Centre of gravity, Centre of Pressure, Floating bodies, Atmospheric pressure.

Centre of gravity of a body

A rigid body may be regarded as made up of an indefinitely large number of particles each of which is attracted towards the centre of the earth by the force of gravity. These forces constitute a system of like parallel forces since the body is very small when compared to the distance of the centre of the earth. These resultant of these approximately parallel forces known as the weight of the body always act through a point which is fixed relative to the body whatever be the position of the body. This fixed point is called the centre of gravity of body.

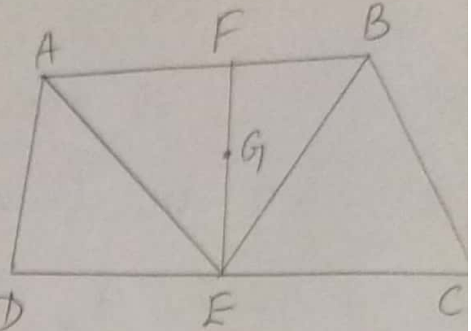
2M

Centre of gravity (Definition)

The centre of gravity of a body may therefore be defined as a point through which the line of action of the weight of the body always passes in what so manner the body is placed.

② Centre of gravity of a trapezoidal lamina :

Let ABCD be a trapezoidal lamina whose the length of the parallel sides AB and CB are $2a$ and $2b$ respectively. Let F & E be the midpoint of AB & CD. Join AE & BE. The trapezoidal lamina is divided into three triangular lamina ADE, AEB and BCE



The weights of the triangular laminae ADE, AEB and BEC are proportional to their areas which in turn are proportional to their bases b, $2a$ and b respectively. Since the altitudes of the triangle are equal.

The weight of the AEB is proportional to $2a$ and is equivalent to $2a/3$, $2a/3$ and $2a/3$ at A, E and B respectively. the weight of the ADE is proportional to b and is equivalent to $b/3$, $b/3$ and $b/3$ at A, D and E respectively. The weight of BEC is proportional to b and is equivalent to $b/3$, $b/3$, $b/3$ at B, E and C respectively.

The weights $\frac{2a}{3} + b/3$ at A and B are equivalent to $\frac{4a+2b}{3}$ at F

Similarly, $\frac{1}{3}b$ at D and $\frac{1}{3}b$ at C are equivalent to $2b/3$ at E.

The total weight at E is proportional to

$$\frac{2a}{3} + \frac{4b}{3} \text{ or } \frac{2a+4b}{3} \text{ at E}$$

The resultant of $\frac{4a+2b}{3}$ at F and $\frac{2a+4b}{3}$ at E will act equally, such that

$$\left(\frac{4a+2b}{3}\right) FG = \left(\frac{2a+4b}{3}\right) GE$$

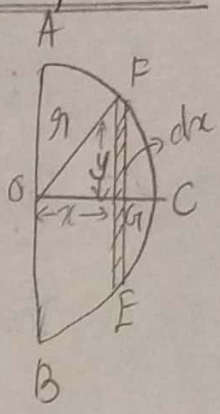
$$\left(\frac{4a+2b}{3}\right) \frac{FG}{GE} = \frac{2a+4b}{3}$$

$$\frac{FG}{GE} = \left[\frac{2a+4b}{\cancel{3}}\right]\left[\frac{\cancel{3}}{4a+2b}\right]$$

$$\frac{FG}{GE} = \frac{2a+4b}{4a+2b} \Rightarrow \frac{FG}{GE} = \frac{\cancel{2}(a+2b)}{\cancel{2}(2a+b)}$$

$$\frac{FG}{GE} = \frac{a+2b}{2a+b}$$

③ Centre of gravity of a Solid hemisphere :



Let ACB represent a section of a solid hemisphere of radius 'r' and centre 'O' and let ω be the weight per unit volume of the material of the hemisphere. Consider a thin slice of the hemisphere of thickness dx at distance x from O. Let y be the radius of the slice.

