UNIT - V

INVARIANT SUBSPACES :

(1) **Definition:**

Let V be a vector space and T a linear operator on V. If W is a subspace of V, we say that W is invariant under T if for each vector α in W the vector $T\alpha$ is in W, i.e., if $T(W)$ is contained in W.

Lemma:

Let W be an invariant subspace for T. The characteristic polynomial for the restriction operator T_W divides the characteristic polynomial for T. The minimal polynomial for T_W divide the minimal polynomial for T.

proof:

We have

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

Where $A = [T]_{\mathcal{B}}$ and $B = [T_W]_{\mathcal{B}'}$.

Because of the block form of the matrix

$$\det(xI - A) = \det(xI - B)\det(xI - D)$$

That proves the statement about characteristic polynomials. Notice that we used I to represent identity matrices of three different sizes.

The kth power of the matrix A has the block form

$$A^k = \begin{bmatrix} B^k & C_k \\ 0 & D^k \end{bmatrix}$$

(2)

Where C_k is some $r \times (n-r)$ matrix. Therefore, any polynomial which annihilates A also annihilates B (and D too). So, the minimal polynomial for B divides the minimal polynomial for A.

Definition:

Let W be an invariant subspace for T and let α be a vector in V. The T- conductor of α into W is the set $S_T(\alpha; W)$, which consists of all polynomials g such that $g(T)\alpha$ is in W.

Lemma:

If W is an invariant subspace for T, then W is invariant under every polynomial in T. Thus, for each α in V, the conductor $S(\alpha; W)$ is an ideal in the polynomial algebra $F[x]$.

proof:

If β is in W, then $T\beta$ is in W. Consequently, $T(T\beta) = T^2\beta$ is in W. By induction, $T^k\beta$ is in W for each k. Take linear combinations to see that $f(T)\beta$ is in W for every polynomial f.

The definition of $S(\alpha; W)$ makes sense if W is any subset of V. If W is a subspace, then $S(\alpha; W)$ is a subspace of $F[x]$, because

$$(cf + g)(T) = cf(T) + g(T)$$

If W is also invariant under T, let g be a polynomial in $S(\alpha; W)$, ie, let $g(T)\alpha$ be in W.

If f is any polynomial, then $f(T)[g(T)\alpha]$ will be in W. Since

$$(fg)(T) = f(T)g(T)$$

fg is in $S(\alpha; W)$.

Thus the conductor absorbs multiplication by any polynomial.

Definition:

The unique monic generator of the ideal $S(\alpha; W)$ is also called the T-conductor of α into W (the T-annihilator in case $W = \{0\}$) The T-conductor of α into W is the monic polynomial g of least degree such that $g(T)\alpha$ is in W.

A polynomial f is in $S(\alpha; W)$ iff g divides f.

Note that the conductor $S(\alpha; W)$ always contains the minimal polynomial for T, hence every T-conductor divides the minimal polynomial for T.

The linear operator T is called tri-angulable if there is an ordered basis in which T is represented by a triangular matrix.

Lemma:

(A) Let V be a finite-dimensional vector space over the field F. Let T be a linear operator on V such that the minimal polynomial for T is a product of linear factors

$$p = (x-c_1)^{r_1} \cdots (x-c_k)^{r_k}, \quad c_i \text{ in } F.$$

Let W be a proper ($W \neq V$) subspace of V which is invariant under T. There exists a vector α in V such that

(a) α is not in W;

(b) $(T - cI)\alpha$ is in W, for some characteristic value c of the operator T.

Proof:

What (a) and (b) say is that the T-conductor of α into W is a linear polynomial. Let β be any vector in V which is not in W. Let g be the T-conductor of β into W. Then g divides p, the minimal polynomial for T. Since β is not in W, the polynomial g is not constant. Therefore,

$$g = (x-c_1)^{e_1} \cdots (x-c_k)^{e_k}$$

Where at least one of the integers e_i is positive. Choose j so that $e_j > 0$. Then $(x-c_j)$ divides g:

$$g = (x - c_j)h$$

By the definition of g, the vector $\alpha = h(T)\beta$ cannot

5 be in W. But,

$$(T - c_j I)\alpha = (T - c_j I)h(T)\beta$$

$$= g(T)\beta \quad \text{is in } W.$$

U.Q Theorem:

Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Then T is triangulable iff the minimal polynomial for T is a product of linear polynomials over F.

proof:

Suppose that the minimal polynomial factors

$$p = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k}$$

By repeated application of the lemma above, we shall arrive at an ordered basis $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ in which the matrix representing T is upper triangular:

$$[T]_{\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \rightarrow (1)$$

Now (1) merely says that

$$T\alpha_j = a_{1j}\alpha_1 + \cdots + a_{jj}\alpha_j, \quad 1 \le j \le n \rightarrow (2)$$

that is, $T\alpha_j$ is in the subspace spanned by $\alpha_1, \ldots, \alpha_j$. To find $\alpha_1, \ldots, \alpha_n$, We start by

10 applying the lemma to the subspace $W = \{0\}$, to obtain the vector α_1.

Then apply the lemma to W_1, the space spanned by α_1, and we obtain α_2. Next apply the lemma to W_2, the space spanned by α_1 and α_2. Continue in that way. One point deserves comment. After $\alpha_1, \ldots, \alpha_i$ have been found, it is the triangular-type relations (2) for $j = 1, \ldots, i$ which ensure that the subspace spanned by $\alpha_1, \ldots, \alpha_i$ is invariant under T.

If T is triangulable, it is evident that the characteristic polynomial for T has the form

$$f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}, \quad c_i \text{ in } F.$$

Just look at the triangular matrix (1). The diagonal entries $a_{11}, \ldots, a_{1n}$ are the characteristic values, with c_i repeated d_i times. But if f can be so factored, so can the minimal polynomial p, because it divides f.

Corollary:

Let F be an algebraically closed field, e.g., the complex number field. Every $n \times n$ matrix over F is similar over F to a triangular matrix.

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(7)

Theorem:

Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Then T is diagonalizable iff the minimal polynomial for T has the form

$$p = (x - c_1) \ldots (x - c_k)$$

Where $c_1, \ldots, c_k$ are distinct elements of F.

proof:

We have noted earlier that, if T is diagonalizable, its minimal polynomial is a product of distinct linear factors. To prove the converse, let W be the subspace spanned by all of the characteristic vectors of T, and suppose $W \neq V$. By the lemma used in the proof of known theorem, there is a vector α not in W and a characteristic value c_j of T such that the vector

$$\beta = (T - c_j I)\alpha$$

lies in W. Since β is in W,

$$\beta = \beta_1 + \ldots + \beta_k$$

Where $T\beta_i = c_i\beta_i$, $1 \le i \le k$, and therefore the vector

$$h(T)\beta = h(c_1)\beta_1 + \ldots + h(c_k)\beta_k$$

is in W, for every polynomial h.

Now $p = (x - c_i) q$, for some polynomial q

Also

$$q - q(c_j) = (x - c_j) h.$$

(8) We have

$$q(T)\alpha - q(c_j)\alpha = h(T)(T - c_j I)\alpha = h(T)\beta$$

But $h(T)\beta$ is in W and, since

$$0 = p(T)\alpha = (T - c_j I) q(T)\alpha$$

the vector $q(T)\alpha$ is in W. Therefore, $q(c_j)\alpha$ is in W. Since α is not in W, we have $q(c_j) = 0$. That contradicts the fact that p has distinct roots.

Simultaneous Triangulation ; Simultaneous Diagonalization :

Definition:

The subspace W is invariant under [the $\mathcal{F}$ family of operators) ~~There exists~~ if W is invariant under each operator in $\mathcal{F}$

Lemma:

Let $\mathcal{F}$ be a commuting family of triangulable linear operators on V. Let W be a proper subspace of V which is invariant under $\mathcal{F}$. There exists a vector α in V such that

(a) α is not in W;

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(b) for each T in $\mathcal{F}$, the vector $T\alpha$ is in the subspace spanned by α and W.

proof:

(a) It is no loss of generality to assume that $\mathcal{F}$ contains only a finite number of operators, because of this observation.

Let $\{T_1, \ldots, T_r\}$ be a maximal linearly independent subset of $\mathcal{F}$, ie) a basis for the subspace spanned by $\mathcal{F}$. If α is a vector such that (b) holds for each T_i, then (b) will hold for every operator which is a linear combination of $T_1, \ldots, T_r$.

