UNIT - V

Half Range Series

Given a function defined in the particular interval $(0,\pi)$ we can expand it in series in two different forms one in terms of Cosines and the other interms of Sines. These expand are respectively called Half - Range Cosines series, Half - Range Sine series.

Half Range cosine Series

The Cosine Series for $f(x)$ in the interval $(0,\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

where
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Half Range Sine Series:-

The Sine Series for $f(x)$ in the interval $(0,\pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

Example 1:-

1. Find a Sine series for $f(x) = c$ in the range of $(0,\pi)$

Sol:-

Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} c\sin nx\,dx$$

$$= \frac{2c}{\pi}\int_0^{\pi} \sin nx\,dx$$

$$= \frac{2c}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= -\frac{2c}{n\pi}\left[\cos n\pi - \cos n0\right]$$

$$= -\frac{2c}{n\pi}\left[(-1)^n - 1\right]$$

$$\boxed{b_n = \frac{2c}{n\pi}\left[1-(-1)^n\right]}$$

n is even, $n=2$

$$b_n = \frac{2c}{n\pi}\left[1-(-1)^2\right]$$

$$= \frac{2c}{n\pi}\left[1-1\right]$$

$$\boxed{b_n = 0}$$

n is odd

$$b_n = \frac{2c}{n\pi}\left[1+1\right] = \frac{4c}{n\pi}$$

$\therefore$ The ~~Seo~~ Sine Series

$$f(x) = \sum_{n=1}^{\infty} \frac{2c}{n\pi}\left[1-(-1)^n\right] \sin nx.$$

$n=1,2,\ldots$ even

$$= \frac{4c}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

H.W

2. Find the half range fourier cosine series for x^2 in $(0,\pi)$

$a_0 =$

3. Find the cosine series for the function $f(x) = x$ in $(0,\pi)$

Example 4:-

4. Find the sine series $f(x) = 1$ in $(0,\pi)$

5. Find the sine series in the interval $(0,\pi)$ for the function.

$$f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & x > \pi/2 \end{cases}$$

Sol:-

Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = 2/\pi \int_0^{\pi} f(x) \sin nx \, dx$$

$$b_n = 2/\pi \left[\int_0^{\pi/2} f(x) \sin nx \, dx + \int^{\pi} f(x) \sin nx \, dx\right]$$

$$= 2/\pi \left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx\right]$$

$u = x$ | $\int dv = \int \sin nx \, dx$ | $u = \pi - x$

$du = dx$ | $v = \frac{-\cos nx}{n}$ | $du = -dx$

$$= \frac{2}{\pi}\left[\left\{\left(\frac{-x\cos nx}{n}\right)_0^{\pi/2} + \int_0^{\pi/2}\frac{\cos nx}{n}\,dx\right\} + \left\{\left(-(\pi - x)\frac{\cos nx}{n}\right)_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\frac{\cos nx}{n}\,dx\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{2}\frac{\cos n\pi/2}{n} + 0 + \left(\frac{\sin nx}{n^2}\right)_0^{\pi/2} + 0 + (\pi - \pi/2)\frac{\cos n\pi/2}{n} - \left(\frac{\sin nx}{n^2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\cancel{\frac{-\pi}{2}\frac{\cos n\pi/2}{n}} + \frac{\sin n\pi/2}{n^2} - 0 + \cancel{\frac{\pi}{2}\frac{\cos n\pi/2}{n}} - 0 + \frac{\sin n\pi/2}{n^2}\right]$$

$$b_n = \frac{2}{\pi}\left[\frac{2\sin n\pi/2}{n^2}\right]$$

$$b_n = \frac{4}{n^2\pi}\left[\sin n\pi/2\right]$$

where n is even $b_n = 0$ $\quad n=2 \Rightarrow \frac{4}{n^2\pi}\left[\sin 2\pi/2\right] (\because \sin\pi = 0)$ $= 0$

where n is odd $b_n = \frac{4}{n^2\pi}$ $\quad n=1 \Rightarrow \frac{4}{n^2\pi}\left[\sin \pi/2\right] = \frac{4}{n^2\pi}$

$\therefore$ The sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty}\frac{4}{n^2\pi}\sin nx$$

$$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \cdots\right]$$

$x = \pi/2$.

$$f(x) = \frac{4}{\pi}\left[\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots\right]$$

6. Find the cosine series in the interval $(0, \pi)$ for the fn. $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases}$

Sol:-

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{(\pi/2)^2}{2} - 0\right] + \left[\pi(\pi) - \frac{\pi^2}{2} - \pi(\pi/2) - \frac{(\pi/2)^2}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2/4}{2}\right] + \left[\pi^2 - \frac{\pi^2}{4} - \frac{\pi^2}{2} - \frac{\pi^2/4}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi^2}{8} - \frac{2\pi^2}{2} + \pi^2\right]$$

$$= \frac{2}{\pi}\left[\pi^2/4 - 2\pi^2/2 + \pi^2\right]$$

$$= \frac{2}{\pi}\left[\pi^2/4 - \pi^2 + \pi^2\right]$$

$$= \cancel{2}/\cancel{\pi}\left[\pi^{\cancel{2}}/\cancel{4}_2\right]$$

$$\boxed{a_0 = \pi/2}$$

$$a_n = 2/\pi \int_0^{\pi/2} x \cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx.$$

$u = x$ | $\int dv = \int \cos nx\,dx$ | $u = \pi - x$

$du = dx$ | $v = \frac{\sin nx}{n}$ | $du = -dx$

$$= 2/\pi\left[x \cdot \frac{\sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin nx}{n}\,dx + \left[(\pi - x)\frac{\sin nx}{n}\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \frac{\sin nx}{n}\,dx$$

$$= 2/\pi\left[\pi/2\,\frac{\sin n(\pi/2)}{n} - (0)\frac{\sin n(0)}{n}\right]_0^{\pi/2} + \left[\frac{\cos nx}{n^2}\right]_0^{\pi/2}$$

$$+ \left[(\pi - \pi)\frac{\sin n\pi}{n} - (\pi - \pi/2)\frac{\sin n\pi/2}{n}\right] + \left[\frac{\cos nx}{n^2}\right]_{\pi/2}^{\pi}$$

$$\cancel{= 2/\pi\left[\left(\pi/2n\right)\right]} = 2/\pi\left[\pi/2 \cdot 1/n\right] + \left[\frac{\cos n\pi/2}{n^2} - \frac{\cos n(0)}{n^2}\right] -$$

$$\left[0 - \pi/2 \cdot 1/n\right] + \left[\frac{\cos n\pi}{n^2} - \frac{\cos n\pi/2}{n^2}\right]$$

$$= 2/\pi\left[\left(\pi/2n\right) + \left(-1/2n^2\right) + \left(\frac{-\pi}{2n}\right) + \frac{(-1)^n}{n^2}\right]$$

$$= \frac{2}{\pi}\left[-1/n^2 + \frac{(-1)^n}{n^2}\right]$$

$$a_n = 2/n^2\pi\left[(-1)^n - 1\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

n is odd

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^1 - 1\right]$$

$$= \frac{2}{\pi},$$

1. Expand $\frac{\pi x}{8}(\pi - x)$ in a sine series when $0 < x < \pi$.

Sol:-

Sine series $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi x}{8}(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\cancel{\pi}} \times \frac{\cancel{\pi}}{8}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx$$

$$= \frac{1}{4}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx$$

$u = \pi x - x^2$ | $\int dv = \int \sin nx\,dx$

$du = (\pi - 2x)dx$ | $v = -\frac{\cos nx}{n}$

$$= \frac{1}{4}\left[\left(-\pi x - x^2)\frac{\cos nx}{n}\right)_0^{\pi} + \int_0^{\pi} (\pi - 2x)\frac{\cos nx}{n}\,dx\right.$$

$$= \frac{1}{4}\left[-\left(\pi x\frac{\cos nx}{n} - x^2\frac{\cos nx}{n}\right)_0^{\pi} + \int_0^{\pi} \pi\frac{\cos nx}{n}\,dx - \int_0^{\pi} 2x\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{1}{4n}\left[\pi\left(\frac{\sin nx}{n}\right)_0^{\pi} - 2\int_0^{\pi} x\cos nx\,dx\right]$$

$u = x$ | $\int dv = \int \cos nx\,dx$

$du = dx$ | $v = \frac{\sin nx}{n}$

$$= \frac{1}{4n}\left[-2\left(\left(\frac{x\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right)\right]$$

$$= -\frac{1}{4n^2}\left[2\left(-\left(-\frac{\cos nx}{n}\right)_0^{\pi}\right)\right]$$

$$= \frac{2}{4n^2}\left[-\frac{\cos n\pi}{n} + \frac{\cos n(0)}{n}\right]$$

$$= \frac{2}{4n^2}\left[-\frac{(-1)^n}{n} + \frac{1}{n}\right]$$

$$= \frac{\cancel{2}}{2\cancel{4}n^3}\left(1 - (-1)^n\right)$$

$$b_n = \frac{1}{2n^3}\left(1 - (-1)^n\right)$$

n is even.

$b_n = 0$

n is odd.

$b_n = \frac{1}{n^3}$

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$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2n^3} (1-(-1)^n) \sin nx$$

$$f(x) = \frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots$$

Homework Sums

2. Find the half range fourier cosine series for x^2 in $(0, \pi)$

Sol:-

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx$$

$$= \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{3\pi} [\pi^3 - 0]$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx$$

$u = x^2$ | $\int dv = \int \cos nx \, dx$

$du = 2x dx$ | $v = \frac{\sin nx}{n}$

$$a_n = 2/\pi \left[x^2 \frac{\sin nx}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} 2x dx .$$

$$= \frac{2}{\pi} \left[\left(x^2 \frac{\sin nx}{n} \right)_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} 2x dx .$$

$$= \frac{2}{\pi} \left[\pi^2 , \frac{\sin n\pi}{n} 0 - \frac{2}{n} \int_0^{\pi} x \sin nx dx \right]$$

$$= \frac{2}{\pi} \left[0 - \frac{2}{n} \left(\left(-x \frac{\cos nx}{n} \right)_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} dx \right.\right.$$

$$= \frac{2}{\pi} \left[\frac{-2}{n} \left(\left(-\pi \cdot \frac{\cos n\pi}{n} + 0 \right) + \left(\frac{\sin nx}{n} \right)_0^{\pi} \right] \right.$$

$$= \frac{2}{\pi} \left[\frac{-2}{n} \left(\left(\frac{-\pi(-1)^n}{n} \right) + \frac{\sin n\pi}{n} - \frac{\sin n(0)}{n} \right) \right]$$

$$= \frac{2}{\pi} \left(\frac{-2}{n} \left(\frac{-\pi(-1)^n}{n} \right) \right)$$

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$$= \frac{2}{\pi}\left(\frac{2\pi(-1)^n}{n^2}\right)$$

$$= \frac{4\pi}{\pi}\frac{(-1)^n}{n^2}$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^n}$$

If n is odd

$$a_n = \frac{4}{n^2}(-1) = \frac{-4}{n^2}$$

If n is even

$$a_n = \frac{4}{n^2}(1) = \frac{4}{n^2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\cancel{2}\pi^2/3}{\cancel{2}} + \frac{4}{1^2}(-1)^1\cos x + \frac{4}{2^2}(-1)^2\cos 2x + \frac{4}{3^2}(-1)^3 \cos 3x + \cdots$$

$$= \frac{\pi^2}{3} - \frac{4}{1^2}\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + \cdots$$

$$f(x) = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$$

3. Find a cosine series for the fun $f(x) = x$ in cosine series.

Sol:-

$\therefore$ cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x.dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x.\cos nx\,dx$$

$u = x$ | $\int dv = \int \cos nx\,dx$

$du = dx$ | $v = \frac{\sin nx}{n}$

$$= \frac{2}{\pi}\left[x.\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}dx$$

$$= \frac{2}{\pi}\left[\pi\frac{\sin n(\pi)}{n} - 0.\frac{\sin n(0)}{n}\right] - \left[\frac{-\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}[0] + \left[\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi} \cdot \frac{1}{n}\left[\frac{\cos n(\pi)}{n} - \frac{\cos n(0)}{n}\right]$$

$$= \frac{2}{\pi} \cdot \frac{1}{n^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{2}{n^2 \pi}\left[(-1)^n - 1\right]$$

n is odd

$$= \frac{2}{\pi 1^2}\left[(-1)^1 - 1\right]$$

$$= \frac{2}{\pi}(-2)$$

n is even.

$$= \frac{2}{\pi 2^2}\left[(-1)^2 - 1\right] = \frac{2}{n2^2}\left[1 - 1\right] = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \pi/2 - \frac{4}{\pi}\cos x - \frac{4}{3\pi}\cos 3x + \cdots$$

$$= \pi/2 - 4\left[\frac{\cos x}{\pi} - \frac{4\cos 3x}{3\pi} + \cdots\right]$$

4. Find Sine series for $f(x) = 1$ in $(0, \pi)$

Sol:-

Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (1)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin nx\, dx$$

$$= \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\cos n(\pi)}{n} + \frac{\cos n(0)}{n}\right]$$

$$= \frac{2}{\pi}\left[-\frac{(-1)^n}{n} + \frac{1}{n}\right]$$

$$b_n = \frac{2}{n\pi}\left[1 - (-1)^n\right]$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{4}{\pi n}\sin nx$$

$$= \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\left(1-(-1)^n\right)\right]\sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin nx$$

$$= \frac{2}{\pi}\left[2/1 \sin x + 2/3 \sin 3x + \cdots\right]$$

$$= \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots\right]$$

10m 5. If $F(x) = \frac{\pi x}{4}$ $(0 < x < \pi/2)$

$= \pi/4(\pi - x)$ $(\pi/2 < x < \pi)$ Prove that for the range $(0 < x < \pi)$ find the expansion in series of sines for the same range.

Sol:- $\therefore$ Sine series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \pi x/4 \sin nx\,dx + \int_{\pi/2}^{\pi} \pi/4(\pi - x)\sin nx\,dx\right]$$

$$= 2/\pi\left[\pi/4\int_0^{\pi/2} x \sin nx\,dx + (\pi/4)\int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$$

$u = x$, $du = dx$ | $\int dv = \sin nx\,dx$, $v = -\frac{\cos nx}{n}$ | $u = \pi - x$, $du = -dx$ | $\int dv = \int \sin nx\,dx$, $v = -\frac{\cos nx}{n}$

$$= 2/\pi\left[\pi/4\left[\left(-x\frac{\cos nx}{n}\right)_0^{\pi/2} - \int_0^{\pi/2} -\frac{\cos nx}{n}dx\right] + \left[\pi/4\left(-(\pi - x)\frac{\cos nx}{n}\right)_{\pi/2}^{\pi} - \int^{\pi}\frac{\cos nx}{n}dx\right]\right.$$

$$= 2/\pi\left[\pi/4\left[-\pi/2\frac{\cos n\pi/2}{n} - (0) + \left[\frac{\sin nx}{n^2}\right]_0^{\pi/2}\right] + \left[\pi/4\left[-(\pi - x)\frac{\cos n\pi}{n} + (\pi - \pi/2)\frac{\cos n\pi/2}{n} - \left(\frac{\sin nx}{n^2}\right)_{\pi/2}^{\pi}\right.\right.\right.$$

$$= 2/\pi\left[\pi/4(0-0) + \left[\frac{\sin n\pi/2}{n^2} - \frac{\sin n(0)}{n^2}\right] + \left[\pi/4(0 + \pi(0) - \frac{\sin n\pi}{n^2} + \frac{\sin n\pi/2}{n^2}\right]\right.$$

$$= 2/\pi\left[\pi/4(1/n^2) + \left[\pi/4\left(0 + 1/n^2\right)\right]\right]$$

$$= \frac{2}{\pi}\left[\pi/4n^2 + \pi/4n^2\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{4n^2}\right] = \frac{4\pi}{4\pi n^2} = 1/n^2$$

$b_n = 1/n^2$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \sin nx$$

$$f(x) = \frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \cdots$$

ii) If $F(x) = \pi x/4 \; (0 < x < \pi/2)$

$= \pi/4(\pi - x) \; (\pi/2 < x < \pi)$ Prove that for the range $(0 < x < \pi)$ find the expansion in series of cosine series for the same range.