At Δ OGF, by pythagoras theorem,

$$x^2+y^2=r^2$$
$$y^2=r^2-x^2$$
$$y=\sqrt{r^2-x^2}$$

Then,

$$\left.\begin{array}{l}\text{volume of the}\\ \text{slice}\end{array}\right\} = \pi\left(\sqrt{r^2-x^2}\right)^2 dx$$
$$= \cancel{r}\,\pi(r^2-x^2)\,dx$$

Weight of the slice $= \pi\omega(r^2-x^2)\,dx$

Distance of the centre of gravity of the slice from O $= x$

Moment of the weight of the slice about O $= \pi\omega(r^2-x^2)\,dx \times x$.

$$= \pi\,\omega(r^2 - x^2)x\,dx$$

Algebraic sum of the moments of all such slice into which the solid hemisphere could be divided

$$= \int_0^r \pi\,\omega(r^2 - x^2)x\,dx \longrightarrow (1)$$

Weight of the hemisphere $= \omega.\,\frac{2}{3}\pi r^3$

Let the centre of gravity of the hemisphere from O be $\bar{x}$

The moment of weight of the hemisphere about O

$$= \frac{2}{3}\pi\,\omega r^3 \times \bar{x} \longrightarrow (2)$$

Then equal the two moments, (1) & (2),

$$\frac{2}{3}\pi\,\omega r^3\bar{x} = \int_0^r \pi\,\omega(r^2 - x^2)x\,dx$$

$$\frac{2}{3}\pi\,\omega r^3\bar{x} = \pi\,\omega\int_0^r (r^2 - x^2)x\,dx$$

$$\frac{2}{3}\pi\,\omega r^3\bar{x} = \pi\,\omega\int_0^r (r^2x - x^3)\,dx$$

$$\frac{2}{3}\pi\,\omega r^3\bar{x} = \pi\,\omega\left[\frac{r^2x^2}{2} - \frac{x^4}{4}\right]_0^r$$

$$\frac{2}{3}\pi\,\omega r^3\bar{x} = \pi\,\omega\left[\frac{r^2.r^2}{2} - \frac{r^4}{4}\right] - 0$$

$$\frac{2}{3}\cancel{\pi} r^3 \cancel{\omega}\bar{x} = \cancel{\pi}\cancel{\omega}\left[\frac{r^4}{2}-\frac{r^4}{4}\right]$$

$$\frac{2}{3} r^3 \bar{x} = \frac{4r^4\ \ 2r^4}{8}$$

$$\frac{2}{3} r^3 \bar{x} = \frac{2r^4}{\cancel{8}\,4}$$

$$\frac{2}{3}\bar{x} = \frac{r}{4}$$

$$\bar{x} = \frac{r}{4}\cdot\frac{3}{2}$$

$$\bar{x} = \frac{3}{8}r$$

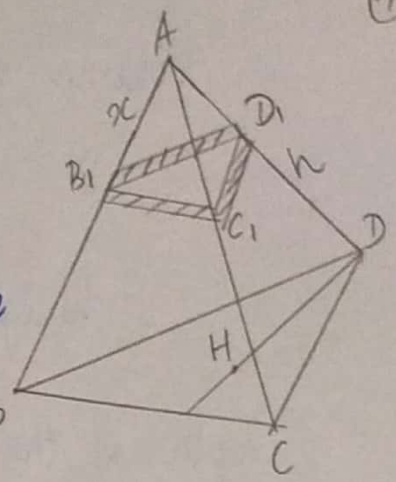
Hence, the centre of gravity of a solid hemisphere is at distance $3/8\,r$ from the centre of hemisphere.

---

④ <u>Centre of gravity of a solid tetrahedron:</u>

Let ABCD be a solid tetrahedron whose base is BCD and whose vertex is A. Let h be the altitude of the tetrahedron and ~~A~~ H the centre of gravity of its triangular base. The centre of gravity of the tetrahedron obviously lies on the line A

Consider a slice $B_1C_1D_1$ of the tetrahedron of thickness dx at a depth x below A, the plane of the slice being parallel to the base BCD. Let S be the area of the triangular base BCD and ω be the weight per unit volume of the tetrahedron.