By the lemma before known theorem, we can find a vector β_1 and a scalar c_1 such that $(T_1 - c_1 I)\beta_1$ is in W. Let V_1 be the collection of all vectors β in V such that $(T_1 - c_1 I)\beta$ is in W. Then V_1 is a subspace of V which is properly larger than W. Furthermore, V_1 is invariant under $\mathcal{F}$. for this reason.

If T commutes with T_1, then

$$(T_1 - c_1 I)(T\beta) = T(T_1 - c_1 I)\beta.$$

If β is in V_1, then $(T_1 - c_1 I)\beta$ is in W. Since W is invariant under each T in $\mathcal{F}$, we have $T(T_1 - c_1 I)\beta$ in W, ie) $T\beta$ in V_1, for all β in V_1 and all T in $\mathcal{F}$

Now W is a proper subspace of V_1. Let U_2 be the linear operator on V_1 obtained by restricting T_2 to the subspace V_1. The minimal polynomial for

10 U_2 divides the minimal polynomial for T_2. Therefore, we may apply the lemma before know theorem to that operator and the invariant subspace W. We obtain a vector β_2 in V_1 and a scalar c_2 such that $(T_2 - c_2 I)\beta_2$ is in W. Note that,

(a) β_2 is not in W;

(b) $(T_1 - c_1 I)\beta_2$ is in W;

(c) $(T_2 - c_2 I)\beta_2$ is in W.

Let V_2 be the set of all vectors β in V_1 such that $(T_2 - c_2 I)\beta$ is in W. Then V_2 is invariant under $\mathcal{F}$. Apply the lemma before known theorem to U_3, the restriction of T_3 to V_2. If we continue in this way, we shall reach a vector $\alpha = \beta_r$ such that $(T_j - c_j I)\alpha$ is in W, $j = 1, \dots, r$.

Theorem:

Let V be a finite-dimensional vector space over the field F. Let $\mathcal{F}$ be a commuting family of triangulable linear operators on V. There exists an ordered basis for V such that every operator in $\mathcal{F}$ is represented by a triangular matrix in that basis.

proof:

Given the lemma which we just proved, this theorem has the same proof as does known theorem, if one replaces T by $\mathcal{F}$.

Corollary:

Let $\mathcal{F}$ be a commuting family of $n \times n$ matrices over an algebraically closed field F. There exists a non-singular $n \times n$ matrix P with entries in F such that $P^{-1}AP$ is upper-triangular, for every matrix A in $\mathcal{F}$.

Theorem:

Let $\mathcal{F}$ be a commuting family of diagonalizable linear operators on the finite-dimensional vector space V. There exists an ordered basis for V such that every operator in $\mathcal{F}$ is represented in that basis by a diagonal matrix.

proof:

We could prove this theorem by adapting the lemma before known theorem to the diagonalizable case, just as we adapted the lemma before known theorem, to the diagonalizable case in order to prove known theorem. However, at this point it is easier to proceed by induction on the dimension of V.

(12) If $\dim V = 1$, there is nothing to prove. Assume the theorem for vector spaces of dimension less than n, and let V be an n-dimensional space. Choose any T in $\mathcal{F}$ which is not a scalar multiple of the identity. Let $c_1, \ldots, c_k$ be the distinct characteristic values of T, and let W_i be the null space of $T - c_i I$.

Fix an index i. Then W_i is invariant under every operator which commutes with T. Let $\mathcal{F}_i$ be the family of linear operators on W_i obtained by restricting the operators in $\mathcal{F}$ to the subspace W_i. Each operator in $\mathcal{F}_i$ is diagonalizable, because its minimal polynomial divides the minimal polynomials for the corresponding operator in $\mathcal{F}$. Since $\dim W_i < \dim V$, the operators in $\mathcal{F}_i$ can be simultaneously diagonalized.

In other words, W_i has a basis $\mathcal{B}_i$ which consists of vectors which are simultaneously characteristic vectors for every operator in $\mathcal{F}_i$. Since T is diagonalizable, the lemma before known theorem tells us that $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_k)$ is a basis for V. That is the basis we seek.