Sol:-

Cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{\pi \cdot x}{4}\,dx + \int_{\pi/2}^{\pi} \frac{\pi}{4}(\pi - x)\,dx\right]$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{4}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{2}\left[\frac{(\pi/2)^2}{2} - 0 + \pi \cdot \pi - \frac{\pi^2}{2} - \pi \cdot \frac{\pi}{2} + \frac{(\pi/2)^2}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi^2/4}{2} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2/4}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\cancel{2}\pi^2}{\cancel{2}} + \frac{\pi^2}{8}\right]$$

$$= \frac{1}{2}\left[\frac{2\pi^2}{8} + \cancel{\pi^2} - \cancel{\pi^2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi^2}{4}\right]$$

$$\boxed{a_0 = \frac{\pi^2}{8}}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{\pi x}{4} \cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x) \cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\int_0^{\pi/2} x \cdot \cos nx\,dx + \frac{\pi}{4}\int_{\pi/2}^{\pi} \cos nx\,dx - \frac{\pi}{4}\int_{\pi/2}^{\pi} x \cos nx\,dx\right]$$

$$= \pi/4 \cdot 2/\pi \left[\int_0^{\pi/2} x \cdot \cos nx\,dx + \pi \int_{\pi/2}^{\pi} \cos nx - \int_{\pi/2}^{\pi} x \cdot \cos nx\,dx \right]$$

$u = x \quad | \quad \int dv = \int \cos nx\,dx^{\pi/2}$

$du = dx \quad | \quad v = \frac{\sin nx}{n}$

$$= \frac{1}{2}\left[\left(x \frac{\sin nx}{n} \right)_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin nx}{n}\,dx + \pi \left(\frac{\sin nx}{n} \right)_{\pi/2}^{\pi} - \left[\left(x \frac{\sin nx}{n} \right)_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \frac{\sin nx}{n}\,dx \right] \right.$$

$$= \frac{1}{2}\left[\pi/2 \frac{\sin n\pi/2}{n} - 0 + \left(\frac{\cos nx}{n^2} \right)_0^{\pi/2} + \pi \left(\frac{\sin n\pi}{n} \cdot \frac{\sin n\pi/2}{n} \right) - \left(\pi \frac{\sin n\pi}{n} \cdot \pi/2 \frac{\sin n\pi/2}{n} + \left(\frac{\cos nx}{n^2} \right)_{\pi/2}^{\pi} \right] \right.$$

$$= \frac{1}{2}\left[\pi/2 \cdot \frac{1}{n} + \frac{\cos n\pi/2}{n^2} - \frac{\cos n(0)}{n^2} + 0 \cdot \pi/n \cdot \left(\pi/2n + \frac{\cos n\pi}{n^2} - \frac{\cos n\pi/2}{n^2} \right) \right]$$

$$= \frac{1}{2}\left[\pi/2n - 1/n^2 - \pi/n + \pi/2n - \frac{(-1)^n}{n^2} \right]$$

$$= \frac{1}{2}\left[\frac{2\pi}{2n} - \frac{1}{n^2} - \frac{\pi}{n} - \frac{(-1)^n}{n^2} \right] = \frac{1}{2}\left[\pi/n - 1/n^2 - \pi/n - \frac{(-1)^n}{n^2} \right]$$

$$= \frac{1}{2}\left[-1/n^2 - \frac{(-1)^n}{n^2} \right] = -1/2n^2 \left[1 + (-1)^n \right]$$

n is odd

$= 1/2n^2 [1 - (-1)^n]$

$= \cancel{\;} \; 1/2(1)^2 [1 - (-1)]$

$= 1/2 [1+1]$

$= 2/2 = 1$

n is even

$= 1/2n^2 [1 - (-1)^n]$

$= 1/2(2)^2 [1 - (-1)^2] = \frac{1}{2(2)^2} \cdot (1-1) = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \pi^2/8/2 + \sum_{n=1}^{\infty} 1/2n^2 \left[1 - (-1)^n \right] \cos nx$$

$$= \pi^2/16 + 1/2 \left[\frac{2\cos x}{1^2} + \frac{2\cos 3x}{3^2} + \dots \right]$$

$$f(x) = \pi^2/16 + 2/2 \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \dots \right]$$

$$f(x) = \pi^2/16 + \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \dots \right]$$

$$= \frac{a}{l}[2l - l] = \frac{a}{l}(l)$$

$$\boxed{a_0 = a}$$

$$a_n = \frac{1}{l}\left[\int_0^l f(x) \cos\frac{n\pi x}{l}\, dx + \int_l^{2l} f(x) \cos\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{1}{l}\int_l^{2l} f(x) \cos\frac{n\pi x}{l}\, dx.$$

$$= \frac{1}{l}\int_l^{2l} a \cos\frac{n\pi x}{l}\, dx.$$

$$= \frac{a}{l}\left[\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_l^{2l}$$

$$= \frac{a}{l} \times \frac{l}{n\pi}\left[\sin\frac{n\pi(2l)}{l} - \sin\frac{n\pi l}{l}\right]$$

$$= \frac{a}{n\pi}\left[\sin 2n\pi - \sin n\pi\right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{1}{l}\left[\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx + \int_l^{2l} f(x) \sin\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{1}{l}\int_l^{2l} f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{1}{l}\int_l^{2l} a \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{a}{l}\left[\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right]_l^{2l}$$

$$= \frac{-a}{n\pi}\left[-\cos n\pi\frac{(2l)}{l} + \cos n\pi\frac{(l)}{l}\right]$$

$$= \frac{-a}{n\pi}\left[+\cos 2n\pi + \cos n\pi\right]$$

$$= \frac{-a}{n\pi}\left[-(-1)^{2n} + (-1)^n\right]$$

$$b_n = \frac{a}{n\pi}\left[(-1)^n - (-1)^{2n}\right]$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{a}{2} + \sum_{n=1}^{\infty}\left[0 + \frac{a}{n\pi}\left[(-1)^n - (-1)^{2n}\right]\sin\frac{n\pi x}{l}\right]$$

$$f(x) = \frac{a}{2} + \sum_{n=1}^{\infty}\left[\frac{a}{n\pi}(-1)^n - (-1)^{2n}\sin\frac{n\pi x}{l}\right]$$

$$= a/2 + a/\pi\left[-2\frac{\sin n\pi x}{l} - 2/3\,\frac{\sin 3\pi}{l} \dots\right]$$

$$= a/2 - 2a/\pi\left[\frac{\sin\pi x}{l} + 1/3\,\sin\frac{3n\pi}{l} \dots\right]$$

2. If $f(x) = 0$ when $-l < x < 0$.

$= 1$ when $0 < x < l$

Express $f(x)$ as a fourier series in the range. $-l < x < l$

Sol:-

$$a_0 = 1/l\int_{l}^{-l} f(x)\,dx = 1/l\int_0^l 1.\,dx = 1/l\,[x]_0^l$$

$$= 1/l\,[l - 0] = 0\,1/l\,[l]$$

$$\boxed{a_0 = l}$$

$$a_n = 1/l\int_{l}^{-l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= 1/l\int_0^l 1.\cos\frac{n\pi x}{l}.\,dx = 1/l\left[\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right]_0^l$$

$$= 1/l \times l/n\pi\left[\frac{\sin n\pi(l)}{l} - \frac{\sin n\pi(0)}{l}\right]$$

$$= 1/n\pi\,[0]$$

$$\boxed{a_n = 0}$$

$$b_n = 1/l\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= 1/l\int_0^l 1.\sin\frac{n\pi x}{l}.\,dx$$

$$= 1/l\left[\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right]_0^l$$

$$= -\frac{1}{l} \times \frac{1}{n\pi} \left[\frac{\cos n\pi x}{l} \right]_0^l$$

$$= \frac{-1}{n\pi} \left[\cos n\pi \frac{l}{l} - \cos \frac{n\pi (0)}{0} \right]$$

$$= \frac{-1}{n\pi} \left[\cos n\pi - \cos n(0) \right]$$

$$= \frac{-1}{n\pi} \left[(-1)^n - 1 \right]$$

$$= \frac{1}{n\pi} \left[1 - (-1)^n \right]$$

$$a_n = \frac{1}{l} \int_0^2 f(x) \frac{\cos n\pi x}{l} \, dx$$

$$= \frac{1}{1} \left[\int_0^1 \pi x \frac{\cos n\pi x}{l} \, dx + \int_1^2 \pi (2-x) \frac{\cos n\pi x}{l} \, dx \right]$$

$$= \pi \int_0^1 x \cos n\pi x \, dx + \int_1^2 (2-x) \cos n\pi x \, dx \Big]$$

$u = x$, $du = dx$ | $\int dv = \int \cos n\pi x \, dx$, $v = \frac{\sin n\pi x}{n\pi}$ | $u = 2-x$, $du = -dx$.

$$= \pi \left[\left(\left(x \cdot \frac{\sin n\pi x}{n\pi} \right)_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi} \, dx \right) + \left(\left((2-x) \frac{\sin n\pi x}{n\pi} \right)_1^2 + \int_1^2 \frac{\sin n\pi x}{n\pi} \, dx \right) \right.$$

$$= \pi \left[\left(1 \cdot \frac{\sin n\pi (1)}{n\pi} - 0 \right) + \frac{1}{n^2\pi^2} \left(\cos n\pi x \right)_0^1 + \left((2-2) \frac{\sin n\pi 2}{n\pi} - (2-1) \frac{\sin n\pi (1)}{n\pi} \right) - \frac{\cos n\pi x}{n^2\pi^2} \Big|_1^2 \right.$$

$$= \pi \left[0 + \frac{1}{n^2\pi^2} \left(\cos n\pi (1) - \cos n\pi (0) \right) + (0) - \frac{\cos n\pi (2)}{n^2\pi^2} + \frac{\cos n\pi (1)}{n^2\pi^2} \right.$$

$$= \pi \left[\frac{1}{n^2\pi^2} (-1)^n - 1 - \frac{1}{n^2\pi^2} \left(1 - (-1)^n \right) \right]$$

$$= \frac{\pi}{n^2\pi^2} \left[(-1)^n - 1 - 1 + (-1)^n \right]$$

$$\boxed{a_n = \frac{1}{n^2\pi} \left[2(-1)^n - 2 \right]}$$

$$b_n = \frac{1}{l} \int_0^2 f(x) \frac{\sin n\pi x}{l} \, dx$$

$$= \frac{1}{1} \left[\int_0^1 \pi x \frac{\sin n\pi x}{l} \, dx + \int_1^2 \pi (2-x) \frac{\sin n\pi x}{l} \, dx \right]$$

$$= \pi \left[\int_0^1 x \sin n\pi x \, dx + \int_1^2 (2-x) \sin \frac{n\pi x}{\cancel{l}} \, dx \right]$$

$u = x$ | $\int dv = \int \sin n\pi x \, dx$ | $u = 2-x$

$du = dx$ | $v = -\frac{\cos n\pi x}{n\pi}$ | $du = -dx$

$$= \pi\left[\left(x - \frac{\cos n\pi x}{n\pi}\right)_0^1 + \int_0^1 \frac{\cos n\pi x}{n}\,dx\right) + \left((2-x) - \frac{\cos n\pi x}{n\pi}\right)_1^2$$

$$= \pi\left[\left(1 - \frac{\cos n\pi(1)}{n\pi} - 0\right) + \left(\frac{\sin n\pi x}{n^2\pi^2}\right)_0^1 + (2-2) - \frac{\cos n\pi(2)}{n\pi} - (2-1)\right.$$

$$\left. - \frac{\cos n\pi(1)}{n\pi} + \left(\frac{\sin n\pi x}{n^2\pi^2}\right)_1^2\right.$$

$$= \pi\left[\frac{-\cos n\pi}{n\pi} + 0 + \left[\frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 + 0 + \frac{\cos n\pi}{n\pi} - \left(\frac{\sin n\pi(2)}{n^2\pi^2} - \frac{\sin n\pi(1)}{n^2\pi^2}\right)\right]$$

$$= \pi\left[\frac{-\cos n\pi}{n\pi} + \left(\frac{\sin n\pi(1)}{n^2\pi^2} - \frac{\sin n\pi(0)}{n^2\pi^2} + \frac{\cos n\pi}{n\pi} - 0\right)\right]$$

$$= \pi\left[\frac{-(-1)^n}{n^2\pi^2} - 0 - 0 + \frac{(-1)^n}{\pi^2 n^2}\right]$$

$$= \pi(0)$$

$$\boxed{b_n = 0}$$

$$f(x) = \pi/2 + \sum_{n=1}^{\infty}\left[\frac{2\cancel{\pi}}{n^2\pi\cancel{\pi}}((-1)^n - 1)\frac{\cos n\pi x}{l} + (0)\frac{\sin n\pi x}{l}\right]$$

$$= \pi/2 + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi}((-1)^n - 1)\cos n\pi x\right] \quad (\because l = 1)$$

$$= \pi/2 + 2/\pi \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\cos n\pi x\right]$$

$$= \pi/2 + 2/\pi\left[\frac{-2}{1^2}\cos n\pi x - \frac{2}{3^2}\cos 3\pi x \cdots\cdots\right]$$

$$f(x) = \pi/2 - 4/\pi\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \cdots\right]$$

2. Find a fourier series with period 3 to represent $f(x) = 2x - x^2$ in the range $(0, 3)$

10marks

Sol:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \frac{\cos n\pi x}{l} + b_n \frac{\sin n\pi x}{l}\right]$$

$$2l = 3$$

$$l = 3/2$$

$$a_0 = 1/l \int_0^3 f(x)\,dx = \frac{1}{3/2}\int_0^3 (2x - x^2)\,dx$$

$$a_0 = 2/3\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= 2/3\left[x^2 - x^3/3 - 0\right]_0^3$$

$$= 2/3\left[(3)^2 - \frac{(3)^3}{3}\right] = 2/3\left[9 - 9/3\right] = 2/3\left[9 - 9\right] = 2/3(0)$$

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$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{l}\int_0^3 f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{3/2}\int_0^3 (2x-x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\cos\frac{n2\pi x}{3}\,dx$$

$u = (2x-x^2)$ | $\int dv = \int\cos\frac{2n\pi x}{3}$

$du = (2-2x)dx$ | $v = \dfrac{\sin\frac{n2\pi x}{3}}{n2\pi/3}$

$$= \frac{2}{3}\left[(2x-x^2)\frac{\sin\frac{n2\pi x}{3}}{n2\pi/3}\right]_0^3 - \int_0^3 \frac{\sin\frac{n2\pi x}{3}}{n2\pi/3}(2-2x)dx$$

$$= \frac{2}{3}\left[(2(3)-3^2)\frac{\sin\frac{2\pi 3}{3}}{n2\pi/3} - 2(0)-(0)^2\frac{\sin 2\pi\, 0/3}{n2\pi/3} - \frac{3}{2\pi n}\int_0^3 \sin\frac{n2\pi}{3}\right.$$

$u = (2-2x)$ | $\int dv = \int \sin\frac{n2\pi x}{3}$

$du = -2dx$ | $v = -\dfrac{\cos\frac{n2\pi x}{3}}{n2\pi/3}$

$$= \frac{2}{3}\left[(6-9)\frac{\sin n2\pi}{n2\pi/3} - 0\right] - \frac{3}{2\pi n}\left[\left((2-2x) - \frac{\cos\frac{n2\pi x}{3}}{\frac{n2\pi}{3}}\right)_0^3 + 2\int_0^3 -\frac{\cos\frac{n2\pi x}{3}}{\frac{n2\pi}{3}}\,dx\right]$$

$$= \frac{2}{3}\left[\frac{-3}{2\pi n}\left((2-2x) - \frac{\cos\frac{n2\pi x}{3}}{\frac{n2\pi}{3}}\right)_0^3 + 2\int_0^3 -\frac{\cos\frac{n2\pi 3}{3}}{n2\pi/3}\,dx\right.$$

$$= \frac{2}{3}\times\frac{-3}{2\pi n}\left[(2-6) - \frac{\cos n2\pi\, 3/3}{n2\pi/3} - 2 - 2(0) - \frac{\cos\frac{n2\pi(0)}{3}}{n2\pi/3} - 2\int_0^3 \frac{\cos\frac{n2\pi}{3}}{n2\pi/3}\,dx\right.$$

$$= \frac{-1}{\pi n}\times\frac{3}{2\pi n}\left[((4\cos n2\pi) + (2)\cos\frac{2\pi(0)}{3}\right]$$

$$= \frac{-3}{2\pi^2 n^2}\left[4(1)+2(1)\right] - 2\times\frac{3}{n2\pi}\int_0^3 \cos\frac{n2\pi x}{3}\,dx$$

$$= \frac{-3}{2\pi^2 n^2}(\overset{3}{\cancel{6}}) - \frac{3}{n\pi}\left[\frac{\sin\frac{n2\pi x}{3}}{2\pi/3}\right]_0^3$$

$$= \frac{-9}{\pi^2 n^2} - \frac{3}{n\pi}\left[\frac{\sin\frac{n2\pi(3)}{3}}{2\pi/3} - \frac{\sin\frac{n2\pi(0)}{3}}{2\pi/3}\right]$$

$$a_n = \frac{-9}{\pi^2 n^2} = 0$$

$$\boxed{a_n = \frac{-9}{\pi^2 n^2}}$$

$$b_n = \frac{1}{l}\int_0^l f(x)\sin\frac{n2\pi}{3}\,dx$$

$$b_n = 2/3\int_0^3 (2x-x^2)\sin\frac{n2\pi}{3}\,dx$$

$u = 2x - x^2$ | $\int dv = \int \sin\frac{n2\pi x}{3}\,dx$

$du = (2-2x)\,dx$ | $v = \frac{-\cos\frac{n2\pi x}{3}}{n2\pi/3}$

$$= 2/3\left[\left[(2(3)-3^2) - \frac{\cos\frac{n2\pi(3)}{3}}{n2\pi/3}\right] - \left[(2(0)-0) - \frac{\cos\frac{n\pi(0)}{3}}{n2\pi/3}\right]\right.$$

$$+ 3/n2\pi\left[(2-2x)\frac{\sin\frac{n2\pi x}{3}}{\frac{n2\pi}{3}}\right]_0^3 - \int_0^3 -2\,dx\,\frac{\sin\frac{2\pi x}{3}}{n2\pi/3}\Bigg]$$