We have, $\frac{B_1C_1}{BC} = \frac{x}{h}$

Also, if a_1, a be the altitudes of triangles $B_1C_1D_1$ and BCD respectively.

$$a_1/a = x/h$$

Area of $\Delta B_1C_1D_1 = \frac{1}{2}$ base $\times$ height

$$= \frac{1}{2} B_1C_1 \times a_1$$

Area of ABCD $= \frac{1}{2} BC \times a = S$

Here,

$$\frac{\text{Area of } \Delta B_1C_1D_1}{S} = \frac{B_1C_1a_1}{BCa}$$

$$\frac{\text{Area of } \Delta B_1C_1D_1}{S} = \frac{x^2}{h^2}$$

$$\text{Area of } \Delta B_1C_1D_1 = \frac{x^2}{h^2} S.$$

Volume of the Slice $B_1C_1D_1 = \frac{Sx^2}{h^2}dx$ [Area × thickness]

Weight of the slice $= \frac{WSx^2}{h^2}dx$

Depth of the C.G of the slice below A $\Big\} = x$

Moment of the slice about A $= \frac{WSx^2}{h^2}dx \cdot x$.

$$A = \frac{WSx^3}{h^2}dx$$

Sum of the moments of the slice in tetrahedron $\Big\} = \int_0^h \frac{WS}{h^2} x^3 dx$

$$= \frac{WS}{h^2}\int_0^h [x^3]dx$$

$$= \frac{WS}{h^2}\left[\frac{x^4}{4}\right]_0^h$$

$$= \frac{WS}{h^2}\left[\frac{h^4}{4} - 0\right]$$

$$= \frac{WSh^4}{h^2 4}$$

Sum of the moments of slice $\Big\} = \frac{WSh^2}{4} \longrightarrow (1).$

$$\left.\begin{array}{r}\text{Volume of the whole}\\ \text{tetrahedron}\end{array}\right\} = \int_0^h \frac{Sx^2}{h^2}\,dx$$

$$= \frac{S}{h^2}\int_0^h [x^2]\,dx$$

$$= \frac{S}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$= \frac{S}{h^2}\left[\frac{h^3}{3} - 0\right]$$

$$= \frac{Sh}{3}$$

Weight of the whole tetrahedron $= \frac{1}{3}Sh \times W$

$$= \frac{1}{3}WSh$$

Let $\bar{x}$ be the depth of C.G of below A

The moment of tetrahedron about $A = \frac{1}{3}WSh \times \bar{x}$

The moment of tetrahedron about $A = \frac{1}{3}WSh\bar{x} \longrightarrow (2)$

Then, equal the moments (1) & (2)

$$\tfrac{1}{3}\cancel{W}\cancel{S}\cancel{h}\,\bar{x} = \tfrac{1}{4}\cancel{W}\cancel{S}h^{\cancel{2}}$$

$$\tfrac{1}{3}\bar{x} = \tfrac{1}{4}h$$

$$\bar{x} = \tfrac{3}{4}h$$

The centre of gravity of the tetrahedron lies at a point G on the line such that $AG : GH = 3:1$

⑤ Centre of gravity of a right solid cone :

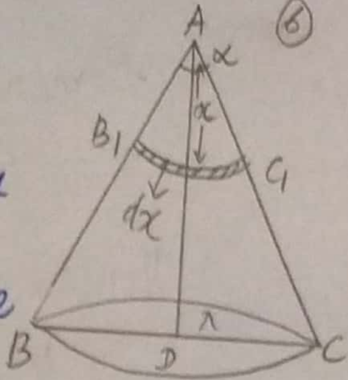
When the number of sides of the base of a pyramid is indefinitely increased the pyramid reduces approximately to a right solid cone. ~~of height~~ Hence, the centre of gravity of a solid cone of height h lies on line joining the vertex to the centre of the circular base at a depth $\frac{3}{4} h$ from the vertex of the cone.