Direct-Sum Decompositions:

Definition:

Let $W_1, \ldots, W_k$ be subspaces of the vector space V. We say that $W_1, \ldots, W_k$ are independent

If

$$\alpha_1 + \cdots + \alpha_k = 0, \quad \alpha_i \text{ in } W_i$$

 implies that each α_i is 0.

Lemma:

Let V be a finite-dimensional vector space. Let $W_1, \ldots, W_k$ be subspaces of V and let $W = W_1 + \cdots + W_k$. The following are equivalent.

(a) $W_1, \ldots, W_k$ are independent

(b) for each j, $2 \leq j \leq k$, we have

$$W_j \cap (W_1 + \cdots + W_{j-1}) = \{0\}$$

(c) If $\mathcal{B}_i$ is an ordered basis for W_i, $1 \leq i \leq k$, then the sequence $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_k)$ is an ordered basis for W.

proof:

Assume (a). Let α be a vector in the intersection $W_j \cap (W_1 + \cdots + W_{j-1})$. Then there are vectors $\alpha_1, \ldots, \alpha_{j-1}$ with α_i in W_i such that $\alpha = \alpha_1 + \cdots + \alpha_{j-1}$

Since,

$$\alpha_1 + \cdots + \alpha_{j-1} + (-\alpha) + 0 + \cdots + 0 = 0$$

and since $W_1, \ldots, W_k$ are independent, it must be that $\alpha_1 = \alpha_2 = \cdots = \alpha_{j-1} = \alpha = 0$

Now, let us observe that (b) implies (a).

Suppose,

$$0 = \alpha_1 + \cdots + \alpha_k, \quad \alpha_i \text{ in } W_i$$

Let j be the largest integer i such that $\alpha_i \neq 0$

Then

(17)

$$0 = \alpha_1 + \cdots + \alpha_j \quad , \quad \alpha_j \neq 0$$

Thus $\alpha_j = -\alpha_1 - \cdots - \alpha_{j-1}$ is a non zero vector in $W_j \cap (W_1 + \cdots + W_{j-1})$.

Now that we know (a) and (b) are the same, let us see why (a) is equivalent to (c). Assume (a). Let $\mathcal{B}_i$ be a basis for W_i, $1 \leq i \leq k$, and let $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_k)$. Any linear relation b/w the vectors in $\mathcal{B}$ will have the form

$$\beta_1 + \cdots + \beta_k = 0$$

where β_i is some linear combination of the vectors in $\mathcal{B}_i$. Since $W_1, \ldots, W_k$ are independent, each β_i is 0. Since each $\mathcal{B}_i$ is independent, the relation we have b/w the vectors in $\mathcal{B}$ is the trivial relation.

We relegate the proof that (c) implies (a) to the exercises.

Definition:

If V is a vector space, a projection of V is a linear operator E on V such that $E^2 = E$.

Theorem:

If $V = W_1 \oplus \cdots \oplus W_k$, then there exists k linear operators $E_1, \ldots, E_k$ on V such that

(i) each E_i is a projection $(E_i^2 = E_i)$;

(ii) $E_i E_j = 0$, if $i \neq j$;

(iii) $I = E_1 + \dots + E_k$;

(iv) the range of E_i is W_i

Conversely, if $E_1, \dots, E_k$ are k linear operators on V which satisfy conditions (i), (ii), and (iii) and if we let W_i be the range of E_i, then $V = W_1 \oplus \dots \oplus W_k$.

Proof:

We have only to prove the converse statement. Suppose $E_1, \dots, E_k$ are linear operators on V which satisfy the first three conditions, and let W_i be the range of E_i. Then certainly

$$V = W_1 + \dots + W_k ;$$

for, by condition (iii) we have,

$$\alpha = E_1\alpha + \dots + E_k\alpha$$

for each α in V, and $E_i\alpha$ is in W_i. This expression for α is unique, because if

$$\alpha = \alpha_1 + \dots + \alpha_k$$

with α_i in W_i, say $\alpha_i = E_i\beta_i$, then using (i) and (ii) we have

$$E_j\alpha = \sum_{i=1}^{k} E_j \alpha_i$$

$$= \sum_{i=1}^{k} E_j E_i \beta_i$$

$$= E_j^2 \beta_j$$

Hence

$= E_j \beta_j$

$= \alpha_j$

16 This shows that V is the direct sum of the W_i

Invariant Direct Sums :