$u = 2-2x$ | $dv = \cos\frac{n2\pi x}{3}$

$du = -2dx$ | $v = \frac{\sin\frac{n2\pi x}{3}}{n2\pi/3}$

$$= \frac{2}{3}\left[(6-9)\left(\frac{-1}{n2\pi/3}\right)\right] - 0 + \frac{3}{n2\pi}\left[(2-2(3))\frac{\sin\frac{n2\pi(3)}{3}}{n2\pi/3}\right.$$

$$\left. - (2-2(0))\frac{\sin\frac{2\pi(0)}{3}}{n2\pi/3}\right] + \frac{2\times3}{n2\pi}\int_0^3 \sin\frac{2\pi x}{3}\Bigg]$$

$$= 2/3\left[9/2n\pi\,(-3-1)\right] + \frac{3}{n2\pi}\left[(0-0) + \frac{3}{n2\pi}\right]\left[\frac{-\cos\frac{n2\pi x}{3}}{\frac{n2\pi}{3}}\right]_0^3$$

$$= 2/3\left[\left[\frac{2\times3}{n2\pi}\right] + \frac{3}{n2\pi}\left[\frac{3}{n2\pi}\left[\frac{-\cos\frac{n2\pi3}{3}}{n2\pi/3} + \frac{\cos\frac{n2\pi0}{3}}{n2\pi/3}\right]\right]\right]$$

$$= 2/3\left[\frac{2\times3}{n2\pi} + \frac{3}{n2\pi}\left[\frac{3}{n2\pi}\left[\frac{-1}{n2\pi/3} + \frac{1}{n2\pi/3}\right]\right]\right]$$

$$= 2/3\left[\frac{2\times3}{n2\pi}\right] + [0-0]$$

$$= 2/3 \cdot 9/n2\pi$$

$$\boxed{b_n = 3/n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\frac{\cos n\pi x}{l} + b_n\frac{\sin n x}{l}\right]$$

$$= 0/2 + \sum_{n=1}^{\infty}\left[\frac{-9}{n^2\pi^2}\frac{\cos n\pi x}{3/2} + \frac{3}{n\pi}\frac{\sin n\pi x}{3/2}\right]$$

$$= \sum_{n=1}^{\infty}\left[\frac{-9}{n^2\pi^2}\frac{\cos 2\pi x}{3} + \frac{3}{n\pi}\sin\frac{n2\pi x}{3}\right]$$

$$= 3/\pi\left[\sum_{n=1}^{\infty}\frac{-3}{n\pi}\cos\frac{n2\pi x}{3} + \frac{1}{n}\sin\frac{n2\pi x}{3}\right]$$

$$= \frac{3}{\pi}\left[-\frac{3}{\pi}\left(\frac{\cos n 2\pi x}{3\times 1} + \frac{\cos 2\pi (2) x}{3\times 2} + \cdots\right]\right.$$

$$+ \left[\frac{\sin 2\pi x}{3\times 1} + \frac{\sin 2\pi 2}{3\times 2} + \cdots\right]$$

COSINE SERIES

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$$

1. Expand $f(x)=x$ as a half range cosine series in the interval $0<x<l$

Sol:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{l}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^2}{2}\right]$$

$$\boxed{a_0 = l}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l x\cdot\cos\frac{n\pi x}{l}\,dx$$

$u = x$ | $dv = \cos\frac{n\pi x}{l}$

$du = dx$ | $v = \sin\frac{n\pi x}{l}$

$$= \frac{2}{l}\left[\left(x\cdot\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right)_0^l - \int_0^l \frac{\sin\frac{n\pi x}{l}}{n\pi/l}\,dx\right]$$

$$= \frac{2}{l}\left[l\cdot\frac{\sin\frac{n\pi l}{l}}{n\pi/l} - 0\cdot\frac{\sin\frac{n\pi(0)}{l}}{n\pi/l}\right] - \frac{l}{n\pi}\int_0^l \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(0) - \frac{1}{n\pi}\left[\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right]_0^l\right.$$

$$= \frac{2}{l}\cdot\frac{l}{n\pi}\cdot\frac{l}{n\pi}\left[\cos n\pi\frac{l}{l} - \cos\frac{n\pi(0)}{l}\right]$$

$$= \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

n is odd

$a_n = \frac{2l}{n^2\pi^2}(-2)$

$= \frac{-4l}{n^2\pi^2}$

n is even

$a_n = \frac{2l}{n^2\pi}(0)$

$a_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]\frac{\cos n\pi x}{l}$$

$$= l/2 + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\frac{n\pi x}{l}$$

$$x = l/2 + \frac{2l}{\pi^2}\left[\frac{-2}{1^2}\cos\frac{n\pi x}{l} - \frac{2}{3^2}\cos\frac{5\pi x}{l} - \frac{2}{5^2}\cos\frac{n5\pi x}{l} + \dots\right]$$

$$x = l/2 - \frac{4l}{\pi^2}\left[\frac{1}{1^2}\cos\frac{n\pi x}{l} + \frac{1}{3^2}\cos\frac{3nx}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \dots\right]$$

Even function

(i) If $F(x)$ is an even function $F(x)$ be expanded as a fourier series consisting of cosine terms only in the interval of length $2d$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where,

$$a_n = \frac{2}{l}\int_0^l F(x)\cos\frac{n\pi x}{l}\,dx$$

Odd function.

(ii) If $F(x)$ is an odd function $F(x)$ be expanded as a fourier series consisting of sine terms only in the interval of length

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}\,dx.$

Sine Series

(iii) $F(x)$ be expanded as a sine series in half range $(0, l)$ with period $2l$ of the form.

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = 2/l\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

Cosine Series

$F(x)$ be expanded as a cosine series in half range $(0, l)$ with period $2l$ of the term.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_n = 2/l\int_0^l F(x)\cos\frac{n\pi x}{l}\,dx$$

1. Express $F(x) = x$ as a fourier series with interval 2 in the range $-1 < x < 1$, $0 < x < 1$.

Sol:-

$$f(x) = x$$

$$-f(-x) = x$$

$$f(x) = -f(-x)$$

$f(x)$ is an odd function.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^1 x \sin \frac{n\pi x}{l} dx$$

$$= 2 \int_0^1 x \sin n\pi x dx$$

$u = x$ | $\int dv = \sin n\pi x \, dx$

$du = dx$ | $v = -\frac{\cos n\pi x}{n\pi}$

$$= 2\left[\left(-x \frac{\cos n\pi x}{n\pi}\right)_0^1 - \int -\frac{\cos n\pi x}{n\pi} dx\right]$$

$$= 2\left[-(-1)\frac{\cos n\pi}{n\pi} + (0)\frac{\cos n\pi(0)}{n\pi} + \left[\frac{\sin n\pi x}{n^2\pi^2}\right]_0^1\right]$$

$$= 2\left[\frac{-(-1)^n}{n\pi} + \frac{\sin\pi(1)}{n^2} - \frac{\sin n\pi(0)}{n^2\pi^2}\right]$$

$$= 2\left[\frac{-(-1)^n}{n\pi}\right]$$

$$\boxed{b_n = \frac{-2(-1)^n}{n\pi}}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n\pi} \sin \frac{n\pi x}{l}$$

$$= \frac{-2(-1)^n}{n\pi} \sin n\pi x$$

$$= \left[\frac{2\sin\pi x}{\pi} - \frac{2\sin 2\pi x}{2\pi} + \frac{2\sin 3\pi x}{3\pi} - \dots\right]$$

$$f(x) = \frac{2}{\pi}\left[\frac{\sin\pi x}{\pi} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \dots\right]$$

2. Express $f(x)=x^2$ as a fourier series with interval 2 in the range $-1<x<1$.

Sol: $f(x)=x^2$

$f(-x)=x^2$

$f(x)=-f(-x)$

$f(x)$ is a even function

$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$a_0=\frac{2}{l}\int_0^l f(x)dx=\frac{2}{l}\int_0^1 x^2dx$$

$$=2\left[\frac{x^3}{3}\right]_0^1=2\left[\frac{1}{3}-\frac{0}{3}\right]=2\cdot\frac{1}{3}$$

$$\boxed{a_0=2/3}$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$a_n=\frac{2}{l}\int_0^1 x^2\cos\frac{n\pi x}{l}dx$$

$u=x^2$ | $\int dv=\cos\frac{n\pi x}{l}dx$

$du=2xdx$ | $v=\frac{\sin n\pi x}{n\pi}$

$$=\frac{2}{1}\int_0^1 x^2\cos\frac{n\pi x}{1}dx$$

$$=2\left[x^2\frac{\sin n\pi x}{n\pi}\right]_0^1-\int_0^1\frac{\sin n\pi x}{n\pi}2xdx.$$

$$=2\left[\left[(1)^2\frac{\sin n\pi}{n\pi}-0\frac{\sin n\pi(0)}{n\pi}\right]-2\int_0^1\frac{\sin n\pi x}{n\pi}x\,dx\right.$$

$u=x$ | $\int dv=\int\frac{\sin n\pi x}{n\pi}dx$

$du=dx$ | $v=-\frac{\cos n\pi x}{n^2\pi^2}$

$$=2\left[\left[0-2\left[2-\left(\frac{\cos n\pi x}{n^2\pi^2}\right)_0^1\right]-\int_0^1\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)dx\right]\right.$$

$$=2\left[(-2)\left[(1)\frac{\cos n\pi}{n^2\pi^2}-(0)-\frac{\cos n\pi(0)}{n^2\pi^2}\right]+\int_0^1\left(-\frac{\cos n\pi x}{n^2\pi^2}dx\right)\right]$$

$$=2\left[\left[-2(1)-\frac{\cos n\pi}{n^2\pi^2}-(0)-\frac{\cos n\pi(0)}{n^2\pi^2}\right]+\int_0^1\frac{\sin n\pi x}{n^3\pi^3}\right]$$

$$=2\left[(-2)\frac{(-1)^n}{n^2\pi^2}-0+\left[\frac{\sin n\pi 3}{n^3\pi^3}\right]_0^1\right]$$

$$= 2\left[-2\left(\frac{-(-1)^n}{n^2\pi^2}\right) + \frac{\sin n\pi(1)}{n^3\pi^3} - \frac{\sin n\pi(0)}{n^3\pi^3}\right]$$

$$= 2\left[-2\left(\frac{-(-1)^n}{n^2\pi^2}\right)\right]$$

$$= \frac{4(-1)^n}{n^2\pi^2}$$

$$a_n = \frac{4(-1)^n}{n^2\pi^2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= \frac{2/3}{2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \frac{\cos n\pi x}{1}$$

$$= \frac{2}{3} \times \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos n\pi x$$

$$= \frac{1}{3} + 4\left[\frac{-\cos\pi x}{\pi^2} + \frac{\cos 2\pi x}{2^2\pi^2} - \frac{\cos 3\pi x}{3^2\pi^2} + \dots\right]$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\left[-\frac{\cos\pi x}{1^2} + \frac{\cos 2\pi x}{2^2} - \frac{\cos 3\pi x}{3^2} + \dots\right]$$

3. Find a half range sine series for $F(x) = 2x/l$, $0 < x < l/2$ $= 2/l(l-x)$ $l/2 < x < l$.

Sol:-

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^{l} f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} 2x/l \,\sin\frac{n\pi x}{l}\, dx + \int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{l}\, dx\right]$$

$u = x$ | $\int dv = \sin\frac{n\pi x}{l}$ | $u = l - x$

$du = dx$ | $v = \frac{-\cos n\pi x/l}{n\pi/l}$ | $du = -dx$.

$$= \frac{4}{l^2}\left[\left(-x\frac{\cos\frac{n\pi x}{l}}{n\pi/l}\right]_{l}^{l/2} + \int_{l/2}^{l} \frac{\cos\frac{n\pi x}{l}}{n\pi/l}\, dx$$

$$= \frac{4}{l^2}\left[\frac{l}{n\pi}\left[-l/2 \frac{\cos n\pi\, l/2}{l} - 0\right] + \left[\frac{\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right]_{l}^{l/2}\right.$$

$$= \frac{4}{l^2}\left[\left(\frac{l^2}{2n\pi}(\cos n\pi/2)\right) + \frac{l^2}{n^2\pi^2} \sin\frac{n\pi x}{l}\right]_{l}^{l/2}$$

$$= \frac{4}{l^2}\left[0 + \left[\frac{l^2}{n^2\pi^2} \frac{\sin n\pi(l/2)}{l} - \frac{l^2}{n^2\pi^2} \frac{\sin n\pi(0)}{l}\right]\right]$$

$$= \frac{4}{l^2}\left[\frac{l^2}{n^2\pi^2}\left(\frac{\sin l\pi/2}{l} - 0/l\right)\right]$$

$$= \frac{4}{l^2}\left[\frac{l^2}{n^2\pi^2}(1)\right]$$

$$= \frac{4}{l^2}\left[\frac{l^2}{n^2\pi^2}\right]$$

First half

$$b_n = \frac{4l^2}{l^2 n^2\pi^2}$$

Second half

$$= \frac{4}{l^2}\left[\int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4}{l^2}\left[-(l-x)\frac{\cos\frac{n\pi x}{l}}{n\pi/l}\right]_{l/2}^{l} - \int_{l/2}^{l}\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}(-dx)\Bigg]$$

$$= \frac{4}{l^2}\left[-(l-x)\frac{\cos\frac{n\pi x}{l}}{n\pi/l}\right)_{l/2}^{l} - \int_{l/2}^{l}\frac{\cos\frac{n\pi x}{l}}{n\pi/l}dx\Bigg)$$

$$= \frac{4}{l^2}\left[-(l-l)\frac{\cos\frac{n\pi(l)}{l}}{n\pi l} + (l-l/2)\frac{\cos n\pi(l/2)}{n\pi/l} - \left[\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_{l/2}^{l}\right.$$

$$= \frac{4}{l^2}\left[\frac{l}{2}\frac{\cos n\pi/2}{n\pi} - \left(\frac{\frac{\sin n\pi l}{n\pi}}{\frac{n^2\pi^2}{l^2}} - \frac{\frac{\sin n(l/2)}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]$$

$$= \frac{4}{l^2}\left[\frac{l^2}{n^2\pi^2}(\sin n\pi/2)\right]$$

$$= \frac{4l^2}{l^2 n^2\pi^2}$$

Second part :

First part + Second part = bn

$$b_n = \frac{4l^2}{l^2n^2\pi^2} + \frac{4l^2}{l^2n^2\pi^2} = \frac{8l^2}{l^2n^2\pi^2}$$

$$\boxed{b_n = \frac{8}{n^2\pi^2}}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}\sin\frac{n\pi x}{l}$$

$$= \frac{8}{l}\left[\frac{\sin \pi x}{l^2 n^2} + \frac{\sin 2\pi x}{2^2} + \cdots\right]$$

$$f(x) = \frac{8}{\pi^2 l}\left[\frac{\sin\pi x}{1^2} + \frac{\sin n\, 2\pi x}{2^2} + \cdots\right]$$

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Combination of Series

Fourier expansion of any linear quardratic function of x over the half range interval $0<x<l$

Half range expansion in the interval $0<x<l$.

