int_0^\infty e^{-st} f(t)\,dt$

$$L\{c\,f(t)\} = c\,L\{f(t)\}$$

(iii) P.T $L\{f'(t)\} = sL\{f(t)\} - f(0)$

Proof: $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$ —— ①

Let $u = e^{-st}$ $\qquad \int dv = \int f'(t)\,dt$

$du = -se^{-st}\,dt$ $\qquad v = f(t)$

$\int u\,dv = uv - \int v\,du$

$$① \Rightarrow L\{f'(t)\} = \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)(-se^{-st})\,dt$$

$$= \left[e^{-\infty} f(\infty) - e^{0} f(0)\right] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= \left[0 - f(0)\right] + sL\{f(t)\}$$

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

(iv) $L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$

Proof: $L\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$

$$= \int_0^\infty e^{-st} f''(t)\,dt \quad \text{—— ①}$$

$u = e^{-st}$ $\qquad \int dv = \int f''(t)\,dt$

$du = -se^{-st}\,dt$ $\qquad v = f'(t)$

$$① \Rightarrow L\{f''(t)\} = \left[e^{-st} f'(t)\right]_0^\infty - \int_0^\infty f'(t)(-se^{-st})\,dt$$

$$= \left[e^{-\infty} f'(\infty) - e^{0} f'(0)\right] + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left[-f'(0)\right] + sL\{f'(t)\}$$

$$= -f'(0) + s\left[sL\{f(t)\} - f(0)\right] \qquad \left[\because L\{f'(t)\} = sL\{f(t)\} - f(0)\right]$$

$$= -f'(0) + s^2 L\{f(t)\} - s f(0)$$

$$L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$$

Hence Proved.

(iv) If $L\{f(t)\} = F(s)$ then

a) $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$

b) $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$.

Proof: We know that,

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

$$L\{f'(t)\} = sF(s) - f(0)$$

Taking lim as $s \to \infty$ on both sides

$$\lim_{s \to \infty} L\{f'(t)\} = \lim_{s \to \infty} \left[sF(s) - f(0)\right]$$

$$\lim_{s \to \infty} \int_0^\infty e^{-st} f'(t)\,dt = \lim_{s \to \infty} sF(s) - f(0)$$

$$0 = \lim_{s \to \infty} sF(s) - f(0)$$

$$\therefore \lim_{s \to \infty} sF(s) = f(0).$$

$\therefore \lim_{s \to \infty} sF(s) = \lim_{t \to 0} f(t)$

This is known as initial value theorem

b) $L\{f'(t)\} = sL\{f(t)\} - f(0)$

Taking limit as $s \to 0$ on both sides

$$\lim_{s \to 0} L\{f'(t)\} = \lim_{s \to 0} \left[sL\{f(t)\} - f(0)\right]$$

$$\lim_{s \to 0} \int_0^\infty e^{-st} f'(t)\, dt = \lim_{s \to 0} \left[sF(s) - f(0)\right]$$

$$\int_0^\infty e^{0t} f'(t)\, dt = \lim_{s \to 0} sF(s) - f(0)$$

$$\int_0^\infty f'(t)\, dt = \lim_{s \to 0} sF(s) - f(0)$$

$$\left[f(t)\right]_0^\infty = \lim_{s \to 0} sF(s) - f(0)$$

$$\left[f(\infty) - f(0)\right] = \lim_{s \to 0} sF(s) - f(0)$$

$$\lim_{t \to \infty} f(t) - \cancel{f(0)} = \lim_{s \to 0} sF(s) - \cancel{f(0)}$$

$$\therefore \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

$$\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t).$$

This is known as final value theorem

Problems: Formula 1 :

1 P.T $L(e^{-at}) = \frac{1}{s+a}$ Provided $s+a>0$

or Find $L[e^{-at}] = \frac{1}{s+a}$

Sol: $L(e^{-at}) = \int_0^\infty e^{-st} e^{-at} dt$ $\left[\because L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt\right]$

Here $f(t) = e^{-at}$

$$= \int_0^\infty e^{-(s+a)t} dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty$$

$$= \frac{-1}{s+a}\left[e^{-(s+a)\infty} - e^{0} \right] = \frac{-1}{s+a}(0-1)$$

$L(e^{-at}) = \frac{1}{s+a}$ Provided $s+a>0$

Similarly $L(e^{at}) = \frac{1}{s-a}$

Ex 1: Find $L(e^{-4t})$

$L(e^{-4t}) = \frac{1}{s+4}$ $\left[\because a=4 \quad L(e^{-at}) = \frac{1}{s+a}\right.$

Ex 2: Find $L(e^{5t})$

$L(e^{5t}) = \frac{1}{s-5}$ $\left[\because L(e^{at}) = \frac{1}{s-a}\right]$

Formula 2:

1) $L(\cosh at) = \frac{s}{s^2-a^2}$

$\left[\cosh x = \frac{e^{x}+e^{-x}}{2}\right]$

Proof $L(\cosh at) = L\left[\frac{e^{at}+e^{-at}}{2}\right]$

$= \frac{1}{2}\left\{L(e^{at}) + L(e^{-at})\right\}$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2}\right]$

$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2-a^2}\right]$

$L(\cosh at) = \frac{s}{s^2-a^2}$

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2 $L(\sinh at) = \frac{a}{s^2-a^2}$

$L(\sinh at) = L\left[\frac{e^{at}-e^{-at}}{2}\right]$

$= \frac{1}{2}\left\{L(e^{at}) - L(e^{-at})\right\}$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-(s-a)}{s^2-a^2}\right]$