Theorem:

Let T be a linear operator on the space V, and let $W_1, \dots, W_k$ and $E_1, \dots, E_k$ be as in before theorem. Then a necessary and sufficient condition that each subspace W_i be invariant under T is that T commute with each of the projections E_i, i.e.,

$$TE_i = E_i T, \quad i = 1, \dots, k.$$

Proof:

Suppose T commutes with each E_i. Let α be in W_j, then $E_j \alpha = \alpha$, and

$$T\alpha = T(E_j \alpha)$$

$$= E_j(T\alpha)$$

which shows that $T\alpha$ is in the range of E_j, i.e., that W_j is invariant under T.

Assume now that each W_i is invariant under T. We shall show that $TE_j = E_j T$. Let α be any vector in V. Then

$$\alpha = E_1\alpha + \dots + E_k\alpha$$

$$T\alpha = TE_1\alpha + \dots + TE_k\alpha$$

Since $E_i\alpha$ is in W_i, which is invariant under T, we must have $T(E_i\alpha) = E_i\beta_i$ for some vector β_i. Then

$$E_jTE_i\alpha = E_jE_i\beta_i$$
$$= \begin{cases} 0, & \text{if } i \neq j \\ E_j\beta_j, & \text{if } i = j \end{cases}$$

Thus

$$E_jT\alpha = E_jTE_1\alpha + \dots + E_jTE_k\alpha$$
$$= E_j\beta_j$$
$$= TE_j\alpha$$

This holds for each α in V, so $E_jT = TE_j$. //

Theorem:

Let T be a linear operator on a finite-dimensional space V. If T is diagonalizable and if $c_1, \dots, c_k$ are the distinct characteristic values of T, then there exists linear operators $E_1, \dots, E_k$ on V such that

(i) $T = c_1E_1 + \dots + c_kE_k$;

(ii) $I = E_1 + \dots + E_k$;

(iii) $E_iE_j = 0$, $i \neq j$;

(iv) $E_i^2 = E_i$ (E_i is a projection);

(v) the range of E_i is the characteristic space for T associated with c_i.

Conversely, if there exist k distinct scalars $c_1, \dots, c_k$ and k non zero linear operators $E_1, \dots, E_k$ which satisfy conditions (i), (ii), and (iii), then T is diagonalizable, $c_1, \dots, c_k$ are the distinct characteristic values of T, and

(18) conditions (iv) and (v) are satisfied also.

proof

Suppose that T is diagonalizable, with distinct characteristic values $c_1, \dots, c_k$. Let W_i be the space of characteristic vectors associated with the characteristic value c_i. As we have seen,

$$V = W_1 \oplus \dots \oplus W_k.$$

Let $E_1, \dots, E_k$ be the projections associated with this decomposition, as in known theorem. Then (i), (ii), (iv) and (v) are satisfied.

To verify (iii), proceed as follows. For each α in V,

$$\alpha = E_1\alpha + \dots + E_k\alpha$$

and so

$$T\alpha = TE_1\alpha + \dots + TE_k\alpha$$

$$= c_1E_1\alpha + \dots + c_kE_k\alpha.$$

In other words, $T = c_1E_1 + \dots + c_kE_k$.

Now suppose that we are given a linear operator T along with distinct scalars c_i and non-zero operators E_i which satisfy (i), (ii) and (iii)

(17)

Since $E_i E_j = 0$ when $i \neq j$, we multiply both sides of $I = E_1 + \dots + E_k$ by E_i and obtain immediately $E_i^2 = E_i$. Multiplying $T = c_1 E_1 + \dots + c_k E_k$ by E_i, we then have $TE_i = c_i E_i$, which shows that any vector in the range of E_i is in the null space of $(T - c_i I)$.

Since we have assumed that $E_i \neq 0$, this proves that there is a non-zero vector in the null space of $(T - c_i I)$, i.e. that c_i is a characteristic value of T. Furthermore, the c_i are all of the characteristic values of T; for, if c is any scalar, then

$$T - cI = (c_1 - c) E_1 + \dots + (c_k - c) E_k$$

So if $(T - cI)\alpha = 0$, we must have $(c_i - c) E_i \alpha = 0$. If α is not the zero vector, then $E_i \alpha \neq 0$ for some i, so that for this i we have $c_i - c = 0$.