Sin formula.

$$1=\frac{4}{\pi}\left[\sin\frac{\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}+\dots\right]$$

$$x=\frac{2l}{\pi}\left[\sin\frac{\pi x}{l}-\frac{1}{2}\sin\frac{2\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\dots\right]$$

$$x^2=\frac{2l^3}{\pi^3}\left\{(\pi^2-4)\sin\frac{\pi x}{l}-\frac{\pi^2}{2}\sin\frac{2\pi x}{l}+\left(\frac{\pi^2}{3}-\frac{4}{3^2}\right)\right.$$

$$\sin\frac{3\pi x}{l}-\frac{\pi^2}{4}\sin\frac{4\pi x}{l}+\left(\frac{\pi^2}{5}-\frac{4}{5^3}\right)$$

Cos formula

$$x=\frac{1}{2}-\frac{4l}{\pi^2}\left(\frac{\cos\pi x}{l}+\frac{1}{3^2}\frac{\cos 3\pi x}{l}+\frac{1}{5^2}\frac{\cos 5\pi x}{l}+\dots\right.$$

$$x^2=\frac{l^2}{3}-\frac{4l^2}{\pi^2}\left[\frac{\cos\pi x}{l}-\frac{1}{2^2}\frac{\cos 2\pi x}{l}+\frac{1}{3^2}\frac{\cos 3\pi x}{l}+\dots\right.$$

1. Find a Sin and a Cosine series for the function $f(x)=3x-2$ in the interval $0<x<l$ and $l=4$

Sol:-

Sine Series

$$1=\frac{4}{\pi}\left[\frac{\sin\pi x}{l}+\frac{1}{3}\frac{\sin 3\pi x}{l}+\frac{1}{5}\frac{\sin 5\pi x}{l}+\dots\right]$$

$$x=\frac{2l}{\pi}\left[\frac{\sin\pi x}{l}-\frac{1}{2}\frac{\sin 2\pi x}{l}+\frac{1}{3}\frac{\sin 3\pi x}{l}+\dots\right]$$

$l=4$.

$$1=\frac{4}{\pi}\left[\frac{\sin\pi x}{4}+\frac{1}{3}\frac{\sin 3\pi x}{4}+\frac{1}{5}\frac{\sin 5\pi x}{4}+\dots\right]$$

$$x=\frac{8}{\pi}\left[\frac{\sin\pi x}{4}-\frac{1}{2}\frac{\sin 2\pi x}{4}+\frac{1}{3}\frac{\sin 3\pi x}{4}+\dots\right]$$

$f(x)=3x-2$.

$$=3\left[\frac{8}{\pi}\left(\frac{\sin\pi x}{4}-\frac{1}{2}\frac{\sin 2\pi x}{4}+\frac{1}{3}\frac{\sin\pi x}{4}+\dots\right)\right]-$$

$$2\left[\frac{4}{\pi}\left(\sin\frac{\pi x}{4}+\frac{1}{3}\sin\frac{3\pi x}{4}+\frac{1}{5}\sin\frac{5\pi x}{4}+\dots\right]\right.$$

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$$= \frac{24}{\pi}\left(\frac{\sin \pi x}{4} - \frac{1}{2}\frac{\sin 2\pi x}{4} + \frac{1}{3}\frac{\sin \pi x}{4} + \cdots \right) - \frac{8}{\pi}\left(\frac{\sin \pi x}{4} + \frac{1}{3}\frac{\sin 3\pi x}{4} + \frac{1}{5}\frac{\sin 5\pi x}{4} + \cdots \right)$$

$$= \frac{24}{\pi}\frac{\sin \pi x}{4} - \frac{24}{\pi \cdot 2}\frac{\sin 2\pi x}{4} + \frac{24}{\pi \cdot 3}\frac{\sin \pi x}{4} + \cdots - \frac{8}{\pi}\frac{\sin \pi x}{4} - \frac{8}{3\cdot\pi}\frac{\sin 3\pi x}{4} - \frac{8}{\pi \cdot 5}\frac{\sin 5\pi x}{4} + \cdots$$

$$= \frac{16}{\pi}\frac{\sin \pi x}{4} - \frac{24}{2\pi}\frac{\sin 2\pi x}{4} + \frac{16}{3\pi}\frac{\sin 3\pi x}{4} - \cdots$$

$$= \frac{8}{\pi}\left[2\sin\frac{\pi x}{4} - 3/2 \sin\frac{2\pi x}{4} + 2/3 \sin\frac{3\pi x}{4} \cdots \right]$$

Cosine Series

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{n\pi}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \cdots \right)$$

$l = 4 \Rightarrow$

$$x = 4/2 - 16/\pi^2 \left(\cos n\pi/4 + 1/3^2 \cos\frac{3\pi x}{4} + \cdots \right)$$

$$x = 2 - \frac{16}{\pi^2}\left(\cos\frac{n\pi}{4} + \frac{1}{3^2}\cos\frac{3\pi x}{4} + \cdots \right)$$

$$f(x) = 3x - 2$$

$$= 3\left(2 - \frac{16}{\pi^2}\left(\cos\frac{n\pi}{4} + \frac{1}{3^2}\cos\frac{3\pi x}{4} + \cdots\right)\right) - 2$$

$$= \left(6 - \frac{48}{\pi^2}\cos\frac{\pi x}{4} - \frac{48}{\pi^2 3^2}\cos\frac{3\pi x}{4} + \cdots\right) - 2$$

$$= 6 - 2 - \frac{48}{\pi^2}\cos\frac{\pi x}{4} - \frac{48}{\pi^2 3^2}\cos\frac{3\pi x}{4} - \frac{48}{n^2\pi^2}\cos\frac{5\pi x}{4}.$$

$$= 4 - \frac{48}{\pi^2}\cos\pi x - \frac{48}{3^2\pi^2}\cos\frac{3\pi x}{4} - \frac{48}{\pi^2 5^2}\cos\frac{5\pi x}{4} + \cdots$$

Find a Sine series and Cosine series $f(x) = 2x-4$ $(0<x<4)$, $l=4$

Sol:- Sine series

$$f(x) = 2x-4.$$

$$1 = \frac{4}{\pi}\left[\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right]$$

$$x = \frac{2l}{\pi}\left[\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \cdots\right]$$

$l = 4$

$$1 = \frac{4}{\pi}\left[\sin\frac{\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} + \frac{1}{5}\sin\frac{5\pi x}{4} + \cdots\right]$$

$$x = \frac{2(4)}{\pi}\left[\sin\frac{\pi x}{4} - \frac{1}{2}\sin\frac{2\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} + \cdots\right]$$

$$x = \frac{8}{\pi}\left[\sin\frac{\pi x}{4} - \frac{1}{2}\sin\frac{2\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} + \cdots\right]$$

$$2x-4 = 2\left[\frac{8}{\pi}\left(\sin\frac{\pi x}{4} - \frac{1}{2}\sin\frac{2\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} + \cdots\right)\right.$$

$$-+\left\{\frac{4}{\pi}\left[\sin\frac{\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} + \frac{1}{5}\sin\frac{5\pi x}{4}\right]\right.$$

$$= \left(\frac{16}{\pi}\sin\frac{\pi x}{4} - \frac{16}{2\pi}\sin\frac{2\pi x}{4} + \frac{16}{3\pi}\sin\frac{3\pi x}{4} + \cdots\right) +$$

$$\left(-\frac{16}{\pi}\sin\frac{\pi x}{4} - \frac{16}{3\pi}\sin\frac{3\pi x}{4} + \cdots\right)$$

$$= \frac{-16}{2\pi} \sin\frac{2\pi x}{4} - \frac{16}{4\pi} \sin\frac{4\pi x}{4} - \dots$$

$$f(x) = -\left[\frac{8}{\pi} \sin\frac{\pi x}{2} + \frac{16}{5\pi} \sin\frac{\pi x}{4}\right]$$

$$= -\frac{8}{\pi}\left[\frac{\sin \pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \dots\right]$$

cosine series

$$f(x) = 2x - 4$$

$$x = l/2 - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \dots\right)$$

$$2x - 4 = 2\left[l/2 - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \dots\right)\right] - 4$$

$$= 2\left(4/2 - \frac{4(4)}{\pi^2}\left(\cos\frac{\pi x}{4} + \frac{1}{3^2}\cos\frac{3\pi x}{4} + \frac{1}{5^4}\cos\frac{5\pi x}{4} + \dots\right)\right) - 4$$

$$= \cancel{4} - \frac{32}{\pi^2}\left(\cos\frac{\pi x}{4} + \frac{1}{3^2}\cos\frac{3\pi x}{4} + \dots\right) - \cancel{4}$$

$$= \frac{-32}{\pi^2}\left(\cos\frac{\pi x}{4} + \frac{1}{3^2}\cos\frac{3\pi x}{4} + \dots\right)$$

Find a sine and cosine series for the fn $f(x) = 3x - 9$, $0 < x < 6$.

Sol:-

Sine series

$$l = 6$$

$$x = \frac{2l}{\pi}\left[\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \dots\right]$$

$$1 = \frac{4}{\pi}\left[\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \dots\right]$$

$$l = 6 \Rightarrow$$

$$\Rightarrow x = \frac{2(6)}{\pi}\left[\sin\frac{\pi x}{6} - \frac{1}{2}\sin\frac{2\pi x}{6} + \frac{1}{3}\sin\frac{3\pi x}{6} - \dots\right]$$

$$x = \frac{12}{\pi}\left[\sin\frac{\pi x}{6} - \frac{1}{2}\sin\frac{2\pi x}{6} + \frac{1}{3}\sin\frac{3\pi x}{6} - \dots\right]$$

$$1 = \frac{4}{\pi}\left[\sin\frac{\pi x}{6} + \frac{1}{3}\sin\frac{3\pi x}{6} + \frac{1}{5}\sin\frac{5\pi x}{6} + \dots\right]$$

$$3x - 9 = 3\left[\frac{12}{\pi}\left(\sin\frac{\pi x}{6} - \frac{1}{2}\sin\frac{2\pi x}{6} + \frac{1}{3}\sin\frac{3\pi x}{6} - \dots\right)\right]$$

$$-9\left(\frac{4}{\pi}\left(\sin\frac{\pi x}{6} + \frac{1}{3}\sin\frac{3\pi x}{6} + \frac{1}{5}\sin\frac{5\pi x}{6} + \dots\right)\right)$$

Cosine Series

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \cdots\right)$$

$f(x) = 3x - 9$

$$= 3\left(\frac{6}{2} - \frac{24}{\pi^2}\left(\cos\frac{\pi x}{6} + \frac{1}{3^2}\cos\frac{3\pi x}{6} + \frac{1}{5^2}\cos\frac{5\pi x}{6} + \cdots\right)\right) - 9$$

$$= 9 - \frac{72}{\pi^2}\left(\cos\frac{\pi x}{6} + \frac{1}{3^2}\cos\frac{3\pi x}{6} + \frac{1}{5^2}\cos\frac{5\pi x}{6} + \cdots\right) - 9$$

$$= \frac{-72}{\pi^2}\left(\cos\frac{\pi x}{6} + \frac{1}{3^2}\cos\frac{3\pi x}{6} + \frac{1}{5^2}\cos\frac{5\pi x}{6} + \cdots\right)$$

The sine series for $f(x)$ in the interval $(0, \pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_n = 0$

$$b_n = 2/\pi \int_0^{\pi} f(x) \sin nx \, dx$$

REMARKS:-

1) If $f(x)$ is even function then fourier series becomes c[...] series

$$f(x) = \frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{l}$$

where

$$a_0 = \frac{2}{l} \int_0^{l} f(x) dx$$

$$a_n = \frac{2}{l} \int_0^{l} f(x) \cos \frac{n\pi x}{l} \, d[...]$$

2) If $f(x)$ is an odd fun[...] interval $(0, 2l)$ [...] series becom[...]

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx.$$

3) sine series in half range $(0, l)$ with period $2l$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}.$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx.$$

4) Cosine series in half ...

Express as a fourier series

in $(0, 2\pi)$ $f(x) = \begin{cases} a, & 0 < x < \pi \\ -a, & \pi < x < 2\pi \end{cases}$

Solution:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \cos nx\, f(x)\, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \sin nx\, f(x)\, dx$$

$$a_0 = \frac{1}{\pi} \left\{ \int_0^{\pi} a\, dx + \int_{\pi}^{2\pi} (-a)\, dx \right\}$$

$$= \frac{1}{\pi} \left\{ a [x]_0^{\pi} + (-a) [x]_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{\pi} \left[a[\pi - 0] + (-a)[2\pi - \pi] \right]$$

$$= \frac{1}{\pi} [a\pi - a[\pi]]$$

$$= \frac{1}{\pi} [0]$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi} \left\{ \int_0^{\pi} a \cos nx\, dx + \int_{\pi}^{2\pi} (-a) c\ldots \right.$$

$$= \frac{1}{\pi} \left\{ a \left[\frac{\sin n\pi}{n} - \sin 0 \right] - \right.$$

$$\left. a \left[\frac{\sin n 2\pi}{n} - \frac{\sin n\pi}{n} \right] \right\}$$

$$= \frac{1}{\pi} [a(0) - a(0)] \qquad [\because \sin n\pi = 0$$

$$\sin 0 = 0]$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi} \left\{ \int_0^{\pi} a \sin nx\,dx + \int_{\pi}^{2\pi} (-a) \sin nx\,dx \right\}$$

$$= \frac{1}{\pi} \left\{ a \left[\frac{-\cos nx}{n} \right]_0^{\pi} - a \left[-\frac{\cos nx}{n} \right]_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ -a \left[\frac{\cos n\pi}{n} - \frac{\cos 0}{n} \right] + \right.$$

$$\left. a \left[\frac{\cos n 2\pi}{} \quad \frac{\cos n\pi}{} \right] \right\}$$

$b_n = \frac{1}{\pi}\left\{\frac{2a}{n}\left[1-(-1)^n\right]\right\}$.

Now,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= 0 + \sum_{n=1}^{\infty} \frac{1}{\pi}\left[0 + \left(\frac{2a}{n}(1-(-1)^n)\right) \sin nx\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{\pi}\left\{\frac{2a}{n}(1-(-1)^n) \sin nx\right\}$$

$$= \frac{2a}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\right)\sin nx$$

n is odd, $1-(-1)^n = 2$ n is even $1-(-1)^n = 0$.

$$f(x) = \frac{2a}{\pi}(2)\left\{\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right\}$$

$$f(x) = \frac{4a}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right\}$$

2) A function $f(x)$ is defined within the range $(0, 2a)$ by the function

$$f(x) = \begin{cases} x & \text{in range } (0,\pi) \\ 2\pi - x & \text{in range } (\pi, 2\pi). \end{cases}$$

Express $f(x)$ is a fourier series the range $(0, 2\pi)$.

Solution:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

given $f(x) = \begin{cases} x & (0, \pi) \\ 2\pi - x & (\pi, 2\pi) \end{cases}$

$$a_0 = \frac{1}{\pi} \left\{ \int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx \right\}$$

$$= \frac{1}{\pi} \left[\left(\frac{x^2}{2} \right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2} \right)_{\pi}^{2\pi} \right]$$

$$= \frac{1}{\pi} \left[\left(\frac{\pi^2}{2} - 0 \right) + \left(2\pi(2\pi) - \frac{4\pi^2}{2} \right) - \left(2\pi(\pi) - \frac{\pi^2}{2} \right) \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + \left(4\pi^2 - \frac{4\pi^2}{2} \right) - \left(2\pi^2 - \frac{\pi^2}{2} \right) \right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + (2\pi^2) - 2\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{\cancel{2}\pi^{\cancel{2}}}{\cancel{2}}\right]$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} x \cos nx \, dx + \int_{\pi}^{2\pi} (2\pi - x)\cos nx\, dx\right]$$

$u = x$ $\quad \int dv = \int \cos nx \, dx$ $\quad u = 2\pi - x$

$du = dx$ $\quad v = \frac{\sin nx}{n}$ $\quad du = -dx$

$u\,dv = uv - \int v\,du$.

$$= \frac{1}{\pi}\left[\left(x \frac{\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n}\, dx\right.$$

$$\left. + (2\pi - x)\frac{\sin nx}{n} - \int_{\pi}^{2\pi} \frac{\sin nx}{n}(-dx)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi \sin n\pi}{n} - 0\right) - \left(-\frac{\cos nx}{n^2}\right)_0^{\pi}\right.$$

$$+ \left[(2\pi - 2\pi)\frac{\sin 2\pi}{n} - (2\pi - \pi)\frac{\sin n\pi}{n}\right]$$

$$+ \left[-\frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}$$

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$$= \frac{1}{\pi}\left[\frac{\pi \sin n\pi}{n} + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right) + \right.$$

$$\left. (0-0) + \left(\frac{\cos n(2\pi)}{n^2} + \frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[0 + \left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right) + 0\left(-\frac{1}{n^2} + \frac{(-1)^n}{n^2}\right.\right.$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^n}{n^2} - \frac{2}{n^2}\right]$$

$$= \frac{2(-1)^n}{n^2\pi} - \frac{2}{n^2\pi}$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$\cos 0 = 1$

$\cos n(2\pi) = (-1)^2$

$\cos n(\pi) = (-1)^n$

$$a_n = \frac{2(-1)^n - 2}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} x \sin nx\,dx + \int_{\pi}^{2\pi} (2\pi - x)\sin nx\,dx\right]$$

$u = x \quad \int dv = \int \sin nx\,dx \quad u = 2\pi - x$

$du = dx \quad v = \frac{-\cos nx}{n} \quad du = -dx$

$\int dv = \int \sin nx\,dx$

$v = -\frac{\cos nx}{n}$. $\quad \int u\,dv = uv - \int v\,du$.

$$= \frac{1}{\pi}\left[\left(x\left(-\frac{\cos nx}{n}\right) \right)_0^{\pi} - \int_0^{\pi} -\frac{\cos nx}{n}\,dx + \right.$$

$$\left. \left((2\pi - x) - \frac{\cos nx}{n} \right)_{\pi}^{2\pi} - \int_{\pi}^{2\pi} -\frac{\cos nx}{n}(-dx) \right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{x\cos nx}{n} \right)_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n}\,dx + \right.$$

$$\left. \left(-(2\pi - x)\frac{\cos nx}{n} \right)_{\pi}^{2\pi} - \int_{\pi}^{2\pi} \frac{\cos nx}{n}\,dx \right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi\cos n\pi}{n} + \frac{0\cos 0}{n} \right) + \left(\frac{\sin nx}{n^2} \right)_0^{\pi} \right.$$

$$+ \left(-(2\pi - 2\pi)\frac{\cos n(2\pi)}{n} + (2\pi - \pi)\frac{\cos n\pi}{n} \right.$$

$$\left. - \left(\frac{\sin nx}{-n^2} \right)_{\pi}^{2\pi} \right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi\cos n\pi}{n} + (0)\frac{1}{n} \right) + \left(\frac{\sin n\pi}{n^2} \right. \right.$$

$$\left. - \frac{\sin n0}{n^2} \right) + 0 + \frac{\pi(-1)^n}{n} -$$

$$\left. \left(\frac{\sin n(2\pi)}{n^2} + \frac{\sin n(\pi)}{n^2} \right) \right]$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n} + 0 + 0 + 0 + \frac{\pi(-1)^n}{n} + 0\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n}\right]$$

$$b_n = 0$$

Now, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n(-2)}{\pi n^2}\cos nx + 0 \sin nx\right]$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)\cos nx\right]$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{-2}{\pi}\left(\frac{1-(-1)^n}{n^2}\right)\cos nx\right]$$

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos nx$$

Hence n is odd, $1-(-1)^n = 2$

n is even, $1-(-1)^n = 0$

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\left[\frac{2}{1^2}\cos x + \frac{2}{3^2}\cos 3x + \cdots\right]$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \cdots\right]$$

$$f(x) = \begin{cases} 0 & , -\pi < x < 0 \\ \pi & , 0 < x < \pi \end{cases}$$

Solution:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

$$a_0 = \frac{1}{\pi} \left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \pi\,dx \right]$$

$$= \frac{1}{\pi} \left[0 + \pi (x)_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\pi (\pi - 0) \right]$$

$$= \frac{\pi^2}{\pi}$$

$$a_0 = \pi$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \cos nx\,dx + \int_{0}^{\pi} \pi \cos nx\,dx \right.$$

$$= \frac{1}{\pi} \left[0 + \pi \int_{0}^{\pi} \cos nx\,dx \right]$$

$$= \frac{\pi}{\pi} \left[\int_{0}^{\pi} \cos nx\,dx \right]$$

$$= \left[\frac{\sin nx}{n} \right]_0^{\pi} = \frac{\sin n\pi}{n} - \frac{\sin 0}{n}$$

$a_n = 0$

$\sin 0 = 0$

$\sin n\pi = 0$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \sin nx\,dx + \int_{0}^{\pi} \pi \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\pi \int_{0}^{\pi} \sin nx\,dx\right]$$

$$= -\frac{\pi}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= -\frac{\cos n\pi}{n} + \frac{\cos 0}{n}$$

$\cos 0 = 1$

$\cos n\pi = (-1)^n$

$$= -\frac{(-1)^n}{n} + \frac{1}{n}$$

$$= \frac{1}{n} - \frac{(-1)^n}{n}$$

$$b_n = \frac{1}{n}(1 - (-1)^n)$$

Now,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \left(0 + \frac{1}{n} - \frac{(-1)^n}{n} \sin nx\right)$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{1}{n} - \frac{(-1)^n}{n}\right)(\sin nx)\right]$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{n}(1 - (-1)^n)\right] \sin nx$$

Hence.