$= \frac{1}{2}\left[\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right] = \frac{\cancel{2}a}{\cancel{2}(s^2-a^2)}$

$L(\sinh at) = \frac{a}{s^2-a^2}$

Formula 3:

$$L(\cos at) = \frac{s}{s^2+a^2}$$

Proof: $L(\cos at) = \int_0^\infty e^{-st} \cos at \, dt$

$$\int_0^\infty e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

Here $a = -s$, $b = a$

$$= \left\{\frac{e^{-st}}{(-s)^2+a^2}\left[-s\cos at + a\sin at\right]\right\}_0^\infty$$

$$= \left\{\left[\frac{e^{-\infty}}{s^2+a^2}\left(-s\cos\infty + a\sin\infty\right)\right] - \left[\frac{e^{0}}{s^2+a^2}\left(-s\cos 0 + a\sin 0\right)\right]\right\}$$

$$= 0 - \frac{1}{s^2+a^2}\left(-s(1)\right) = \frac{s}{s^2+a^2}$$

$$L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(\sin at) = \frac{a}{s^2+a^2}$$

Proof: $L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt$

$$\left[\because \int_0^\infty e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]\right]$$

$$= \left[\frac{e^{-st}}{s^2+a^2}\left(-s\sin at - a\cos at\right)\right]_0^\infty$$

$$= \left[0 - \frac{e^{-s(0)}}{s^2+a^2}\left(-s\sin 0 - a\cos 0\right)\right]$$

$$= \frac{e^{-0}}{s^2+a^2}(-a) = \frac{a}{s^2+a^2}$$

$$\therefore L(\sin at) = \frac{a}{s^2+a^2}$$

Formula :

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$$

Proof : $L(t^n) = \int_0^\infty e^{-st} t^n \, dt$

put $st = x \Rightarrow t = x/s$

$s(1) = dx/dt$

$dt = dx/s$

$$L(t^n) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \int_0^\infty \frac{x^n}{s^{n+1}} e^{-x} \, dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx$$

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$$

when n is a positive integer $\Gamma(n+1) = n!$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

Ex 1:

Find $L(t^2+2t+3)$

$$L(t^2+2t+3) = L(t^2) + 2L(t) + 3L(1)$$

$$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}.$$

Corollary:

$$L(1) = \frac{1}{s}$$

$$L(t) = \frac{1}{s^2}$$

$$L(t^2) = \frac{2}{s^3}$$

P.T

$$L(t^{1/2}) = \frac{\sqrt{\pi}}{2s^{3/2}}$$

we know that

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$L(t^{1/2}) = \frac{\Gamma(1/2+1)}{s^{1/2+1}}$$

$$= \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\Gamma(1+1/2)}{s^{3/2}}$$

$$= \frac{1/2 \, \Gamma(1/2)}{s^{3/2}} = 1/2 \frac{\sqrt{\pi}}{s^{3/2}}$$

$$L(t^{1/2}) = \frac{\sqrt{\pi}}{2s^{3/2}} \quad [\because \Gamma(1/2) = \sqrt{\pi}]$$

Similarly

$$L(t^{-1/2}) = \frac{\sqrt{\pi}}{2s^{1/2}}$$

Ex2: Find $L(\sin^2 2t)$

Sol: $\sin^2 t = \dfrac{1-\cos 2t}{2}$

$\sin^2 2t = \dfrac{1-\cos 4t}{2}$

$L(\sin^2 2t) = L\left[\dfrac{1-\cos 4t}{2}\right]$

$= \frac{1}{2} L(1) - \frac{1}{2} L(\cos 4t)$ $\quad \because L(1) = \frac{1}{s}$

$L(\cos at) = \dfrac{s}{s^2+a^2}$

$= \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \dfrac{s}{s^2+4^2}$

$= \frac{1}{2}\left[\dfrac{1}{s} - \dfrac{s}{s^2+16}\right]$

$= \frac{1}{2}\left[\dfrac{s^2+16-s^2}{s(s^2+16)}\right] = \frac{1}{\cancel{2}} \dfrac{\cancel{16}\,8}{s(s^2+16)}$

$= \dfrac{8}{\cancel{8}(s^2+16)s}$

$L(\sin^2 2t) = \dfrac{8}{s(s^2+16)}$

Ex3: Find $L(\sin^3 2t)$

Sol $\sin 3t = 3\sin t - 4\sin^3 t$

$4\sin^3 t = 3\sin t - \sin 3t$

$\sin^3 t = \dfrac{3\sin t - \sin 3t}{4}$

$\sin^3 2t = \dfrac{3\sin 2t - \sin 6t}{4}$

$L(\sin^3 2t) = L\left[\dfrac{3\sin 2t - \sin 6t}{4}\right]$

$$L(\sin^3 2t) = \tfrac{3}{4} L(\sin 2t) - \tfrac{1}{4} L(\sin 6t)$$

$$= \tfrac{3}{4} \cdot \frac{2}{s^2+2^2} - \tfrac{1}{4} \cdot \frac{6}{s^2+6^2} \qquad \left[\because L(\sin at) = \frac{a}{s^2+a^2}\right]$$

$$= \frac{6}{4} \frac{1}{s^2+4} - \frac{6}{4(s^2+36)}$$

$$= \frac{6}{4}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right]$$

$$= \tfrac{6}{4}\left[\frac{s^2+36-s^2-4}{(s^2+4)(s^2+36)}\right]$$

$$= \tfrac{6}{\cancel{4}}\left[\frac{\cancel{32}\,8}{(s^2+4)(s^2+36)}\right] = \frac{48}{(s^2+4)(s^2+36)}$$

$$L(\sin^3 2t) = \frac{48}{(s^2+4)(s^2+36)}$$

EX 4: Find $L\{f(t)\}$ where $f(t) = \begin{cases} 0, & \text{when } 0 < t \leq 2 \\ 3, & \text{when } t > 2 \end{cases}$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st}(0)\,dt + \int_2^\infty e^{-st}(3)\,dt = 3\int_2^\infty e^{-st}\,\cancel{f(t)}\,dt$$

$$= 3\left[\frac{e^{-st}}{-s}\right]_2^\infty = \frac{-3}{s}\left[e^{-st}\right]_2^\infty = \frac{-3}{s}\left[e^{-\infty} - e^{-2s}\right]$$

$$L\{f(t)\} = \frac{3e^{-2s}}{s}$$

Some general Theorem:

(1) If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

Proof: $L\{f(at)\} = \int_0^\infty e^{-st} f(at)\, dt$

put $at = y \Rightarrow t = y/a$

$a(1) = dy/dt$

$dt = dy/a$

$$L\{f(at)\} = \int_0^\infty e^{-(y/a)s} f(y)\, dy/a$$

$$= 1/a \int_0^\infty e^{-sy/a} f(y)\, dy$$

$$= 1/a\, F\left(\frac{s}{a}\right)$$

Hence Proved.

Result:

1) $L(\cos at) = 1/a\, F(s/a)$

$L(\cos t) = \frac{s}{s^2+1}$

2) $L(\sin at) = \frac{1}{a} F(s/a)$

$L(\sin t) = \frac{1}{s^2+1}$

3) $L(\sinh t) = \frac{1}{s^2-1}$

4) $L(\cosh t) = \frac{s}{s^2-1}$

Formula:

1) $L(e^{-at}\cos bt) = \dfrac{s+a}{(s+a)^2+b^2}$

2) $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$

3) $L(e^{-at}\cdot t^n) = \dfrac{n!}{(s+a)^{n+1}}$

4) $L(e^{at}t^n) = \dfrac{n!}{(s-a)^{n+1}}$.

P.T $L\{f(t)\} = F(s)$, then $L\{t\,f(t)\} = -\dfrac{d}{ds}F(s)$

Proof: $F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$\therefore \frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\right] dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt = -t\int_0^\infty e^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\, t\, f(t)\,dt$$

$$= -L\{t\,f(t)\}$$

$$L\{t\,f(t)\} = -\frac{d}{ds}F(s)$$

Corollary:

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[L\{f(t)\}]$$

Sol: ~~$L\{f(t)\}$~~

Put $n=1$ we have

$$L\{t\, f(t)\} = (-1)^1 \frac{d}{ds} L\{f(t)\}$$

Put $n=2$ we have

$$L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} L\{f(t)\}$$

Ex 1: Find $L\{t\, e^{-at}\}$

We know that

$$L\{t\, f(t)\} = -1 \frac{d}{ds}[L\{f(t)\}]$$

Here $f(t) = e^{-at}$

$$L\{t\, e^{-at}\} = -\frac{d}{ds} L[e^{-at}]$$

$$= -\frac{d}{ds}\left[\frac{1}{s+a}\right]$$

$$= -\frac{1}{(s+a)^2}$$

Ex2: Find $L\{t^2 e^{-3t}\}$

Sol:

$$L\{t^2 e^{-3t}\} = (-1)^2 \frac{d^2}{ds^2} L[e^{-3t}]$$

$$= -1 \cdot \frac{d^2}{ds^2}\left[\frac{1}{(s+3)}\right]$$

$$= \frac{2}{s+3}$$

Ex3: Find $L\{t \sin at\}$

Sol

$$L\{t \sin at\} = -\frac{d}{ds} L\{\sin at\}$$

$$= -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]$$

$$= -a \frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)$$

$$= -a \left(\frac{-2s}{(s^2+a^2)^2}\right)$$

$$= \frac{2as}{(s^2+a^2)^2}$$

$\frac{d}{ds}(s^2+a^2)^{-1}$

$= -1(s^2+a^2)^{-2} \cdot 2s$

$= -2s/(s^2+a^2)^2$

Ex4: Find $L\{t e^{-t} \sin t\}$

$\left[\because L(e^{at} \sin bt) = \frac{b}{(s-a)^2+b^2}\right]$

Sol

$$L(t e^{-t} \sin t) = \frac{-d}{ds} L(e^{-t} \sin t)$$

Here $a=-1$, $b=1$

$$= \frac{-d}{ds}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= \frac{-d}{ds}\left[\frac{1}{(s^2+2s+1+1)}\right] = \frac{-d}{ds}\left[(s^2+2s+2)\right]^{-1}$$

$$= \frac{-d}{ds}\left[(s^2+2s+2)\right]^{-1}$$

$$= -(-1)\left[s^2+2s+2\right]^{-1-1}\cdot(2s+2)$$

$$= \frac{2s+2}{(s^2+2s+2)^2} = \frac{2(s+1)}{(s^2+2s+2)^2}$$

Result:

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\, ds$$

Ex1: Find $L\left(\frac{1-e^t}{t}\right)$

$$L\left(\frac{1-e^t}{t}\right) = \int_s^{\infty} L(1-e^t)\,dt \quad [\because F(s) = 1-e^t]$$

$$= \int_s^{\infty}\left(L(1) - L(e^t)\right) ds$$

$$= \int_s^{\infty}\left[\frac{1}{s} - \frac{1}{s-1}\right] ds$$

$$= \int_s^{\infty} \frac{1}{s}\, ds - \int_s^{\infty} \frac{1}{s-1}\, ds$$

$$= \left[\log s - \log(s-1)\right]_s^{\infty} = \left[\log\frac{s}{s-1}\right]_s^{\infty}$$

$$= \left[\log\frac{\infty}{\infty-1} - \log\frac{s}{s-1}\right] = -\log\left(\frac{s}{s-1}\right)$$

$$= -\log s + \log(s-1)$$

$$= \log \frac{s-1}{s}.$$

Ex2: Find $L\left(\frac{\sin at}{t}\right)$

$$L\left(\frac{\sin at}{t}\right) = \int_s^\infty L(\sin at)\, ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2}\, ds = \cancel{\tfrac{1}{a} \tan^{-1} s/a}$$

$\because \int \frac{1}{s^2+a^2} = \frac{1}{a}\tan^{-1} s/a$

$$= a \int_s^\infty \frac{1}{s^2+a^2}\, ds$$

$$= a\left[\frac{1}{a}\tan^{-1} \frac{s}{a}\right]_s^\infty$$

$$= \frac{a}{a}\left[\tan^{-1} \frac{\infty}{a} - \tan^{-1} \frac{s}{a}\right]$$

$$= \tan^{-1}\infty - \tan^{-1} s/a$$

$$= \pi/2 - \tan^{-1} s/a$$

$[\because \tan^{-1}x + \cot^{-1}x = \pi/2$
$\pi/2 - \tan^{-1}x = \cot^{-1}x]$

$$= \cot^{-1}[s/a]$$

$$L\left(\frac{\sin at}{a}\right) = \cot^{-1}(s/a)$$

Evaluate $\int_0^\infty e^{-2t} \sin 3t \, dt$

Sol: By definition of Laplace Transform

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt$$

$$\int_0^\infty e^{-2t} \sin 3t \, dt = L(\sin 3t) = \frac{3}{s^2+3^2}$$ $\left[\because L(\sin at) = \frac{a}{s^2+a^2}\right]$

Put $s=2$, we get

$$\int_0^\infty e^{-2t} \sin 3t \, dt = \frac{3}{2^2+9} = \frac{3}{4+9} = \frac{3}{13}$$

Evaluate: $\int_0^\infty t \, e^{-3t} \cos t \, dt$.

Sol: $\int_0^\infty e^{-3t} (t \cos t) \, dt = L\{t \cos t\}$ ——(1)

$$L\{t \cos t\} = -\frac{d}{ds} L[\cos t]$$ $\left[\because L(t f(t)) = -\frac{d}{ds} L\{f(t)\}\right]$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$$

$$= -\left[\frac{(s^2+1)(1) - (s)(2s)}{(s^2+1)^2}\right]$$

$d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

$u = s$, $v = s^2+1$

$u' = 1$, $v' = 2s$

$$= -\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

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$$= \frac{-(1-s^2)}{(s^2+1)^2}$$

$$= \frac{s^2-1}{(s^2+1)^2}$$

$\therefore$ Put $s=3$, we have

$$\int_0^\infty e^{-3t}\, t\cos t) = \frac{3^2-1}{(3^2+1)^2} = \frac{9-1}{(9+1)^2} = \frac{8}{100}$$

$$= \frac{4}{50} = \frac{2}{25}$$

Evaluate: $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}\, dt$

$$\int_0^\infty e^{-st}\left[\frac{e^{-t}-e^{-2t}}{t}\right] dt = L\left[\frac{e^{-t}-e^{-2t}}{t}\right]$$

$$= \int_s^\infty L(e^{-t}-e^{-2t})\, ds \quad \left[\because L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)\, ds\right.$$

$$= \int_s^\infty \left[\frac{1}{s+1} - \frac{1}{s+2}\right] ds$$

$$= \Big[\log(s+1) - \log(s+2)\Big]_s^\infty$$

$$= \left[\log\left[\frac{s+1}{s+2}\right]\right]_s^\infty = \left[0 - \log\frac{s+1}{s+2}\right]$$

$$= -\log(s+1) + \log(s+2) = \log\left(\frac{s+2}{s+1}\right)$$

Inverse Laplace Transforms:

Let the symbol $L^{-1}\{F(s)\}$ denote the function, whose Laplace Transforms is $F(s)$.

Thus if $L\{f(t)\} = F(s)$ then $f(t) = L^{-1}\{F(s)\}$.

Formula For Inverse Laplace Transforms:

$L^{-1}[F(s)]$	$f(t)$	$L^{-1}[F(s)]$	$f(t)$
1) $\frac{1}{s-a}$	e^{at}	10) $\frac{2}{(s-a)^3}$	$t^2 e^{at}$
2) $\frac{s}{s^2-a^2}$	$\cosh at$	11) $\frac{n!}{(s-a)^{n+1}}$	$t^n e^{at}$
3) $\frac{a}{s^2-a^2}$	$\sinh at$	12) $\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
4) $\frac{s}{s^2+a^2}$	$\cos at$	13) $\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
5) $\frac{a}{s^2+a^2}$	$\sin at$	14) $\frac{2as}{(s^2+a^2)^2}$	$t\sin at$
6) $1/s$	1	15) $\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$
7) $1/s^2$	t	16) $\frac{1}{s^2+a^2}$	$\frac{\cos at}{s}$
8) $\frac{n!}{s^{n+1}}$	t^n	17) $\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
9) $\frac{1}{(s-a)^2}$	te^{at}		

Result 1:

$$L^{-1}\{F(s+a)\} = e^{-at} L^{-1} F(s)$$

Ex 1: Find $L^{-1}\left[\frac{1}{(s+a)^2}\right]$

$$L^{-1}\left[\frac{1}{(s+a)^2}\right] = e^{-at} L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{-at}\, t.$$