The centre of gravity of a right solid cone can derived from the first principles as follows

Let ABC be a section of a right solid cone of height h and let the radius of the base be r. Let D be the centre of the base. Join AD. Then $AD = h$ is the altitude of the cone. Consider a slice B_1C_1 of the cone parallel to the plane base of width dx at depth x below the vertex A. If y be the radius of slice, then $y = x \tan\alpha$

)

⑥



Volume of the slice $= \pi y^2 dx$

$= \pi x^2 \tan^2\alpha \, dx$

If W be the weight per unit volume of the cone, the weight of the slice $=$

$\pi w x^2 \tan^2\alpha \, dx$

Moment of the slice about A $= \pi w x^3 \tan^2\alpha \, dx$.

Sum of the moments of all such slices in the cone about A

$$= \int_0^h \pi w x^3 \tan^2\alpha \, dx$$

$$\cancel{\int_0^h \pi w}$$

$$= \pi w \tan^2\alpha \int_0^h x^3 \, dx$$

$$= \pi w \tan^2\alpha \left[\frac{x^4}{4}\right]_0^h$$

$$= \pi w \tan^2\alpha \frac{h^4}{4}$$

$$= \frac{1}{4}\pi w h^4 \tan^2\alpha \longrightarrow (1).$$

Volume of the whole cone $= \frac{1}{3} w \pi h^3 \tan^2\alpha$

Weight of the cone $= \frac{1}{3} w \pi h^3 \tan^2\alpha \, x$

Let $\bar{x}$ be the depth of the centre of gravity below A.

Moment of the weight of the cone about A $\Big\} = \frac{1}{3}\pi w \tan^2\alpha\, h^3 \bar{x}$

Hence,

$$\frac{1}{3}\pi w h^3 \tan^2\alpha \cdot \bar{x} = \frac{1}{4} w \pi h^4 \tan^2\alpha$$

$$\frac{1}{3}\bar{x} = \frac{1}{4} h$$

$$\bar{x} = \frac{3}{4} h.$$

6) Centre of Pressure:

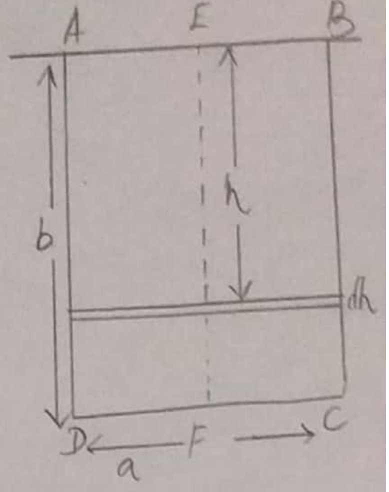
Centre of pressure of a plane surface incontact with a fluid is a point on the surface through ~~within~~ which the line of action of the resultant of the thrusts on the various element of the area passes.

7) Rectangular lamina:

Centre of pressure of a rectangular lamina. immersed in a homogeneous liquid at rest with one side on the surface not subjected to any rexternal pressure

Suppose a rectangle ABCD is immersed in a liquid of density ρ with the side AB in the surface

Let $AB = a$ and $AD = b$. If rectangle is divided into a number of narrow strips are parallel to AB of width dh. Then considering one such strip at a depth h below the surface of the liquid. We have the area of the strip is dh and the thrust acting on it is $h\rho g a\,dh$



The moment of this about AB is $h^2 \rho g a\,dh$

The sum of the moments on all the strips $\Big\} = \int_0^b h^2 \rho g a\,dh.$

The resultant of all the thrusts on all the strips $\Big\} = \int_0^b h^2 \rho g a\,dh \longrightarrow (1)$

The moment of this resultant about AB $\Big\} = H\int_0^b h\rho g a\,dh \longrightarrow (2)$

where, H is the depth of the centre of pressure below AB.