N is odd, $1-(-1)^n = 2$

N is even. $1-(-1)^n = 0$

$$f(x) = \frac{\pi}{2} + 2\sin x + \frac{2}{3}\sin 3x + \frac{2}{5}\sin 5x + \dots$$

$$f(x) = \frac{\pi}{2} + 2\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right]$$

show that the range 0 to 2π the expression of e^x as a fourier series,

$$e^x = \frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2+1} - \sum_{n=1}^{\infty}\frac{n\sin nx}{n^2+1}\right]$$

Find the fourier series of $f(x) = e^x$ in $(0, 2\pi)$

Solution:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n \sin nx]$$

where.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

Given $f(x) = e^x$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^x\,dx$$

$$= \frac{1}{\pi}\left[e^{x}\right]_0^{2\pi} = \frac{1}{\pi}\left[e^{2\pi} - e^{0}\right]$$

$$= \frac{e^{2\pi} - e^{0}}{\pi} = \frac{e^{2\pi} - 1}{\pi}$$

$$a_0 = \frac{e^{2\pi} - 1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{x}\cos nx \, dx$$

$$\left[\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{x}}{1^2+n^2}(\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}}{1^2+n^2}(\cos n(2\pi) + n\sin(2\pi)) - \frac{e^{0}}{1^2+n^2}(\cos n(0) + n\sin(0))\right]$$

$\cos 2\pi = (-1)^{n}$

$\sin n\pi = 0$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}}{1^2+n^2}(1+0) - \frac{1}{1+n^2}(1+0)\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2} - \frac{1}{n^2+1}\right]$$

$$= \frac{1}{\pi(1+n^2)}(e^{2\pi} - 1)$$

$$= \frac{e^{2\pi} - 1}{\pi(1+n^2)}.$$

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$$a_n = \frac{e^{2\pi}-1}{\pi} \cdot \frac{1}{(1+n^2)}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} e^{x} \sin nx \, dx$$

$$\left[\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos x) \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{x}}{1+n^2} (\sin nx - n \cos nx) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{2\pi}}{1+n^2} (\sin(2\pi) - n \cos n(2\pi)) \right.$$

$$\left. - \frac{e^{0}}{1+n^2} (\sin n(0) - n \cos n(0)) \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{2\pi}}{1+n^2} (0 - n(1)) - \frac{1}{1+n^2} (0 - n(1)) \right]$$

$\sin \pi = 0$

$\cos n\pi = (-1)^n$

$\cos n(2\pi) = 1$

$$= \frac{1}{\pi} \left[\frac{e^{2\pi}}{1+n^2} (-n) - \frac{1}{n^2+1} (-n) \right]$$

$$= \frac{1}{\pi} \left[\frac{-ne^{2\pi}}{1+n^2} + \frac{n}{1+n^2} \right]$$

$$= \frac{1}{\pi (1+n^2)} (n - ne^{2\pi})$$

$$= \frac{1}{\pi(1+n^2)} \, n(1-e^{2\pi})$$

$$= \frac{1-e^{2\pi}}{\pi} \cdot \frac{n}{1+n^2}$$

$$= -\frac{(e^{2\pi}-1)}{\pi} \left(\frac{n}{1+n^2}\right)$$

$$b_n = \frac{-n}{1+n^2} \cdot \frac{e^{2\pi}-1}{\pi}$$

Now,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$e^x = \frac{\frac{e^{2\pi}-1}{\pi}}{2} + \sum_{n=1}^{\infty} \left(\frac{e^{2\pi}-1}{\pi} \cdot \frac{1}{1+n^2} \cos nx + \frac{e^{2\pi}-1}{\pi} \cdot \frac{(-n)}{1+n^2} \sin nx \right)$$

$$e^x = \frac{e^{2\pi}-1}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2+1} - \frac{n \sin nx}{n^2+1} \right]$$

Express as a fourier series in $(0, 2\pi)$

$$f(x) = \begin{cases} x & ; 0 < x < \pi \\ 0 & ; \pi < x < 2\pi \end{cases}$$

Solution :-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \cos nx \, f(x) dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \sin nx \, f(x) dx.$$

$$a_0 = \frac{1}{\pi} \left[\int_0^{\pi} x dx + \int_{\pi}^{2\pi} 0 dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x dx \right] = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} - 0 \right]$$

$$a_0 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \left[\int_0^{\pi} x \cos nx \cdot dx + \int_{\pi}^{2\pi} 0 \cos nx dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x \cos nx \, dx \right]$$

$u = x \qquad \int dv = \int \cos nx \, dx$

$du = dx \qquad v = \frac{\sin nx}{n}$

$$\int u dv = uv - \int v du$$

$$= \frac{1}{\pi}\left[\left(\frac{x\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n}\left(\frac{\cos nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] \quad \cos n\pi = (-1)^n \quad \cos 0 = 1$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{1}{n^2\pi}\left[\frac{(-1)^n}{n^2\pi} - \frac{1}{n^2\pi}\right]$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi} 0 \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx\right]$$

$u = x \qquad \int dv = \int \sin nx\,dx$

$du = dx \qquad v = \frac{-\cos nx}{n}$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{1}{\pi}\left[\left(\frac{-x\cos nx}{n}\right)_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-x\cos nx}{n}\right)_0^{\pi} + \left(\frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n} + \frac{0\cos n\cos}{n} + 0\right]$$

$\sin n\pi = 0$

$\sin 0 = 0$

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$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n}\right]$$

$$= \frac{-1(-1)^n}{n}$$

$$b_n = \frac{-(-1)^n}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\left(\frac{(-1)^n}{n^2\pi} - \frac{1}{n^2\pi}\right)\cos nx - \frac{(-1)^n}{n}\sin nx\right]$$

n is odd,

$$\left[\frac{(-1)^1}{1^2\pi} - \frac{1}{1^2\pi}\right]\cos x - \frac{(-1)^1}{1}\sin nx$$

$$= \frac{-1-1}{\pi}\cos x + \sin x$$

n is even.

$$\left[\frac{(-1)^2}{2^2\pi} - \frac{1}{2^2\pi}\right]\cos 2x - \frac{(-1)^2}{2}\sin 2x = 0 - \frac{1}{2}$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

$$+ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots$$

6) If $f(x) = \begin{cases} -\pi & (-\pi < x < 0) \\ x & (0 < x < \pi) \end{cases}$

Find fourier series in $(-\pi, \pi)$.

Solution :-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

$$a_0 = \frac{1}{\pi} \left[\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx \right]$$

$$= \frac{1}{\pi} \left[-\pi (x)_{-\pi}^{0} + \left(\frac{x^2}{2} \right)_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[-\pi (0+\pi) + \frac{\pi^2}{2} - 0 \right]$$

$$= \frac{1}{\pi} \left[0 - \pi^2 + \frac{\pi^2}{2} - 0 \right]$$

$$= \frac{1}{\pi} \left[-\pi^2/2 \right]$$

$$= -\pi/2$$

$$a_0 = -\pi/2.$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0} \cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$u = x$ $\quad\quad$ $\int dv = \int \cos nx\,dx$

$du = dx$ $\quad\quad$ $v = \frac{\sin nx}{n}$

$$\int u\,dv = uv - \int v\,du.$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{x\sin nx}{n}\right)_{0}^{\pi} - \int_{0}^{\pi}\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[0+0+\left(\frac{\cos nx}{n^2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos n(0)}{n^2}\right]$$

$\sin n\pi = 0$
$\sin 0 = 0$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx\,dx + \int_{0}^{\pi} x\sin nx\right.$$

$$= \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0} \sin nx\,dx + \int_{0}^{\pi} x\sin nx\right.$$

$u = x \qquad \int dv = \int \sin nx\, dx$

$du = dx \qquad v = -\frac{\cos nx}{n}$

$$= \frac{1}{\pi}\Big[-\pi\Big(\frac{-\cos nx}{n}\Big)_{-\pi}^{0} +$$

$$\Big(\frac{-x\cos nx}{n}\Big)_{0}^{\pi} + \int_{0}^{\pi}\frac{\cos nx\, dx}{n}\Big]$$

$$= \frac{1}{\pi}\Big[-\pi\Big(\frac{\cos 0}{n} - \frac{\cos n\pi}{n}\Big) +$$

$$\Big(-\frac{\pi\cos n\pi}{n} + \frac{0\cos 0}{n}\Big) + 0\Big]$$

$$= \frac{1}{\pi}\Big[\pi\Big(\frac{1}{n} - \frac{(-1)^n}{n}\Big) + \frac{-\pi(-1)^n}{n}$$

$$+ \frac{1}{n}(0)\Big]$$

$$= \frac{1}{\pi}\Big[\frac{\pi}{n} - \frac{\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n} + \frac{1}{n}(0)\Big]$$

$$= \frac{1}{\pi}\Big[\frac{\pi}{n} - \frac{2\pi(-1)^n}{n}\Big]$$

$$= \frac{\pi(1 - 2(-1)^n)}{n\pi}$$

$$b_n = \frac{1 - 2(-1)^n}{n}$$

Now, $f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \left[\left(\frac{1}{\pi n^2} ((-1)^n - 1) \right) \cos nx \right.$

$\left. + \left(\frac{1 - 2(-1)^n}{n} \right) \sin nx \right]$

n is odd.

$[(-1)^1 - 1] \cos x = \frac{-2}{\pi} \cos x$

$1 - 2(-1)^1 \sin x = 3 \sin x.$

n is even.

$[(-1)^2 - 1] \cos 2x = 0$

$\frac{1 - 2(-1)^2}{2} \sin 2x = \frac{-1}{2} \sin 2x.$

$f(x) = -\frac{\pi}{4} - \frac{2}{\pi} \left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right]$

$+ \left[3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \cdots \right]$

Express $f(x) = \frac{1}{2}(\pi - x)$ as a fourier series with period 2π to be valid in the interval 0 to 2π.

Solution:-

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where, $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx$.

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx.$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx.$$

Given $f(x) = \frac{1}{2}(\pi - x)$.

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2}(\pi - x)\,dx$$

$$= \frac{1}{2\pi} \left[\int_0^{2\pi} \pi \, dx - \int_0^{2\pi} x \, dx \right]$$

$$= \frac{1}{2\pi} \left[[\pi x]_0^{2\pi} - \left(\frac{x^2}{2}\right)_0^{2\pi} \right]$$

$$= \frac{1}{2\pi} \left[(\pi x)_0^{2\pi} - \left(\frac{x^2}{2}\right)_0^{2\pi} \right]$$

$$= \frac{1}{2\pi} \left[2\pi^2 - \frac{4\pi^2}{2} \right]$$

$$= \frac{1}{2\pi} \left[2\pi^2 - 2\pi^2 \right]$$

$$a_0 = 0.$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2}(\pi - x) \cos nx \, dx.$$

$\int u\,dv = uv - \int v\,du$.

$$= \frac{1}{2\pi}\left\{\left[(\pi - x)\left(\frac{\sin nx}{n}\right)\right]_0^{2\pi} - \int_0^{2\pi}\frac{\sin nx}{n}(-dx)\right\}$$

$$= \frac{1}{2\pi}\left\{\left[(\pi - x)\left(\frac{\sin nx}{n}\right)\right]_0^{2\pi} + \int_0^{2\pi}\frac{\sin nx}{n}\,dx\right\}$$

$$= \frac{1}{2\pi}\left\{0 + \left[-\frac{\cos nx}{n^2}\right]_0^{2\pi}\right\}$$

$$= \frac{1}{2\pi}\left(-\frac{\cos n(2\pi)}{n^2} - \frac{\cos 0}{n^2}\right).$$

$$= \frac{1}{2\pi}\left(-\frac{1}{n^2} + \frac{1}{n^2}\right).$$

$\sin n\pi = 0$

$\sin 0 = 0$

$\cos n\pi = (-1)^n$

$\cos 2\pi = (-1)^2 = 1$

$a_n = 0$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\sin nx\,dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\sin nx\,dx$$

$u = (\pi - x)$ $\int dv = \int \sin nx\,dx$

$du = -dx$

$v = -\frac{\cos nx}{n}$

$= \frac{1}{2\pi}\left\{\left[(\pi - x)\left(-\frac{\cos nx}{n}\right)\right]_0^{2\pi} - \int_0^{2\pi} -\frac{\cos nx}{n}(-dx)\right\}$

$= \frac{1}{2\pi}\left\{\left[-(\pi - x)\left(\frac{\cos nx}{n}\right)\right]_0^{2\pi} - \int_0^{2\pi}\frac{\cos nx}{n}dx\right\}$

$= \frac{1}{2\pi}\left\{-(\pi - 2\pi)\left(\frac{\cos n(2\pi)}{n}\right) - (\pi - 0)\left(\frac{-\cos n(0)}{n}\right) - \int_0^{2\pi}\frac{\cos nx}{n}dx\right\}$

$= \frac{1}{2\pi}\left\{-(-\pi)\frac{\cos n(2\pi)}{n} - \pi\left(\frac{-\cos n(0)}{n}\right) - \int_0^{2\pi}\frac{\cos nx}{n}dx\right\}$

$= \frac{1}{2\pi}\left\{\frac{\pi}{n} + \frac{\pi}{n} - \left[\frac{\sin nx}{n^2}\right]_0^{2\pi}\right\}$

$\cos n(2\pi) = (-1)^2 = 1$

$\cos 0 = 1$

$\sin n\pi = 0$

$\sin 0 = 0$

$= \frac{1}{2\pi}\left\{\frac{2\pi}{n} - 0\right\}$

$= \frac{1}{2\pi}\left[\frac{2\pi}{n}\right]$ $b_n = \frac{1}{n}$

Now,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\tfrac{1}{2}(\pi - x) = 0/2 + \sum_{n=1}^{\infty} (0 + 1/n \sin nx)$$

$$= \sum_{n=1}^{\infty} (1/n \sin nx)$$

$$= 1/1 \sin x + 1/2 \sin 2x + 1/3 \sin 3x + \dots$$

$$\tfrac{1}{2}(\pi - x) = \sin x + 1/2 \sin 2x + 1/3 \sin 3x + \dots$$

DIRICHLET CONDITION :-

$$f(x) = \begin{cases} -1 & , (-\pi, 0) \\ 1 & , (0, \pi) \end{cases}$$

solution :-

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx.$$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -1\,dx + \int_{0}^{\pi} 1\,dx \right]$$

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$$= \frac{1}{\pi}\left[(-x)_{-\pi}^{0} + (x)_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[-\pi + \pi \right]$$

$$a_0 = 0$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-1)\cos nx\,dx + \int_{0}^{\pi} 1\cos nx\,dx \right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{\sin nx}{n}\right)_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}(0+0)$$