Ex 2: Find $L^{-1}\left[\frac{1}{(s+2)^2+16}\right]$

Sol

$$L^{-1}\left[\frac{1}{(s+2)^2+16}\right] = e^{-2t} L^{-1}\left[\frac{1}{s^2+16}\right] = e^{-2t} L^{-1}\left[\frac{1}{s^2+4^2}\right]$$

$$= e^{-2t} \cdot \frac{\sin 4t}{4}$$ [use 17 formula]

Ex 3: $L^{-1}\left[\frac{s-3}{(s-3)^2+4}\right]$

$$L^{-1}\left[\frac{s-3}{(s-3)^2+4}\right] = e^{3t} L^{-1}\left[\frac{s}{s^2+4}\right] = e^{3t} L^{-1}\left[\frac{s}{s^2+2^2}\right]$$

$$= e^{3t} \cos 2t$$

Ex 4: Find $L^{-1}\left[\frac{s}{s^2+2s+5}\right]$

$\frac{2s}{2} = 1$

Divide coefficient of s by 2.

Sol

$$L^{-1}\left[\frac{s}{s^2+2s+5}\right] = L^{-1}\left[\frac{s}{(s+1)^2-1+5}\right]$$

$$L^{-1}\left[\frac{s}{s^2+2s+5}\right] = L^{-1}\left[\frac{s}{(s+1)^2+4}\right]$$

$$L^{-1}\left[\frac{s+1-1}{(s+1)^2+2^2}\right] = \cancel{e^{-t}\, L^{-1}\left[\frac{s+1}{s^2+2^2}\right] - L^{-1}[1]}$$

$$= e^{-t}$$

$$L^{-1}\left[\frac{s}{(s+1)^2+2^2}\right] = L^{-1}\left[\frac{s+1-1}{(s+1)^2+2^2}\right]$$

$$= L^{-1}\left[\frac{s+1}{(s+1)^2-2^2}\right] - L^{-1}\left[\frac{1}{(s+1)^2-2^2}\right]$$

$$= e^{-t} L^{-1}\left[\frac{s}{s^2+2^2}\right] - e^{-t}\cdot L^{-1}\left[\frac{1}{s^2-2^2}\right]$$

$$= e^{-t}\cos 2t - e^{-t}\,\frac{\sin 2t}{2}$$

$$= \frac{e^{-t}}{2}\left[2\cos 2t - \sin 2t\right]$$

Result 2:

$$L^{-1}\left[F'(s)\right] = -t\, L^{-1}\left[F(s)\right]$$

Ex1: Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Sol

$$F'(s) = \frac{s}{(s^2+a)^2}$$

$$F(s) = \int \frac{s}{(s^2+a^2)^2}\, ds = -\frac{1}{2(s^2+a^2)}$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = -t\, L^{-1}\left[-\frac{1}{2(s^2+a^2)}\right]$$

$$= \frac{t}{2} L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{t}{2}\cdot\frac{\sin at}{a}$$

Ex 2: Find $L^{-1}\left[\frac{s}{(s^2-1)^2}\right]$

$s.\ \frac{(s^2-1)^{-2+1}}{-2+1\cdot 2s}$

Sol

$$F'(s) = \frac{s}{(s^2-1)^2}$$

$$F(s) = \int \frac{s}{(s^2-1)^2}\, ds$$

$$= -\frac{1}{2(s^2-1)}$$

$$L^{-1}\left[\frac{s}{(s^2-1)^2}\right] = -t\, L^{-1}\left[\frac{1}{-2(s^2-1)}\right]$$

$$= t/2\ L^{-1}\left[\frac{1}{s^2-1}\right]$$

$$= t/2 \cdot \sinh t.$$

Ex 3: Find $L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right]$

Sol:

$$F'(s) = \frac{s+2}{(s^2+4s+5)^2}$$

$$F(s) = \int \frac{s+2}{(s^2+4s+5)^2}\, ds$$

$$= -\frac{1}{2(s^2+4s+5)}$$

$$L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = -t\, L^{-1}\left[\frac{1}{-2(s^2+4s+5)}\right]$$

$$= \frac{t}{2} L^{-1}\left[\frac{1}{s^2+4s+5}\right]$$

$$= \frac{t}{2} L^{-1}\left[\frac{1}{(s+2)^2+5-4}\right]$$

$$= t/2\, L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= t/2 \cdot e^{-2t} L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

$$= t/2\, e^{-2t} \sin t$$

Result 3:

$$L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)]$$

Ex: Find $L^{-1}\left[\frac{s}{s^2+k^2}\right]$

Sol: $L^{-1}\left[\frac{s}{s^2+k^2}\right] = \frac{d}{dt} L^{-1}\left[\frac{1}{s^2+k^2}\right]$

$$= \frac{d}{dt}\left(\frac{\sin kt}{k}\right)$$

$$= \cancel{1/k} \cdot \cos kt \cdot \cancel{k}$$

$$= \cos kt$$

Here $\frac{\sin kt}{k} = 0$ when $t = 0$.

Ex 2: Find $L^{-1}\left[\frac{s}{(s+3)^2+4}\right]$.