Then equate (1) & (2).

$$\int_0^b h^2 \rho g a\,dh = H\int_0^b h\rho g a\,dh$$

$$\rho g a\int_0^b h^2 dh = H\,\rho g a\int_0^b h\,dh$$

$$\rho g a \left[\frac{h^3}{3}\right]_0^b = H \rho g a \left[\frac{h^2}{2}\right]_0^b$$

$$\frac{b^3}{3} = H\frac{b^2}{2}$$

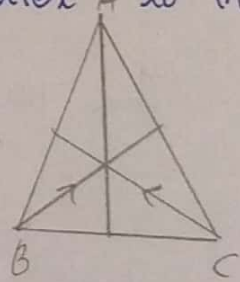
$$\frac{b}{3} = \frac{H}{2}$$

$$H = \frac{2b}{3}$$

It is clear from symmetry that the centre of pressure will lie on EF ~~without~~ where E and F are midpoints of AB and DC

8. <u>Triangular lamina</u> :

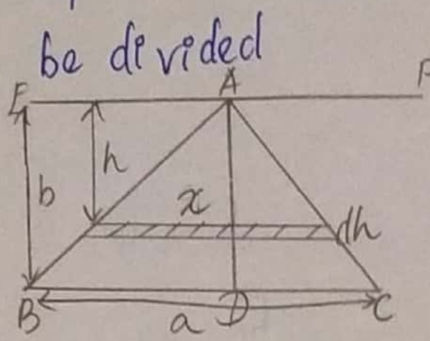
Triangular lamina which means the median of a triangle is a line from a vertex to the midpoint of the opposite side.



5m

Centre of pressure of triangular lamina immersed in a liquid with its vertex on the surface and base horizontal nor subjected to any external pressure:

Consider a triangular lamina of base a immersed vertically in a liquid of density ρ. Its vertex A in the surface EF of the liquid the base BC horizontal at a depth b below the surface. Let the triangle be divided into a number of elementary strips of width dh all parallel to the base BC.



Consider a strip of length x at a depth h below the surface EF. the area of strip is x dh and the thrust on it $h\rho g x\,dh$ and the moment of this about EF is $h^2 \rho g x\,dh$.

By similar triangle.

$$\frac{x}{h} = \frac{a}{b}$$

$$x = \frac{ah}{b}.$$

Substituting this value in the above expression,

We have the moment $= \dfrac{ah^3 \rho g\,dh}{b} \longrightarrow (1).$

The sum of the moment due to all the strips

$$= \int_0^b \frac{ah^3 \rho g\,dh}{b}$$

The resultant of the thrusts on all the strips

$$= \int_0^b \frac{ah^2 \rho g\,dh}{b}.$$

The moment of this resultant thrust about EF

$$= H\int_0^b \frac{ah^3 \rho g\, dh}{b}$$

where H is the depth of the centre of pressure below EF.

Hence,

$$\int_0^b \frac{ah^3\rho g\, dh}{b} = H\int_0^b \frac{ah^2\rho g\, dh}{b}$$

$$\frac{a\rho\cancel{g}}{\cancel{b}} \int_0^b (h^3)\, dh = H\frac{a\rho\cancel{g}}{\cancel{b}} \int_0^b (h^2)\, dh$$

$$\left[\frac{h^4}{4}\right]_0^b = H\left[\frac{h^3}{3}\right]_0^b$$

$$\frac{b^{\cancel{4}}}{4} = H\frac{\cancel{b^3}}{3}$$

$$\frac{b}{4} = \frac{H}{3}$$

$$\boxed{H = \frac{3b}{4}}$$

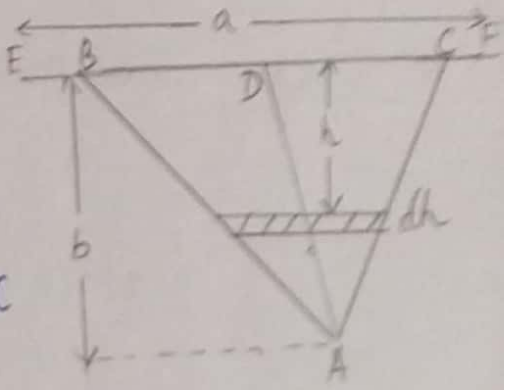
Hence, the centre of pressure lies ~~ø~~ in the median AD at a depth $\frac{3}{4}b$ below the surface at a distance of $\frac{3}{4}$ AD from A.