$\sin \pi = 0$

$\sin 0 = 0.$

$$a_n = 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -1\sin nx\,dx + \int_{0}^{\pi} 1\sin nx\,dx \right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(\frac{-\cos nx}{n}\right)_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{\pi} - \frac{\cos n\pi}{n}\right) - \left(\frac{\cos n\pi}{n} - \frac{1}{n}\right) \right]$$

$$= \frac{1}{\pi}\left[\frac{2}{n} - \frac{2(-1)^n}{n}\right]$$

$$= \frac{2[1-(-1)^n]}{n\pi}$$

$$bn = \frac{2}{n\pi}[1-(-1)^n]$$

$$bn = \frac{2}{n\pi}[1-(-1)^n]$$

Now,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \sum_{n=1}^{\infty} \frac{2}{n\pi}[1-(-1)^n] \sin nx.$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right] \sin nx$$

$$f(x) = \frac{2}{\pi}\left[\frac{2}{1}\sin x + \frac{2}{3^2}\sin 3x + \frac{2}{5^2}\sin 5x + \right.$$

$$f(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x + \ldots\right]$$

10) If $f(x) = x+x^2$ in $(-\pi < x < \pi)$ find fourier series.

solution :-

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx\, f(x)\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx.$$

Given $f(x) = x + x^2$.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+x^2)\,dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}.$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{(-\pi)^2}{2} + \frac{(-\pi)^3}{3}\right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi} \left[\frac{2\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_0 = \frac{2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx.$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx\,dx.$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\,dx + \int_{-\pi}^{\pi} x^2\cos nx\,dx\right]$$

$u = x$ $\qquad \int dv = \int \cos nx\,dx$

$du = dx$ $\qquad v = \frac{\sin nx}{n}$

$u = x^2$ $\qquad \int dv = \int \cos nx\,dx$

$du = 2x\,dx$ $\qquad v = \frac{\sin nx}{n}$

$$= \frac{1}{\pi}\left[\left(x\frac{\sin nx}{n}\right)_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin nx}{n}\,dx\right] +$$

$$\frac{1}{\pi}\left[\left(x^2\frac{\sin nx}{n}\right)_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin nx}{n}\,2x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi\sin n\pi}{n} - \frac{(-\pi)\sin(-\pi)}{n}\right) + \left(\frac{\cos nx}{n^2}\right)_{-\pi}^{\pi}\right]$$

$$+ \frac{1}{\pi}\left[\left(\frac{\pi^2\sin n(\pi^2)}{n} - \frac{(-\pi)^2\sin n(-\pi)}{n} - \frac{2}{n}\int_{-\pi}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \left(\frac{\cos nx}{n^2}\right)_{-\pi}^{\pi}\right] + \frac{1}{\pi}\left[-\frac{2}{n}\int_{-\pi}^{\pi} x\sin nx\,dx\right]$$

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$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos n(-\pi)}{n^2}\right] + \frac{1}{\pi}\left[\frac{-2}{n}\int_{-\pi}^{\pi} x \sin nx\, dx\right.$$

$$= \frac{-2}{n\pi}\left[\left(-\frac{x\cos nx}{n}\right)_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -\frac{\cos nx}{n}\, dx\right]$$

$$= \frac{2}{n\pi}\left[\frac{\pi\cos n\pi}{n} - \frac{(-\pi)\cos n(-\pi)}{n}\right] + \left[\frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}$$

$\sin n\pi = 0$

$\cos n\pi = \cos n\pi$

$$= \frac{2}{n\pi}\left[\frac{2\pi\cos n\pi}{n} + 0\right]$$

$$= \frac{4\pi\cos n\pi}{n^2\pi} = \frac{4\cos n\pi}{n^2} = \frac{4(-1)^n}{n^2}$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\, dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx$$

$u = x \qquad u = x^2 \qquad \int dv = \int \sin nx\, dx$

$du = dx \qquad du = 2x\,dx \qquad v = -\frac{\cos nx}{n}$

$$= \frac{1}{\pi}\left[\left(-\frac{x\cos nx}{n}\right)_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx}{n}\, dx\right] +$$

$$\frac{1}{\pi}\left[\left(-\frac{x^2\cos nx}{n}\right)_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx}{n}\, 2x\, dx\right]$$

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$$= \frac{1}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} - \frac{(-\pi)\cos n(-\pi)}{n}\right) + \int_{-\pi}^{\pi}\frac{\cos nx}{n}\,dx\right]$$

$$+ \left[\left(\frac{-\pi^2\cos n\pi}{n} - \frac{(-\pi)^2\cos n\pi}{n}\right) + \frac{2}{n}\int_{-\pi}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\pi\cos n\pi}{n} + 0\right] + \frac{1}{\pi}\left[\left(\frac{-2\pi^2\cos n\pi}{n}\right) + \frac{2}{n}\int_{-\pi}^{\pi} x\cos nx\right)$$

$u = x \qquad \int dv = \int \cos nx\,dx$

$du = dx \qquad v = \frac{\sin nx}{n}$

$$= \frac{-2\cos n\pi}{n} + \frac{1}{\pi}\left[\frac{-2\pi^2\cos n\pi}{n} + \frac{2}{n}\left(\left(\frac{x\sin nx}{n}\right)_{-\pi}^{\pi}\right.\right.$$

$$\left.\left. - \int_{-\pi}^{\pi}\frac{\sin nx}{n}\,dx\right)\right]$$

$$= \frac{-2\cos n\pi}{n} + \frac{1}{\pi}\left[\frac{-2\pi^2\cos n\pi}{n} + \frac{2}{n}\cancel{}0\left(\frac{\cos nx}{n^2}\right)_{-\pi}^{\pi}\right]$$

$$= \frac{-2\cos n\pi}{n} + \frac{1}{\pi}\left[\frac{-2\pi^2\cos n\pi}{n} + \frac{2}{n^3}\left[\cos n\pi - (\cos n(-\pi))\right]\right]$$

$$= \frac{-2\cos n\pi}{n} + \frac{1}{\pi}\left[\frac{-2\pi^2\cos n\pi}{n} + \frac{2}{n^3}(0)\right]$$

$$= -\frac{2\cos n\pi}{n} - \frac{2\pi\cos n\pi}{n}$$

Now,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$= \frac{2\pi^2/3}{2} + \sum_{n=1}^{\infty} \left[\frac{4(-1)^n}{n^2} \cos nx + \frac{2(-1)^{n+1}}{n} \sin nx \right]$$

$$= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$$f(x) = \frac{\pi^2}{3} + 4 \left[\frac{-1}{1^2} \cos x + \frac{1}{2^2} \cos 2x - \frac{1}{3^2} \cos 3x \ldots \right]$$

$$+ 2 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots \right]$$

11) $f(x) = x(2\pi - x)$ in $(0, 2\pi)$

Solution :-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\,dx$$

given $f(x) = x(2\pi - x)$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} x(2\pi - x)\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} (2\pi x - x^2) dx$$

$$= \frac{1}{\pi} \left[\frac{2\pi x^2}{2} - \frac{x^3}{3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\pi x^2 - \frac{x^3}{3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\pi 4\pi^2 - \frac{(2\pi)^3}{3} \right]$$

$$= \frac{1}{\pi} \left[4\pi^3 - \frac{8\pi^3}{3} \right]$$

$$= \frac{1}{\pi} \left[\frac{12\pi^3 - 8\pi^3}{3} \right]$$

$$= \frac{1}{\pi} \left[\frac{4\pi^3}{3} \right]$$

$$a_0 = \frac{4\pi^3}{3}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x(2\pi - x) \cos nx \, dx.$$

$$= \frac{1}{\pi} \int_0^{2\pi} (2\pi x - x^2) \cos nx \, dx.$$

$$= \frac{1}{\pi} \left[\int_0^{2\pi} 2\pi x \cos nx dx - \int_0^{2\pi} x^2 \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[2\pi \int_0^{2\pi} x \cos nx dx - \int_0^{2\pi} x^2 \cos nx \, dx \right]$$

$$= \frac{1}{\pi}\left[2\pi\int_0^{2\pi} x\cos nx\,dx - \int_0^{2\pi} x^2\cos nx\,dx\right]$$

$$= \frac{2\pi}{\pi}\int_0^{2\pi} x\cos nx\,dx - \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

$u = x \qquad u = x^2 \qquad \int dv = \int \cos nx\,dx$

$du = dx \qquad du = 2x\,dx \qquad v = \frac{\sin nx}{n}$

$$= 2\left[\left(\frac{x\sin nx}{n}\right)_0^{2\pi} - \int_0^{2\pi}\frac{\sin nx}{n}\,dx\right]$$

$$- \frac{1}{\pi}\left[\left(x^2\frac{\sin nx}{n}\right)_0^{2\pi} - \int_0^{2\pi}\frac{\sin nx}{n}\,2x\,dx\right]$$

$$= 2\left[\left(\frac{2\pi\sin n(2\pi)}{n} - 0\right) - \frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right]$$

$$- \frac{1}{\pi}\left[\left(\frac{(2\pi)^2\sin n(2\pi)}{n} - 0\right) - \frac{2}{\pi}\int_0^{2\pi} x\sin nx\,dx\right]$$

$[\sin n\pi = 0]$

$$= 2(0) + \frac{1}{n}\left[\frac{\cos nx}{n}\right]_0^{2\pi} - \frac{1}{\pi}\left[0 - \frac{2}{n}\int_0^{2\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{n}\left[\cos n(2\pi) - \cos 0\right] - \frac{2}{n\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{n^2}\left[(0)\right] - \frac{2}{n\pi}\int_0^{2\pi} x\sin nx\,dx$$

$u = x$ $\quad\quad \int dv = \int \sin nx \, dx$

$du = dx$ $\quad\quad v = -\frac{\cos nx}{n}$.

$= -\frac{2}{n\pi}\left[\left(\frac{-2\pi \cos n(2\pi)}{n} - 0\right) - \int -\frac{\cos nx}{n} dx\right]$

$= -\frac{2}{n\pi}\left[\left(\frac{-2\pi \cos n(2\pi)}{n} - 0\right) + \left(\frac{\sin nx}{n^2}\right)_0^{2\pi}\right]$

$= \frac{-2}{n\pi}\left[\frac{-2\pi}{n} + \left(\frac{\sin n(2\pi)}{n^2} - \frac{\sin n(0)}{n^2}\right)\right]$

$\sin n(2\pi) = 0$

$\sin 0 = 0$

$= \frac{-2}{n\pi}\left[\frac{-2\pi}{n} + 0\right]$

$= \frac{4\pi}{n^2\pi} = \frac{4}{n^2}$.

$a_n = \frac{4}{n^2}$ $\quad\quad b_n = 0$.

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx$

$= \frac{4\pi^2/3}{2} + \sum_{n=1}^{\infty} \left(\frac{-4}{n^2}\right) \cos nx$.

$= \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2}\right)$

$f(x) = \frac{2\pi^2}{3} - 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right]$

UNIT - V

Characteristic equation & cayley Hamilton theorem:

Def:

An expression of the form $A_0 + A_1 x + A_2 x^2 + \dots + A_n x^n$ where $A_0, A_1, \dots, A_n$ are square matrices of the same order and $A_n \neq 0$ is called a matrix polynomial of degreen

Ex:

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} + x \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} + x^2 \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$ is a matrix polynomial of degree 2 and it is symply the matrix

$$\begin{bmatrix} 1+x+2x^2 & 2+x \\ 2x+3x^2 & 3+x+x^2 \end{bmatrix}$$

Characteristic Matrix

Definition :-

Let A be any square matrix of order n and let I be the identity matrix of order n. Then the matrix polynomial given by $A - xI$ is called the Chacteristic matrix of A.

Characteristic polynomial

The determinant $|A - xI|$ determinate which is an ordinary polynomial in x degree n is called the characteristic polynomial of A. (The equation $|A - xI| = 0$ is called the characteristic equation of A.)

Ex:1 5 mark

Find characteristic matrix characteristic polynomial and characteristic equation of the following matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Sol:

The characteristic matrix of A is given by

$$A - xI = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$$

$$A - xI = \begin{bmatrix} 1-x & 2 \\ 3 & 4-x \end{bmatrix}$$

The characteristic polynomial is given by

$$|A - xI| = \begin{vmatrix} 1-x & 2 \\ 3 & 4-x \end{vmatrix}$$

$$= (1-x)(4-x) - 6$$

$$= 4 - x - 4x + x^2 - 6$$

$$= 4 - 5x - x^2 - 6$$

$$|A - xI| = x^2 - 5x - 2$$

The characteristic equation is given by

$$|A - xI| = 0$$

$$\therefore x^2 - 5x - 2 = 0.$$

Ex: 2

Find characteristic Martric, characteristic Polynomial and characteristic equation of the following Matrix $= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

Sol:-

The characteristic matrix of A is given by

$$A - xI = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} - x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} - \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}$$

$$A - xI = \begin{bmatrix} 1-x & 0 & 2 \\ 0 & 1-x & 2 \\ 1 & 2 & x \end{bmatrix}$$

The characteristic polynomial is given by

$$|A-xI| = \begin{vmatrix} 1-x & 0 & 2 \\ 0 & 1-x & 2 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= (1-x)\left[(1-x)(-x)-4\right] - 0(0-2) + 2(0-(1-x))$$

$$= (1-x)\left[-x+x^2-4\right] + 2 + 2(-1+x)$$

$$= -x + x^2 - 4 + x^2 - x^3 + 4x - 2 + 2x$$

$$|A-xI| = -x^3 + 2x^2 + 5x - 6$$

$$-x^3 + 2x^2 + 5x - 6 = 0$$

The characteristic equation is $|A-xI| = 0$

$$-x^3 + 2x^2 + 5x - 6 = 0$$

$$x^3 - 2x^2 - 5x + 6 = 0$$

3. Find the characteristic polynomial for the following matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Sol:-

The characteristic matrix of A is given by

$$A - xI = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$$

$$A - xI = \begin{bmatrix} -x & 1 \\ 0 & -x \end{bmatrix}$$

The characteristic polynomial is given by

$$|A-xI| = \begin{vmatrix} -x & 1 \\ 0 & -x \end{vmatrix}$$

$$= x^2 - 0$$

$$= x^2$$

The characteristic equation is $|A-xI| = 0$

$$x^2 = 0$$

Theorem 7.31

Any Square Matrix its characteristic equation.

i.e) if $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is the Characteristic polynomial of degree n of A then

$$a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0$$

Proof:

Let A be a square matrix of order n.

let $|A - xI| = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \rightarrow (1)$

be the characteristic polynomial of A.

Now, $\operatorname{adj}(A - xI)$ is a matrix polynomial of degree $n-1$ since each entry of the matrix $\operatorname{adj}(A - xI)$ is a confactor of $A - xI$ and hence is a polynomial of degree $\leq n-1$

$\therefore$ Let $\operatorname{adj}(A - xI) = B_0 + B_1 x + B_2 x^2 + \cdots + B_{n-1} x^{n-1} \rightarrow (2)$

Now $(A - xI)\operatorname{adj}(A - xI) = |A - xI| I$

$(\because (\operatorname{adj} A) A = A(\operatorname{adj} A) = |A| I)$

$\therefore (A - xI)(B_0 + B_1 x + \cdots + B_{n-1} x^{n-1})$

$= (a_0 + a_1 x + \cdots + a_n x^n)$ using (1) and (2)

$\therefore$ Equating the coefficients of the corresponding powers of x we get.

$$A B_0 = a_0 I$$

$$A B_1 - B_0 = a_1 I$$

$$A B_2 - B_1 = a_2 I$$

$$\cdots \quad \cdots$$

$$A B_{n-1} - B_{n-2} = a_{n-1} I$$

$$- B_{n-1} = a_n I$$

Pre multiplying the above equations by $I, A, A^2, \dots A^n$ respectively and adding we get.

$$a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n = 0.$$

NOTE

The inverse of a non-singular matrix can be calculated by using the cayley Hamilton theorem as follows.

Let $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ be the characteristic polynomial of A.

Then by theorem 1.1 we have

$$a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n = 0 \rightarrow (3)$$

Since $|A - xI| = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ we get

$a_0 = |A|$ (by putting $x = 0$)

$\therefore a_0 \neq 0$ ($\because$ A is a non singular matrix)

$$\therefore I = -\frac{1}{a_0}\left[a_1 A + a_2 A^2 + \dots + a_n A^n\right] \text{ (by 3)}$$

$$\boxed{A^{-1} = \frac{-1}{a_0}\left[a_1 I + a_2 A + \dots + a_n A^{n-1}\right]}$$

Solved Problems.

Problem 1:

Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Soln:

The characteristic equatio of A is given by

$|A - \lambda I| = 0$

$$\text{i.e.,} \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)[(7-\lambda)(3-\lambda) - 16] + 6[-6(3-\lambda) + 8] + 2[24 - 2(7-\lambda)] = 0$$

i.e., $(8-\lambda)(\lambda^2 - 10\lambda + 5) + 6(6\lambda - 10) + 2(2\lambda + 10) = 0$

i.e., $(8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda) + (36\lambda - 60) + (4\lambda + 20) = 0$

i.e., $\lambda^3 - 18\lambda^2 + 45\lambda = 0$ which represents the characteristic equation of A.

Problem 2

Show that the non-singular matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ satisfies the equation $A^2 - 2A - 5I = 0$ Hence evaluate A^{-1}.