Sol: $L^{-1}\left[\frac{s}{(s+3)^2+4}\right] = \frac{d}{dt} L^{-1}\left[\frac{1}{(s+3)^2+4}\right]$

$= \frac{d}{dt}\left[e^{-3t} L^{-1}\left[\frac{1}{s^2+2^2}\right]\right]$

$= \frac{d}{dt}\left[e^{-3t} \cdot \frac{\sin 2t}{2}\right]$

$= \frac{1}{2}\frac{d}{dt}\left[e^{-3t}\cdot \sin 2t\right]$

$= \frac{1}{2}\left[e^{-3t}\cdot 2\cos 2t + \sin 2t(-3)e^{-3t}\right]$

$= \frac{e^{-3t}}{2}\left[2\cos 2t - 3\sin 2t\right]$

Ex 3: $L^{-1}\left[\frac{s-3}{s^2+4s+13}\right]$

Sol: $L^{-1}\left[\frac{s-3}{(s^2+4s+13)}\right] = L^{-1}\left[\frac{s}{(s^2+4s+13)}\right] - 3L^{-1}\left[\frac{1}{s^2+4s+13}\right]$

$= \frac{d}{dt} L^{-1}\left[\frac{1}{s^2+4s+13}\right] - 3L^{-1}\left[\frac{1}{s^2+4s+13}\right]$

$= \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2-4+13}\right] - 3L^{-1}\left[\frac{1}{(s+2)^2-4+13}\right]$

$= \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2+9}\right] - 3L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$

$= \frac{d}{dt}\left[e^{-2t}\cdot\frac{\sin 3t}{3}\right] - 3e^{-2t}\cdot\frac{\sin 3t}{3}$ $\qquad 9 = 3^2$

$$= \frac{1}{3}\left[e^{-2t}\cos 3t \cdot 3 + \sin 3t \cdot (-2)e^{-2t}\right] - 3e^{-2t}\frac{\sin 3t}{3}$$

$$= \frac{1}{3}\left[3e^{-2t}\cos 2t - 2e^{-2t}\sin 3t - 3e^{-2t}\sin 3t\right]$$

$$= \frac{1}{3}\cdot e^{-2t}\left[\cos 2t - 5\sin 3t\right]$$

EX 4: Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

Sol

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = \frac{d}{dt}L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{d}{dt}\left\{e^{-2t}\cdot L^{-1}\left[\frac{1}{s^2}\right]\right\}$$

$$= \frac{d}{dt}\left(e^{-2t}t\right)$$

$$= \left[e^{-2t}\cdot(1) + t(-2)e^{-2t}\right]$$

$$= e^{-2t}(1-2t)$$

EX5: Find $L^{-1}\left[\frac{s^2}{(s-1)^3}\right]$

Sol: $L^{-1}\left[\frac{s^2}{(s-1)^3}\right] = \frac{d}{dt}L^{-1}\left[\frac{s}{(s-1)^3}\right]$

$$= \frac{d}{dt}\cdot\frac{d}{dt}L^{-1}\left[\frac{1}{(s-1)^3}\right]$$

$$= \frac{d^2}{dt^2}L^{-1}\left[\frac{1}{(s-1)^3}\right] = \frac{d^2}{dt^2}\left[e^{-t}L^{-1}\left(\frac{1}{s^3}\right)\right]$$

$$= \frac{d^2}{dt^2}\left[e^{-t}\cdot\frac{t^2}{2}\right] = \frac{1}{2}\frac{d}{dt}\left\{\frac{d}{dt}\left[e^{-t}t^2\right]\right\}$$

$$= \frac{1}{2}\frac{d}{dt}\left[e^{t}.2t + t^{2}.e^{t}\right]$$

$$= \frac{1}{2}\frac{d}{dt}\left[2te^{t} + t^{2}e^{t}\right]$$

$$= \frac{1}{2}\left[2\left(t.e^{t} + e^{t}.(1)\right) + \left(2t.e^{t} + e^{t}.t^{2}\right)\right]$$

$$= \frac{1}{2}\left[2te^{t} + 2e^{t} + 2te^{t} + e^{t}.t^{2}\right]$$

$$= \frac{1}{2}\left[e^{t}.t^{2} + 4t\,e^{t} + 2e^{t}\right]$$

$$= \frac{1}{2}e^{t}\left[t^{2} + 4t + 2\right]$$

Result 4:

$$L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}\left[F(s)\right]dt.$$

Ex1: Find $L^{-1}\left[\frac{1}{s(s+a)}\right]$

Sol $L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t \left(\frac{1}{s+a}\right)dt$

$$= \int_0^t e^{-at}\,dt = \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= -\frac{1}{a}\left[e^{-at} - e^{0}\right]$$

$$= \frac{1}{a}\left[1 - e^{-at}\right].$$

Ex 2: Find $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

Sol $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2+a^2}\right] dt$

$= \int_0^t \frac{\sin at}{a} dt$

$= 1/a \left[\frac{-\cos at}{a}\right]_0^t$

$= -1/a^2 \left[\cos at - \cos 0\right] = 1/a^2\left[-\cos at + 1\right]$

$= 1/a^2 (1 - \cos at)$

Ex 3: Find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$

Sol: $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right]$

$= \int_0^t L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] dt$

$= \int_0^t t\cdot\frac{\sin at}{2a} dt$

$= 1/2a\left[\int_0^t t\sin at\, dt\right]$ $u = t$, $u' = 1$ $\int dv = \int \sin at\, dt$

$= \frac{1}{2a}\left\{\left[t\left(\frac{-\cos at}{a}\right)\right]_0^t - \int_0^t \frac{-\cos at}{a}.dt\right]$ $V = \frac{-\cos at}{a}$

$$= \frac{1}{2a}\left\{\left[\frac{-t\cos at}{a} - 0\right] + \frac{1}{a}\int_0^t \cos at\, dt\right]$$

$$= \frac{1}{2a}\left[\frac{-t\cos at}{a} + \frac{1}{a}\left[\frac{\sin at}{a}\right]_0^t\right]$$

$$= \frac{1}{2a}\left[\frac{-t\cos at}{a} + \frac{1}{a^2}(\sin at - 0)\right]$$

$$= \frac{1}{2a}\left[\frac{-t\cos at}{a} + \frac{\sin at}{a^2}\right]$$

$$= \frac{1}{2a}\left(\frac{-at\cos at + \sin at}{a^2}\right)$$

$$= \frac{1}{2a^3}\left[\sin at - at\cos at\right]$$

Methods of partial fraction:

Ex 1 Find $L^{-1}\left[\frac{1}{S(S+1)(S+2)}\right]$

Sol: Let $\frac{1}{S(S+1)(S+2)} = \frac{A}{S} + \frac{B}{S+1} + \frac{C}{S+2}$ —— (1)

$$\frac{1}{S(S+1)(S+2)} = \frac{A(S+1)(S+2) + BS(S+2) + CS(S+1)}{S(S+1)(S+2)}$$

$$1 = A(S+1)(S+2) + BS(S+2) + CS(S+1) \quad \text{—— (2)}$$

Put $S=0$ in (2)

$$1 = A(0+1)(0+2) + B\,0(0+2) + C\,0(0+1)$$

$$1 = A\,2$$

$$\boxed{A = \tfrac{1}{2}}$$

Put $S=-1$ in (2)

$$1 = A(-1+1)(-1+2) + B(-1)(-1+2) + C(-1)(-1+1)$$

$$1 = A(0) + B(-1)(1) + C(0)(-1)$$

$$1 = -B$$

$$\boxed{B = 1}$$

Put $S=-2$ in (2)

$$1 = A(-1+1)(-2+2) + B(-2)(-2+2) + C(-2)(-2+1)$$

$$1 = A(0) + B(0) + C(-2)(-1)$$

$$1 = 2C$$

$$\boxed{C = \tfrac{1}{2}}$$

Substitute $A=\frac{1}{2}$, $B=-1$, $C=\frac{1}{2}$ in (1)

$$\frac{1}{S(S+1)(S+2)} = \frac{1}{2S} - \frac{1}{S+1} - \frac{1}{2}\frac{1}{S+1}$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2} L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2}(1) - e^{-t} + \frac{1}{2} e^{-2t}$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}.$$

Ex2: Find $L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right]$

Sol: $$\frac{1}{(s+1)(s^2+2s+1)} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+2s+2)} \quad \text{--- (1)}$$

$$= \frac{A(s^2+2s+2) + (Bs+C)(s+1)}{(s+1)(s^2+2s+2)}$$

$$1 = A(s^2+2s+2) + (Bs+C)(s+1) \quad \text{--- (2)}$$

Put $s=-1$ in (2)

$$1 = A\left((-1)^2 + 2(-1) + 2\right) + \left[B(-1)+C\right](-1+1)$$

$$1 = A(1-2+2) + B(0)$$

$$1 = A$$

Put $s=0$ in (2)

$$1 = A(0+0+2) + (B0+C)(0+1)$$

$$1 = 2A + C$$

$$1 = 2(1) + C \Rightarrow \quad C = 1-2$$

$$C = -1$$

Put $s = 1$

$$1 = A(1+2+2) + (B+C)(1+1)$$

$$1 = A(5) + (B+C)(2)$$

$$1 = 5A + 2B + 2C$$

$$1 = 5(1) + 2B + 2(-1)$$

$$1 = 3 + 2B$$

$$1 - 3 = 2B$$

$$-2 = 2B$$

$$B = -1$$

Substitute $A = 1$, $B = -1$, $C = -1$ in ①

$$\frac{1}{(s+1)(s^2+2s+1)} = \frac{1}{s+1} - \left[\frac{s+1}{s^2+2s+2}\right]$$

$$L^{-1}\left[\frac{1}{(s+1)(s^2+2s+1)}\right] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{s+1}{s^2+2s+2}\right]$$

$$= e^{-t} - L^{-1}\left[\frac{s+1}{(s+1)^2-1+2}\right]$$

$$= e^{-t} - L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]$$

$$= e^{-t} - e^{-t} L^{-1}\left[\frac{s}{s^2+1^2}\right]$$

$$= e^{-t} - e^{-t}\cos t$$

$$= e^{-t}(1 - \cos t)$$

Ex3: Find $L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right]$

Sol

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2} \quad \text{--- (1)}$$

$$= \frac{A(s+2)(s-1)^2 + B(s-1)^2 + C(s-1)(s+2)^2 + D(s+2)^2}{(s+2)^2(s-1)^2}$$

$$1+2s = A(s+2)(s-1)^2 + B(s-1)^2 + C(s-1)(s+2)^2 + D(s+2)^2 \quad \text{--- (2)}$$

Put $s=-2$

$$1+2(-2) = A(0) + B(-2-1)^2 + C(0) + D(0)$$

$$-3 = B\,9$$

$$-3 = 9B$$

$$B = -\tfrac{1}{3}.$$

Put $s=1$ in (2)

$$1+2 = A(0) + B(0) + C(0) + D(1+2)^2$$

$$3 = 9D$$

$$D = \tfrac{1}{3}.$$

Put $s=0$

$$1 = A(2)(-1)^2 + B(-1)^2 + C(-1)(2)^2 + D(2)^2$$

$$1 = 2A + B - 4C + 4D$$

$$1 = 2A - \tfrac{1}{3} - 4C + \tfrac{4}{3} = 2A - 4C + \frac{3}{3}$$

$1 = 2A - 4C + 1$

$2A - 4C = 0$ —③

put $s = 2$

$1 + 2(2) = A(4)(1)^2 + B(1)^2 + C(1)(4)^2 + D(4)^2$

$5 = 4A + B + 16C + 16D$

$5 = 4A + \frac{1}{3} + 16C + \frac{16}{3}$

$5 = 4A + 16C + \frac{15}{3}$

$5 = 4A + 16C + 5$

$4A + 16C = 0$ —④

solving ③ & ④ we get $A = 0,\ C = 0$

substitute $A = 0,\ B = -\frac{1}{3},\ C = 0,\ D = \frac{1}{3}$ in ①

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{-1}{3(s+2)^2} + \frac{1}{3(s-1)^2}$$