---

(9) Centre of pressure of a triangular lamina immersed in a liquid with one side on the surface not subject to any external pressure:

9

Consider triangular lamina of base a immersed vertically in a liquid of density ρ with the side BC in the surface of the liquid. Let b be the depth of the vertex A below the surface. Suppose the area is divided into an ~~ind~~ infinitely large number of narrow strips each of width dh parallel to the base BC.



Consider a strip of length x at a depth h below BC. The thrust on it is $h\rho g x\, dh$ and the moment of this about BC is $h^2 \rho g x\, dh$.

But, by similar triangles,

$$\frac{x}{a} = \frac{b-h}{b}$$

$$x = \frac{a(b-h)}{b}$$

The moment of the thrust on the strip

$$= \frac{h^2 \rho g a (b-h)\, dh}{b}$$

The sum of the moments of the thrusts on all the strips

$$= \int_0^b \frac{ah^2 \rho g (b-h)\, dh}{b}$$

The resultant of the thrusts on all the strips

$$= \int_0^b h \rho g x \, dh$$

and moment of this about BC

$$= H \int_0^b \frac{ah(b-h)\rho g \, dh}{b}$$

where H is the depth of the centre of pressure below the base BC

Hence,

$$\int_0^b \frac{ah^2(b-h)\rho g \, dh}{b} = H \int_0^b \frac{ah(b-h)\rho g \, dh}{b}$$

$$\frac{a\cancel{\rho g}}{\cancel{b}} \int_0^b h^2(b-h) \, dh = H \cancel{\frac{a\rho g}{b}} \int_0^b h(b-h) \, dh$$

$$\int_0^b (h^2 b - h^3) \, dh = H \int_0^b (hb - h^2) \, dh$$

$$\left[\frac{bh^3}{3} - \frac{h^4}{4}\right]_0^b = H \left[\frac{bh^2}{2} - \frac{h^3}{3}\right]_0^b$$

$$\left[\frac{b^4}{3} - \frac{b^4}{4}\right] = H \left[\frac{b^3}{2} - \frac{b^3}{3}\right]$$

$$\frac{b^{\cancel{4}}}{12} = H \frac{\cancel{b^3}}{6}$$

$$b/12 = H/6$$

$$H = \frac{6b}{12}$$

$$\boxed{H = b/2}$$

The resultant thrust on each strip passes through its midpoint and since AD is the median. the resultant of the thrusts on all the strips must pass through the point in AD. Hence the centre of pressure lies on the median AD at a depth $\frac{1}{2}b$ below BC or at a distance ~~of~~ $\frac{1}{2}$ AD from D.

In all the cases discussed above, the We have taken the lamina to be immersed vertically. However, the result is same even if the lamina is inclined at angle θ with the vertical. If H & h reckoned as distances measured along the plane of the lamina, the vertical components of H and h are ~~are~~ H cos θ and h cos θ and the expression becomes,

$$\int h \cancel{\cos\theta} g \, ds = H \int h^2 \cancel{\cos\theta} g \, ds$$

$$\int h \, ds = H \int h^2 \, ds$$

Hence, the result is same.

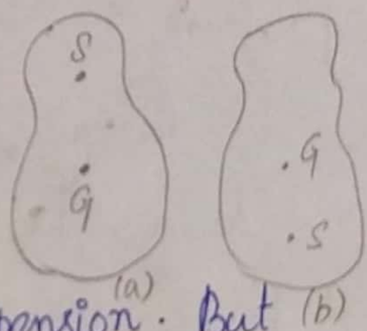
⑩ Conditions of equilibrium of a floating body:

⇒ the weight of the liquid displaced by the immersed part must be equal to the weight of the body

⇒ the centre of gravity and the centre of buoyancy of the body should be along the same vertical line

⑪ Stability of ~~α~~ equilibrium of a floating body:

A freely suspended body is in equilibrium, If the centre of gravity of the body is vertically below or above ~~on~~ the point of suspension. But the equilibrium is stable, only if the centre of gravity of the body is below. the point of suspension and unstable [fig (a)]. If the centre of gravity of the body is above the point of ~~its~~ suspension.