Sol:-

The characteristic polynomial of A is

$$|A - xI| = \begin{vmatrix} 1-x & 2 \\ 3 & 1-x \end{vmatrix} = x^2 - 2x - 5.$$

By cayley Hamilton theorem $A^2 - 2A - 5I = 0$

$$= \frac{1}{5}\left[\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{5}\begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}$$

Problem 3.

Show that the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$ Satisfies the equation $A(A-I)(A+2I) = 0$

Sol:-

The characteristic polynomial of A is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{vmatrix}$$

$$= -\lambda^3 - \lambda^2 + 2\lambda \text{ (verify)}$$

By cayley Hamilton theorem:- $A^3 - A^2 + 2A = 0$

i.e., $A^3 + A^2 - 2A = 0$ Hence $A(A^2 + A - 2I) = 0$

$\therefore A(A+2I)(A-I) = 0$

Problem 4:

Using cayley Hamilton theorem find the inverse of the matrix.

$$\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

Sol:-

Let $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic polynomial of $A = |A - xI|$

$$= \begin{vmatrix} 7-x & 2 & -2 \\ -6 & -1-x & 2 \\ 6 & 2 & -1-x \end{vmatrix}$$

$$= (7-x)[(1+x)^2 - 4] - 2[6(1+x) - 12] - 2[-12 + 6(1+x)]$$

$$= (7-x)(x^2 + 2x - 3) - 12(x-1) - 12(x-1)$$

$$= 7x^2 + 14x - 21 - x^3 - 2x^2 + 3x - 12x + 12 - 12x + 12$$

$$= -x^3 + 5x^2 - 7x + 3.$$

$\therefore$ By Cayley Hamilton theorem.

$$-A^3 + 5A^2 - 7A + 3I_3 = 0$$

$$\therefore A^3 - 5A^2 + 7A - 3I_3 = 0$$

$$\therefore 3I_3 = A^3 - 5A^2 + 7A$$

$$I_3 = \frac{1}{3}(A^3 - 5A^2 + 7A$$

Pre multiplying by A^{-1} on both side we get

$$A^{-1} = \frac{1}{3}[A^2 - 5A + 7I_3] \quad \rightarrow (1)$$

Now, $A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \text{verify}$$

$\therefore$ From (1)

$$A^{-1} = \frac{1}{3}\left[\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - \begin{bmatrix} 35 & 10 & -10 \\ 30 & -5 & 10 \\ 30 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}\right]$$

$$= \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

Problem 5.

Find Inverse of the matrix $\begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ using Cayley-Hamilton theorem.

Sol:-

The characteristic polynomial of A.

$$= |A - xI| = \begin{vmatrix} 3-x & 3 & 4 \\ 2 & -3-x & 4 \\ 0 & -1 & 1-x \end{vmatrix}$$

$$= -x^3 + x^2 + 11x - 11 \text{ (verify)}$$

$\therefore$ By cayley Hamilton theorem

$$-A^3 + A^2 + 11A - 11I_3 = 0$$

$$\therefore A^3 - A^2 - 11A + 11I_3 = 0$$

Hence $11I_3 = -(A^3 - A^2 - 11A)$

$$\therefore I_3 = -\frac{1}{11}\left[A^3 - A^2 - 11A\right]$$

Pre multiplying by A^{-1} on both sides we get

$$A^{-1} = -\frac{1}{11}\left[A^2 - A - 11I_3\right]$$

$$= -\frac{1}{11}\left[\begin{bmatrix} 15 & -4 & 28 \\ 0 & 11 & 0 \\ -2 & 2 & -3 \end{bmatrix} - \begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\right.$$

$$\left. -11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] = \begin{bmatrix} \frac{-1}{11} & \frac{7}{11} & -\frac{24}{11} \\ \frac{2}{11} & \frac{-3}{11} & \frac{4}{11} \\ \frac{2}{11} & \frac{-3}{11} & \frac{15}{11} \end{bmatrix}$$

Problem 6:

Verify cayley Hamilton's theorem for the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

Sol:-

The characteristic equation of A is $|A - \lambda I| = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore (1-\lambda)(3-\lambda) - 8 = 0$$

$$\therefore \lambda^2 - 4\lambda - 5 = 0$$

By cayley Hamilton's theorem A satisfies its characteristic equation.

$\therefore$ we have $A^2-4A-5I=0$

Now, $A^2=\begin{bmatrix}1&2\\4&3\end{bmatrix}\begin{bmatrix}1&2\\4&3\end{bmatrix}$

$=\begin{bmatrix}9&8\\16&17\end{bmatrix}$

$4A=\begin{bmatrix}4&8\\16&12\end{bmatrix}$ and $5I=\begin{bmatrix}5&0\\0&5\end{bmatrix}$

$\therefore A^2-4A-5I=\begin{bmatrix}9&8\\16&17\end{bmatrix}-\begin{bmatrix}4&8\\16&12\end{bmatrix}-\begin{bmatrix}5&0\\0&5\end{bmatrix}$

$=\begin{bmatrix}0&0\\0&0\end{bmatrix}=0$

Thus cayley Hamilton theorem is verified.

Problem 7:

Using cayley Hamilton's theorem for the matrix $A=\begin{bmatrix}1&0&-2\\2&2&4\\0&0&2\end{bmatrix}$

find i) A^{-1} ii) A^4

Sol:-

The characteristic equation of A is $|A-\lambda I|=0$

$\therefore \begin{vmatrix}1-\lambda&0&-2\\2&2-\lambda&4\\0&0&2-\lambda\end{vmatrix}=0$

i.e., $\lambda^3-5\lambda^2+8\lambda-4=0$ (verify)

$\therefore$ By cayley Hamilton's theorem.

$A^3-5A^2+8A-4I=0 \rightarrow (1)$

$\therefore 4I=A^3-5A^2+8A.$

i) To find A^{-1} pre multiplying by A^{-1} we get

$4A^{-1}=A^{-1}A^3-5A^{-1}A^2+8A^{-1}A.$

$=A^2-5A+8I$

$\therefore A^{-1}=\frac{1}{4}\left[A^2-5A+8I\right] \rightarrow (2)$

Now, $A^2=\begin{bmatrix}1&0&-2\\2&2&4\\0&0&2\end{bmatrix}\begin{bmatrix}1&0&-2\\2&2&4\\0&0&2\end{bmatrix}$

$=\begin{bmatrix}1&0&-6\\6&4&12\\0&0&4\end{bmatrix}$

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From (2)

$$A^{-1} = \frac{1}{4}\left[\begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & -10 \\ 10 & 10 & 20 \\ 0 & 0 & 10 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right]$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 4 \\ 4 & 2 & -8 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1/2 & -2 \\ 0 & 0 & -1/2 \end{bmatrix}$$

ii) To find A^4

From (1) $A^3 = 5A^2 - 8A + 4I$

$$\therefore A^4 = 5A^3 - 8A^2 + 4A$$

$$= 5[5A^2 - 8A + 4I] - 8A^2 + 4A \quad [\text{using } (1)]$$

$$= 17A^2 - 36A + 20I$$

$$= 17\begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} - 36\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 0 & -102 \\ 102 & 68 & 204 \\ 0 & 0 & 68 \end{bmatrix} - \begin{bmatrix} 36 & 0 & -72 \\ 72 & 72 & 144 \\ 0 & 0 & 72 \end{bmatrix} + \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

$$\therefore A^4 = \begin{bmatrix} 1 & 0 & -30 \\ 30 & 16 & 60 \\ 0 & 0 & 16 \end{bmatrix}$$

EIGEN VALUES AND EIGEN VECTORS

Definition:

Let A be an $n \times n$ matrix. A number λ is called an eigen value of A if there exists a non-zero vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ such that $AX = \lambda X$ and X is called an eigen vector corresponding to the eigen value λ.

Remark 1:

If X is an eigen vector corresponding to the eigen value λ of A, then αX where α is any non-zero number, is also an eigen vector corresponding to λ.

Remark 2:

Let X be an eigen vector corresponding to the eigen value λ of A. Then $AX = \lambda X$ So that $(A - \lambda I)X = 0$. Thus X is a non trivial solution of the System of homogeneous linear equations $(A - \lambda I)X = 0$. Hence $|A - \lambda I| = 0$ which is the Characteristic polynomial of A.

[Let $|A - \lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_n$.

The roots of this polynomial give the eigen values of A. Hence eigen values are also called characteristic roots.]

Properties of Eigen Values.

Property 1:

Let X be an eigen vector corresponding to the eigen values λ_1 and λ_2. Then $\lambda_1 = \lambda_2$

Proof:

By definition $X \neq 0$, $AX = \lambda_1 X$ and $AX = \lambda_2 X$

$\therefore \lambda_1 X = \lambda_2 X$

$\therefore (\lambda_1 - \lambda_2)X = 0$

Since $X \neq 0$, $\lambda_1 = \lambda_2$

Property 2:

Let A be a Square matrix.

Then (i) the sum of the ~~region~~ eigen values of A is equal to the sum of the digonal elements (trace) of A.

(ii) Product of eigen values of A is $|A|$

Proof:

i) Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \vdots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$

The eigen values of A are the roots of the Characteristic equation.

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0. \rightarrow (1)$$

Let $|A-\lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n \rightarrow (2)$

From (1) and (2) we get

$a_0 = (-1)^n;\ a_1 = (-1)^{n-1}(a_{11}+a_{22}+\cdots+a_{nn}) \rightarrow (3)$

Also by putting $\lambda = 0$ in (2) we get $a_n = |A|$

Now let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the given values of A

$\therefore \lambda_1, \lambda_2, \ldots, \lambda_n$ ~~be the eigen~~ are the roots of (2)

$\therefore \lambda_1 + \lambda_2 + \cdots + \lambda_n = -\dfrac{a_1}{a_0}$

$= a_{11} + a_{22} + \cdots + a_{nn}$ (using (3))

($\therefore$ Sum of the eigen values = trace of A) 2m

ii) Product of the eigen values = product of the roots

$= \lambda_1 \lambda_2 \ldots \lambda_n$

$= (-1)^n \dfrac{a_n}{a_0} = \dfrac{(-1)^n a_n}{(-1)^n}$

$= a_n$

$= |A|$

Property 3:

The eigen values of A and its transpose A^T are the same.

Proof:

It is enough if we prove that A and A^T have the same characteristic polynomial. Since for any square matrix M, $|M| = |M^T|$ we have

$|A-\lambda I| = |(A-\lambda I)^T| = |A^T - (\lambda I)^T|$

$= |A^T - \lambda I|$

Hence the result.

Property 4:

If λ is an eigen values of ~~a~~ ~~and~~ its non singular matrix A then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Proof:

Let X be an eigen vector corresponding to λ.

Then $AX = \lambda X$, Since A is non singular A^{-1} exists.

$$\therefore A^{-1}(AX) = A^{-1}(\lambda X)$$

$$IX = \lambda A^{-1}X$$

$$\boxed{A^{-1}X = \left(\frac{1}{\lambda}\right)X}$$

$\therefore \frac{1}{\lambda}$ is an eigen value of A^{-1}.

COROLLARY

If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of a non singular matrix A then.

$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$ are the eigen values of A^{-1}

Property 5:

If λ is an eigen value of A then $k\lambda$ is eigen value of kA where k is a scalar.

Proof:

Let X be an eigen vector corresponding to λ

Then $AX = \lambda X$. $\rightarrow$ (1)

Now, $(kA)X = k(AX)$

$= k(\lambda X)$ by (1)

$= (k\lambda)X$.

$\therefore k\lambda$ is an eigen value of kA.

Property 6

If λ is an eigen value of A then λ^k is an eigen value of A^k where k is any positive integer.

Proof:

Let X be an eigen vector corresponding to λ.

Then $AX = \lambda X$. $\rightarrow$ (1)

Now, $A^2X = (AA)X = A(AX)$

$= A(\lambda X)$ by (1)

$= \lambda(Ax)$
$= \lambda(\lambda x)$ by (1)
$= \lambda^2 x$.

$\therefore \lambda^2$ is an eigen value of A^2.

Proceeding like this we can prove that λ^k is an eigen value of A^k for any positive integer ~~to~~

COROLLARY.

If $\lambda_1, \lambda_2, \dots \lambda_n$ are eigen values of A then $\lambda_1^k, \lambda_2^k, \dots \lambda_n^k$ are eigen values of A^k for any positive integer k.

Property 7:

Eigen vectors corresponding to distinct eigen values of a matrix are linearly independent

Proof:

Let $\lambda_1, \lambda_2, \dots \lambda_k$ be distinct eigen values of a matrix and let x_i be the eigen vector corresponding to λ_i.

Hence $Ax_i = \lambda_i x_i \quad (i = 1, 2, \dots, k) \rightarrow (1)$

Now, Suppose $x_1, x_2, \dots x_k$ are linearly dependent. Then there exist real numbers $\alpha_1, \alpha_2 \dots \alpha_k$ not all zero Such that $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_k x_k = 0$. Among all Such relations, we choose one of shortest length, Say j.

By rearranging the vectors $x_1, x_2, \dots, x_k$ we may assume that.

~~$\alpha_1 x_1 + \alpha_2 x_2 +$~~

$\alpha_1 x_1 + \alpha_2 x_2 + \dots \alpha_j x_j = 0 \rightarrow (2)$

$\therefore A(\alpha_1 x_1) + A(\alpha_2 x_2) + \dots + A(\alpha_j x_j) = 0$

$\therefore \alpha_1 (Ax_1) + \alpha_2 (Ax_2) + \dots + \alpha_j (Ax_j) = 0$

$\therefore \alpha_1 \lambda_1 x_1 + \alpha_2 \lambda_2 x_2 + \dots + \alpha_j \lambda_j x_j = 0 \rightarrow (3)$

Multiplying ② by λ_1 and Subtracting from ③ we get.

$$\alpha_2(\lambda_1-\lambda_2)x_2+\alpha_3(\lambda_1-\lambda_3)x_3+\dots+\alpha_j(\lambda_1-\lambda_j)x_j=0 \rightarrow ④$$

and Since $\lambda_1, \lambda_2, \dots, \lambda_j$ are distinct and $\alpha_2, \dots \alpha_j$ are non-zero we have

$$\alpha_i(\lambda_1-\lambda_i)\neq 0;\ i=2,3,\dots,j.$$

Thus ④ gives a relation whose length is $j-1$, giving a contradiction.

Hence $x_1, x_2, \dots, x_k$ are linearly independent.

Property 8:

The characteristic roots of a Hermitian matrix are all real.

Proof:

Let A be a Hermitian matrix.

Hence,

$$A=\bar{A}^T \quad (\text{by theorem } 7.13). \rightarrow ①$$

Let λ be a characteristic root of A and let X be a characteristic vector corresponding to λ.

$$\therefore\ AX=\lambda X.. \rightarrow ②$$

Now,

$$AX=\lambda X \Rightarrow \bar{X}^T AX=\lambda\bar{X}^T X$$

$$\Rightarrow (\bar{X}^T AX)^T=\lambda\bar{X}^T X \quad (\text{Since } X^T AX \text{ is a } 1\times 1 \text{ matrix})$$

$$\Rightarrow X^T A^T (\bar{X}^T)^T=\lambda\bar{X}^T X$$

$$\Rightarrow X^T A^T \bar{X}=\lambda\bar{X}^T X$$

$$\Rightarrow \overline{X^T A^T . \bar{X}}=\overline{\lambda\bar{X}^T X}$$

$$\Rightarrow \bar{X}^T \bar{A}^T X=\bar{\lambda}X^T\bar{X} \quad (\text{using } 1)$$

$$\Rightarrow \bar{X}^T \lambda X=\bar{\lambda}X^T\bar{X} \quad (\text{using } 2)$$

$$\Rightarrow \lambda(\bar{X}^T X)=\bar{\lambda}(X^T\bar{X}).. \rightarrow ③$$

Now,

$$[\because \bar{A}^T=A]$$

$$\bar{X}^T X=(\overline{X^T\bar{X}})^T=(\bar{X}^T)(\bar{\bar{X}}^T)=\bar{X}X$$

$$=\bar{x}_1x_1+\bar{x}_2x_2+\dots+\bar{x}_nx_n$$

$$=|x_1|^2+|x_2|^2+\dots+|x_n|^2$$

$$\neq 0.$$

$\therefore$ From ③ we get $\lambda=\bar{\lambda}$

Hence λ is real.

Corollary.

The characteristic roots of a real Symmetric matrix are real.

Proof:

We know that any real Symmetric matrix is Hermitian. Hence the result follows from the above property.

Property 9.

The characteristic roots of a Skew Hermitian matrix are either purely Imaginary or zero.

Proof:

Let A be a Skew Hermitian matrix and λ be a characteristic root of A.

$\therefore |A-\lambda I| = 0$

$\therefore |iA - i\lambda I| = 0$

$\therefore i\lambda$ is a characteristic root of A.

$\therefore |A-\lambda I| = 0$

$\therefore |iA - i\lambda I| = 0$

$\therefore i\lambda$ is a characteristic root of iA

Hence A is Skew Hermitian iA is Hermitian (refer result theorem 7.14)

$\therefore$ By theorem 7.32 $i\lambda$ is real. Hence λ is Purely imaginary or zero.