$$= \frac{1}{3}\left[\frac{-1}{(s+2)^2} + \frac{1}{(s-1)^2}\right]$$

$$= \frac{1}{3}\frac{1}{(s-1)^2} - \frac{1}{3}\frac{1}{(s+2)^2}$$

$$L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right] = \frac{1}{3}L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{1}{3}e^{t}L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{3}e^{-2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= \frac{1}{3} e^{t} \cdot t - \frac{1}{3} e^{-2t} \cdot t$$

$$= \frac{t}{3} \left[e^{t} - e^{-2t} \right]$$

10 Marks :

Ex1: Solve the equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$

given that $y = \frac{dy}{dt} = 0$ when $t=0$. $dy/dt = y'$

Sol $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$, Here $y(0)=0$, $y'(0)=0$

This equation can be written as.

$$y'' + 2y' - 3y = \sin t$$

Applying Laplace Transform on both sides

$$L(y'' + 2y' - 3y) = L(\sin t)$$

$$L(y'') + 2L(y') - 3L(y) = L(\sin t)$$

$$s^2 L(y) - s\,y(0) - y'(0) + 2\{sL(y) - y(0)\} - 3L(y) = L(\sin t)$$

$$s^2 L(y) - s(0) - (0) + 2sL(y) + 2(0) - 3L(y) = \frac{1}{s^2+1}$$

$$s^2 L(y) + 2sL(y) - 3L(y) = \frac{1}{s^2+1}$$

$$L(y)\left[s^2 + 2s - 3\right] = \frac{1}{s^2+1}$$

Put $s = 0$

$$1 = A(3)(1) + B(-1)(1) + (0+D)(-3)$$

$$1 = 3A - B - 3D$$

$$1 = 3/8 - (-1/40) - 3D$$

$$1 = \frac{3}{8} + \frac{1}{40} - 3D$$

$$1 = \frac{15+1}{40} - 3D$$

$$1 = 16/40 - 3D$$

$$1 - 16/40 = -3D$$

$$\frac{40-16}{40} = -3D$$

$$\frac{24}{40} = -3D$$

$$\frac{\cancel{24}\,8}{-\cancel{3} \times 40} = D$$

$$D = \frac{-8}{40} = \frac{-1}{5}$$

Put $s = -1$

$$1 = A(2)(2) + B(-2)(2) + [-\cancel{1}C + D](-2)2$$

$$1 = 4A - 4B + \cancel{4}C - \cancel{4}D$$

$\div$ by 4,

$$1/4 = A - B + C - D$$

$$\frac{1}{8} + \frac{1}{40} + \frac{1}{5}$$

$$= \frac{5+1+}{40}$$

$$L(y)\left[(s-1)(s+3)\right] = \frac{1}{s^2+1}$$

$$L(y) = \frac{1}{(s-1)(s+3)(s^2+1)}$$

$\cancel{L(y)}$ 7

$$y = L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right]$$

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{(s-1)} + \frac{B}{(s+3)} + \frac{Cs+D}{s^2+1} \quad \text{—(1)}$$

$$= \frac{A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3)}{(s-1)(s+3)(s^2+1)}$$

$$1 = A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3) \quad \text{—(2)}$$

Put $s = 1$

$$1 = A(1+3)(1^2+1) + B(0) + [C(1)+D](0)$$

$$1 = 8A + 0B + 0C + 0D$$

$$A = 1/8$$

Put $s = -3$

$$1 = A(0) + B(-3-1)[(-3)^2+1] + [C(-3)+D](0)$$

$$1 = B(-4)(10) + 0$$

$$1 = -40B$$

$$B = -1/40$$

$$\frac{1}{4} = \frac{1}{8} + \frac{1}{40} + C + \frac{1}{5}$$

$$\frac{1}{4} = \frac{5+1+8}{40} + C$$

$$\frac{1}{4} = \frac{14}{40} + C$$

$$C = \frac{1}{4} - \frac{14}{40}$$

$$C = \frac{10-14}{40} = \frac{-4}{40} = \frac{-1}{10}$$

Substitute $A = \frac{1}{8}$, $B = -\frac{1}{40}$, $C = -\frac{1}{10}$, $D = -\frac{1}{5}$ in ①

$$\frac{1}{(S-1)(S+3)(S^2+1)} = \frac{1}{8(S-1)} - \frac{1}{40}\,\frac{1}{S+3} + \frac{S\left(-\frac{1}{10}\right) - \frac{1}{5}}{S^2+1}$$

$$= \frac{1}{8(S-1)} - \frac{1}{40(S+3)} - \frac{1}{10}\,\frac{S}{(S^2+1)} - \frac{1}{5}\cdot\frac{1}{S^2+1}$$

$$L^{-1}\left[\frac{1}{(S-1)(S+3)(S^2+1)}\right] = \frac{1}{8}L^{-1}\left[\frac{1}{S-1}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{S+3}\right] - \frac{1}{10}L^{-1}\left[\frac{S}{S^2+1}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{S^2+1}\right]$$

$$y = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t$$