We have already seen that a floating body will be in equilibrium provided is the weight of the displaced liquid is equal to the weight of the floating body and that of ~~on~~ the displaced liquid lie in the same vertical line but the equilibrium and may or may not be stable. Another condition has got to be satisfied for the equilibrium to be stable.

When a body floats freely in a liquid, the resultant thrust acts through the centre of gravity of the liquid displaced. This point is called the centre of buoyancy.

When a body floats in a liquid, the section in which the surface on the liquid intersects the floating body is called plane of floatation.

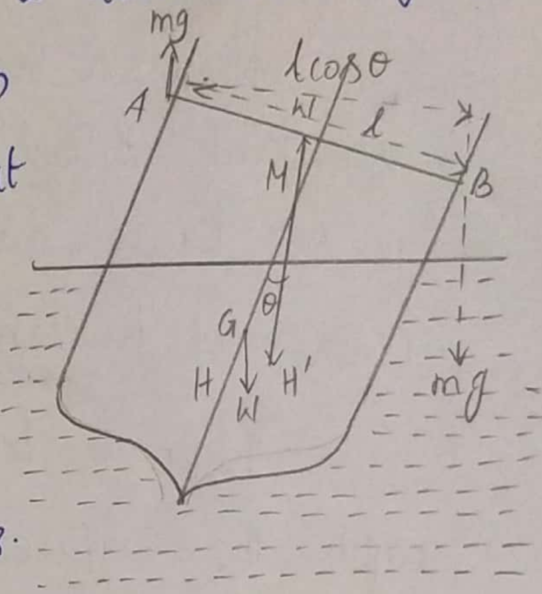
If a fluid floating body rotates about in such a way that it takes up in succession every position in which the volume of the liquid displaced by it remains the same, the centre of buoyancy traces a surface called the surface of buoyancy.

12 Metacentre:

If a floating body be slightly displaced such that the volume of liquid displaced by it remains the same, then the point in which the vertical line through the new centre of buoyancy meets the line joining the centre of gravity of the body to the original centre of buoyancy is called the metacentre.

(13) Experimental determination of a metacentric height of a ship:

The weight of a ship W is determined by the displacement method. This ship has two boats on the deck at a distance l apart and the mass of the walter filling the ~~board~~ boat is determined from a knowledge of its cubical contents.



If A & B represents the boats at distance l apart on the deck filling A & B alternately with water is equivalent to moving a weight from A to B across the deck. The water filling the boat B with the same mass of waster as in A, turns the ship through a very small angle θ. This angle θ is determined by means of a plumb line suspended in the ship.

Since the ship slightly inclined, the centre of buoyancy is altered. Let H & H' be altered positions of the centre of buoyancy, G the centre of gravity of the ship, M the metacentre and GM the metacentric height.

The deflecting couple due to filling the boats alternately with water

$$= mg \times l \cos\theta \quad [\because W = mg]$$

$$C = Wl\cos\theta \rightarrow ①$$

The restoring couple due to the weight of the ship acting at G and the force of buoyancy acting through

$$H'C = W \times GM \sin\theta \rightarrow ②.$$

Equate ① & ②

$$WGM\sin\theta = Wl\cancel{\sin}\overset{\cos}{}\theta$$

$$GM = \frac{Wl\cancel{\sin}\overset{\cos}{}\theta}{W\sin\theta}$$

$$GM = \frac{Wl}{W}\frac{\cos\theta}{\sin\theta} \qquad \because \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$GM = \frac{Wl}{W\tan\theta}$$

$$\boxed{GM = \frac{Wl}{W\theta}}$$ Since θ is small.