Corollary.

The characteristic roots of a real Skew Symmetric matrix are either purely imaginary or zero.

Proof:

We know that any real Skew Symmetric matrix Skew Hermitian.

Hence the result follows from the above Property.

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Property 10:

Let λ be a characteristic root of an unitary matrix A. Then $|\lambda| = 1$. (i.e) the characteristic roots of a unitary matrix are all the unit modulus.

Proof:

Let λ be a characteristic root of an unitary matrix A and x be a characteristic vector corresponding to λ.

$\therefore AX = \lambda X \longrightarrow (1)$

Taking conjugate and transpose in (1) we get

$(\overline{AX})^T = (\overline{\lambda X})^T$

$\therefore \overline{X}^T \overline{A}^T = \overline{\lambda} \overline{X}^T \longrightarrow (2)$

Multiplying (1) and (2) we get

$(\overline{X}^T \overline{A}^T)(AX) = (\overline{\lambda}\, \overline{X}^T)(\lambda X)$

$\therefore \overline{X}^T (\overline{A}^T A) X = \overline{\lambda} \lambda (\overline{X}^T X)$

Now, since A is an unitary matrix $\overline{A}^T A = I$.

Hence $\overline{X}^T X = (\overline{\lambda} \lambda) \overline{X}^T X$

Since X is non-zero vector $\overline{X}^T$ is also non-zero vector and $\overline{X}^T X = |x_1|^2 + |x_2|^2 + \dots + |x_n|^2 \neq 0$

we get $\lambda \overline{\lambda} = 1$

Hence $|\lambda|^2 = 1$ Hence $|\lambda| = 1$

Corollary.

Let λ be a characteristic root of an orthogonal matrix A. Then $|\lambda| = 1$.

Since any orthogonal matrix is unitary the result follows from property 10.

Property 11.

Zero is an eigen value of A if and only if A is a singular matrix.

Proof:

The eigen values of A are the roots of the characteristic equation $|A - \lambda I| = 0$. Now, 0 is an eigen value of $A \Leftrightarrow |A - 0I| = 0$

$\Leftrightarrow |A| = 0$

$\Leftrightarrow A$ is a singular matrix.

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Sol:.

Since X_1 and X_2 are given vectors corresponding to λ, we have

$$AX_1 = \lambda X_1 \text{ and } AX_2 = \lambda X_2.$$

Hence $A(aX_1) = \lambda(aX_1)$ and $A(bX_2) = \lambda(bX_2)$

$$\therefore A(aX_1 + bX_2) = \lambda(aX_1 + bX_2)$$

$\therefore$ $aX_1 + bX_2$ is an eigen vector corresponding to λ.

Problem 2.

If the eigen values of

$$A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

are 2, 2, 3 find the eigen values of A^{-1} and A^2.

Sol:-

Since 0 is not an eigen value of A, A is non singular matrix and hence A^{-1} exists. using the property 4.

Eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$ and eigen values of A^2 are $2^2, 2^2, 3^2$.

Problem 3

Find the eigen values of A^5 when

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$$

Sol:-

$$|A - \lambda I| = \begin{vmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 5 & 4-\lambda & 0 \\ 3 & 6 & 1-\lambda \end{vmatrix} = 0$$

The characteristic equation of A is obviously $(3-\lambda)(4-\lambda)(1-\lambda) = 0$

Hence the eigen values of A are 3, 4, 1

$\therefore$ The eigen values of A^5 are $3^5, 4^5, 1^5$.

Problem 4:

Find the sum ~~of~~ and product of the eigen values of the matrix $\begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$ without actually finding the eigen values.

Sol:-

Let $A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

Sum of the eigen values = trace of $A = 3+(-2)+3 = 4$,

Product of the eigen values = $|A|$

Now $|A| = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{vmatrix}$

$= 3(-6+4) + 4(3-4) - 4(-1+2)$

$= -6 - 4 - 4 = -14$

$\therefore$ Product of the eigen values $= -14$.

Problem 5.

Find the characteristic roots of the matrix

$\begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Sol:-

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

The characteristic equation of A is given by

$|A - \lambda I| = 0$.

$\therefore \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$

$\therefore (\cos\theta - \lambda)^2 - \sin\theta = 0$

$\therefore (\cos\theta - \lambda - \sin\theta)(\cos\theta - \lambda + \sin\theta) = 0$

$\therefore [\lambda - (\cos\theta - \sin\theta)][\lambda - (\cos\theta + \sin\theta)] = 0$.

$\therefore$ The characteristic roots, (the two eigen values) of the matrix are $(\cos\theta - \sin\theta)$ and $(\cos\theta + \sin\theta)$

Problem 6

~~The chara~~ Find the characteristic roots of the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$

Sol:-

The characteristic equation of A is given by

$|A - \lambda I| = 0$.

(i.e) $\begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ -\sin\theta & -\cos\theta-\lambda \end{vmatrix}=0$

$\therefore -(\cos^2\theta-\lambda^2)-\sin^2\theta=0$

$\therefore \lambda^2-(\cos^2\theta+\sin^2\theta)=0$

$\therefore \lambda^2-1=0$

$\therefore$ The characteristic roots are 1 and -1.

Problem 7.

Find the sum and product of the eigen values of the matrix.

$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ without finding the roots of the characteristic equation.

Sol:-

Sum of the eigen values of $A=$ trace of $A=a_{11}+a_{22}$.

Product of the eigen values of $A=|A|=a_{11}a_{22}-a_{12}a_{21}$

Problem 8

Verify the statement that the sum of elements in the diagonal of a matrix is the sum of eigen values of the matrix.

$$A=\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol:-

The characteristic equation of A is $|A-\lambda I|=0$

(i.e) $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix}=0$

(i.e) $(-2-\lambda)[(1-\lambda)(-\lambda)-12]-2[-2\lambda-6]-3[-4+(1-\lambda)]=0$

(i.e) $(-2-\lambda)(\lambda^2-\lambda-12)+4(\lambda+3)+3(\lambda+3)=0$

(i.e) $-2\lambda^2+2\lambda+24-\lambda^3+\lambda^2+12\lambda+4\lambda+12+3\lambda+9=0$

(i.e) $-\lambda^3-\lambda^2+21\lambda+45=0$

(i.e) $\lambda^3+\lambda^2-21\lambda-45=0.$

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This is a cubic equation in λ and hence it has 3 roots and the three roots are the three eigen values of the matrix.

The Sum of the eigen values $= -\left(\dfrac{\text{coefficient of } \lambda^2}{\text{coefficient of } \lambda^3}\right)$

$= -1.$

The Sum of the elements on the diagonal of the matrix

$A = -2+1+0 = -1.$

Hence the result.

Problem 9.

The Product of two eigen values of the matrix A

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$ is 16. Find the third eigen value, what is the sum of the eigen values of A?

Sol:-

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

Given, product of 2 eigen values (say) λ_1, λ_2 is 16.

$\therefore \lambda_1 \lambda_2 = 16$

we know that the product of the eigen values is $|A|$.

(i.e) $\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

(i.e) $16\lambda_3 = 6(9-1)+2(-6+2)+2(2-6)$

$= 48-8-8$

$= 32$

$\therefore \lambda_3 = 2$

$\therefore$ The third eigen value is 2.

Also we know that the sum of the ~~region~~ eigen values of $A =$ trace of $A = 6+3+3 = 12$.

Problem 10:

The product of two eigen values of the matrix

$$A = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$ is -12. Find the eigen values of A

Sol:-

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

Given product of 2 eigen values, say, λ_1 and λ_2 is -12

$$\therefore \lambda_1 \lambda_2 = -12 \rightarrow (1)$$

we know that the product of the eigen values is $|A|$

$$\therefore \lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{vmatrix}$$

$$(i.e)\ 12 \lambda_3 = -12$$

$$\therefore \lambda_3 = 1 \rightarrow (2)$$

Also we know sum of the eigen values = Trace of A.

$$\therefore \lambda_1 + \lambda_2 + \lambda_3 = 2+1-3 = 0$$

$$\therefore \lambda_1 + \lambda_2 = -1 \quad (\text{using } (2)) \rightarrow (3)$$

using (3) in (1) we get $\lambda_1(-1-\lambda_1) = -12$

$$\lambda_1^2 + \lambda_1 - 12 = 0$$

$$(\lambda_1+4)(\lambda_1-3) = 0$$

$$\therefore \lambda_1 = 3 \text{ or } -4.$$

Putting $\lambda_1 = 3$ in (1) we get $\lambda_2 = -4$. or putting $\lambda_1 = -4$ in (1) we get $\lambda_2 = 3$.

Thus the three eigen values are $3, -4, 1$.

Problem 11.

Find the sum of the squares of the eigen values of $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Sol:-

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

we know that $\lambda_1^2, \lambda_2^2, \lambda_3^2$ are the eigen values of A^2.

$$\therefore A^2 = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 5 & 38 \\ 0 & 4 & 42 \\ 0 & 0 & 25 \end{bmatrix}$$

$\therefore$ Sum of the eigen values of A^2 = Trace of A^2

$= 9+4+25$.

(i.e) $\lambda_1^2+\lambda_2^2+\lambda_3^2 = 38$

$\therefore$ Sum of the squares of the eigen values of $A = 38$

Problem 12:

Find the eigen values and eigen vectors of the matrix.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol:

The characteristic equation of A is

$|A-\lambda I| = 0$.

$$\therefore \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\therefore (1-\lambda)[(5-\lambda)(1-\lambda)-1] - [(1-\lambda)-1] - [(1-\lambda)-3]$
$+ 3[1-3(5-\lambda)] = 0$

$(1-\lambda)(\lambda^2-6\lambda+4)+(\lambda^2+2)+3(3\lambda-14) = 0$

$\lambda^2-6\lambda+4-\lambda^3+6\lambda^2-4\lambda+\lambda+2+9\lambda-42=0$

$\therefore -\lambda^3+7\lambda^2-36=0$ Hence $\lambda^3-7\lambda^2+36=0$.

$\therefore (\lambda+2)(\lambda^2-9\lambda+18)=0$

Hence $(\lambda+2)(\lambda-6)(\lambda-3)=0$.

$\therefore \lambda = -2, 3, 6$ are the three eigen values.

Case (i)

Eigen vector corresponding to $\lambda = -2$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector corresponding to

$\lambda = -2$.

Hence $AX = -2X$

(i.e) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ -2x_2 \\ -2x_3 \end{bmatrix}$

$\therefore x_1+x_2+3x_3 = -2x_1$

$x_1+5x_2+x_3 = -2x_2$

$3x_1+x_2+x_3 = -2x_3$

x coefficient ① & ②

$\therefore 3x_1 + x_2 + 3x_3 = 0 \rightarrow ①$

$x_1 + 7x_2 + x_3 = 0 \rightarrow ②$ $\quad \frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1} = k$

$3x_1 + x_2 + 3x_3 = 0 \rightarrow ③$ $\quad \frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2} = k$

Clearly this System of three equations reduces to two equations only. From ① and ② we get.

$\therefore x_1 = -2k;\ x_2 = 0;\ x_3 = 2k.$

$\therefore$ It has only one independent Solution can be obtained by given any value to k Say $k=1$

$\therefore (-2,0,2)$ is an eigen vector corresponding to $\lambda = -2$

Case (ii) Eigen vector corresponding to $\lambda = 3$.

Then $AX = 3X$ gives.

$-2x_1 + x_2 + 3x_3 = 0 \rightarrow ①$

$x_1 + 2x_2 + x_3 = 0 \rightarrow ②$

$3x_1 + x_2 - 2x_3 = 0 \rightarrow ③$

Taking the first 2 equations we get.

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} = k(\text{say})$$

$\therefore x_1 = -k;\ x_2 = k;\ x_3 = -k.$

Taking $k=1$ (say) $(-1,1,-1)$ is an eigen vector corresponding to $\lambda = 3$.

Case (iii)

Eigen vector corresponding to $\lambda = 6$

we have $AX = 6X$

Hence $-5x_1 + x_2 + 3x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$3x_1 + x_2 - 5x_3 = 0$

Taking the first two equations we get.

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} = k.$$

$\therefore x_1 = k;\ x_2 = 2k;\ x_3 = k.$ It Satisfies the third equation also.

Taking $k=1$ (say) $(1,2,1)$ is an eigen vector Corresponding to $\lambda = 6$.

Problem 13.

Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solu:

The characteristic equation of A is $|A-\lambda I|=0$

$$\therefore \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore (6-\lambda)[(3-\lambda)^2-1]+2[(2\lambda-6)+2]+2(2-6+2\lambda)=0$$

$$\therefore 48+6\lambda^2-36\lambda-8\lambda-\lambda^3+6\lambda^2+8\lambda-16=0$$

$$-\lambda^3+12\lambda^2-36\lambda+32=0$$

Hence

$$\lambda^3-12\lambda^2+36\lambda-32=0$$

$$\therefore (\lambda-2)(\lambda-2)(\lambda-8)=0$$

$\therefore$ The eigen values are $2,2,8$.

We now find the eigen vectors.

Case(i)

$\lambda=2$

The eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is got from

$$AX=2X$$

$$\therefore 6x_1-2x_2+2x_3=2x_1$$

$$-2x_1+3x_2-x_3=2x_2$$

$$2x_1-x_2+3x_3=2x_3$$

$$\therefore 4x_1-2x_2+2x_3=0$$

$$-2x_1+x_2-x_3=0$$

$$2x_1-x_2+x_3=0$$

The above three equations are equivalent to the single equation $2x_1-x_2+x_3=0$

The independent eigenvectors can be obtained by given arbitary values to any two of the unknowns x_1, x_2, x_3.

Giving $x_1=1$; $x_2=2$ we get $x_3=0$

Giving $x_1=3$; $x_2=4$ we get $x_3=-2$

$\therefore$ Two independent vectors corresponding to $\lambda=2$ are $(1,2,0)$ and $(3,4,-2)$.

Case (iii)

$\lambda=8$

The eigen vector $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is got from.

$AX=8X$.

$\therefore -2x_1-2x_2+2x_3=0 \rightarrow (1)$

$-2x_1-5x_2-x_3=0 \rightarrow (2)$

$2x_1-x_2-5x_3=0 \rightarrow (3)$

From (1) and (2) we get.

$$\frac{x_1}{12}=\frac{x_2}{-6}=\frac{x_3}{6}=k(\text{say})$$

$\therefore x_1=2k$; $x_2=-k$; $x_3=k$.

Giving $k=1$ we get an eigen vector corresponding to 8 as $(2,-1,1)$.

10 mark

Problem 14.

Find the eigen values and eigen vectors of the matrix

$$A=\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Sol:-

The characteristic equation of A is $|A-\lambda I|=0$

(i.e) $$\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix}=0$$

$\therefore (2-\lambda)[-(1-\lambda)(1+\lambda)-3]+2[-(1+\lambda)-1]+2[3-(1-\lambda)]=0$

$\therefore (2-\lambda)(\lambda^2-4)-2(2+\lambda)+2(2+\lambda)=0$

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$\therefore 2\lambda^2 - 8 - \lambda^3 + 4\lambda - 4 - 2\lambda + 4 + 2\lambda = 0$

$\therefore -\lambda^3 + 2\lambda^2 + 4\lambda - 8 = 0$

Hence $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

$\therefore (\lambda - 2)(\lambda^2 - 4) = 0$

Hence $(\lambda - 2)(\lambda - 2)(\lambda + 2) = 0$

$\therefore \lambda = 2, 2, -2$ are the three eigen values.

case (i)

$X = (x_1, x_2, x_3)$ be an eigen vector corresponding to $\lambda = 2$, X is got from $AX = 2X$.

(i.e) $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$

$\therefore$ The eigen vector corresponding to $\lambda = 2$ is given by the equations.

$$2x_1 - 2x_2 + 2x_3 = 2x_1$$
$$x_1 + x_2 + x_3 = 2x_2$$
$$x_1 + 3x_2 - x_3 = 2x_3$$

(i.e) $-x_2 + x_3 = 0 \rightarrow (1)$

$x_1 - x_2 + x_3 = 0 \rightarrow (2)$

$x_1 + 3x_2 - 3x_3 = 0 \rightarrow (3)$

Taking (1) and (2) we get $\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1} = k$ (Say)

$\therefore x_1 = 0;\ x_2 = k;\ x_3 = k.$

Taking $k = 1$, we get $(0, 1, 1)$ as an eigen vector corresponding to $\lambda = 2$.

case (ii)

$\lambda = -2$

Corresponding to $\lambda = -2$ we have $AX = -2X$

$\therefore 2x_1 - 2x_2 + 2x_3 = -2x_1$

$x_1 + x_2 + x_3 = -2x_2$

$x_1 + 3x_2 - x_3 = -2x_3$

$\therefore 2x_1 - x_2 + x_3 = 0$

$x_1 + 3x_2 + x_3 = 0$

$x_1 + 3x_2 + x_3 = 0.$

$\therefore$ Taking the first two equations we get,

$$\frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7} = k \text{ (say)}$$

$\therefore x_1 = -4k;\ x_2 = -k;\ x_3 = 7k$

Taking $k=1$ we get $(-4,-1,7)$ as an eigen vector corresponding to the eigen value $\lambda = -2$