Certainly T is diagonalizable, since we have shown that every non-zero vector in the range of E_i is a characteristic vector of T, and the fact that $I = E_1 + \dots + E_k$ shows that these characteristic vectors span V. All that remains to be demonstrated is that the null space of $(T - c_i I)$ is exactly the range of E_i. But this is clear, because if $T\alpha = c_i \alpha$, then

$$\sum_{j=1}^{k} (c_j - c_i) E_j \alpha = 0$$

hence

$$(c_j - c_i) E_j \alpha = 0 \quad \text{for each } j$$

and then

$$E_j \alpha = 0 , \quad j \neq i$$

Since $\alpha = E_1\alpha + \cdots + E_k\alpha$ and $E_j\alpha = 0$ for $j \neq i$, we have $\alpha = E_i\alpha$, which proves that α is in the range of E_i.

The Primary Decomposition Theorem:

Theorem [Primary Decomposition Theorem]

Statement:

Let T be a linear operator on the finite-dimensional vector space V over the field F. Let p be the minimal polynomial for T,

$$p = p_1^{r_1} \cdots p_k^{r_k}$$

where the p_i are distinct irreducible monic polynomials over F and the r_i are positive integers. Let W_i be the null space of $p_i(T)^{r_i}$, $i = 1, \ldots, k$. Then

$V = W_1 \oplus \cdots \oplus W_k$;

each W_i is invariant under T;

if T_i is the operator induced on W_i by T, then the minimal polynomial for T_i is $p_i^{r_i}$.

Proof

The idea of the proof is this. If the

direct sum decomposition (i) is valid, how can we get hold of the projections $E_1, \ldots, E_k$ associated with the decomposition? The projection E_i will be the identity on W_i and zero on the other W_j.

(2) We shall find a polynomial h_i such that $h_i(T)$ is the identity on W_i and is zero on the other W_j, and so that $h_1(T) + \cdots + h_k(T) = I$, etc.

For each i, let

$$f_i = \frac{P}{P_i^{r_i}} = \prod_{j \neq i} P_j^{r_j}$$

Since $P_1, \ldots, P_k$ are distinct prime polynomials, the polynomials $f_1, \ldots, f_k$ are relatively prime. Thus there are polynomials $g_1, \ldots, g_k$ such that

$$\sum_{i=1}^{n} f_i g_i = 1.$$

Note also that if $i \neq j$, then $f_i f_j$ is divisible by the polynomial P, because $f_i f_j$ contains each $P_m^{r_m}$ as a factor. We shall show that the polynomials $h_i = f_i g_i$ behave in the manner described in the first paragraph of the proof.

Let $E_i = h_i(T) = f_i(T) g_i(T)$. Since $h_1 + \cdots + h_k = 1$ and P divides $f_i f_j$ for $i \neq j$, we have

$$E_1 + \cdots + E_k = I$$

$$E_i E_j = 0, \quad \text{if } i \neq j.$$

22

Thus the E_i are projections which correspond to some direct sum decomposition of the space V. We wish to show that the range of E_i is exactly the subspace W_i. It is clear that each vector in the range of E_i is in W_i, for if α is in the range of E_i, then $\alpha = E_i\alpha$ and so

$$p_i(T)^{r_i}\alpha = p_i(T)^{r_i} E_i\alpha$$

$$= p_i(T)^{r_i} f_i(T) g_i(T)\alpha$$

$$= 0$$

because $p^{r_i} f_i g_i$ is divisible by the minimal polynomial p. Conversely, suppose that α is in the null space of $p_i(T)^{r_i}$. If $j \neq i$, then $f_j g_j$ is divisible by $p_i^{r_i}$ and so $f_j(T) g_j(T)\alpha = 0$, i.e., $E_j\alpha = 0$ for $j \neq i$. But then it is immediate that $E_i\alpha = \alpha$, i.e., that α is in the range of E_i. This completes the proof of statement (i).

It is certainly clear that the subspaces W_i are invariant under T. If T_i is the operator induced on W_i by T, then evidently $p_i(T_i)^{r_i} = 0$, because by definition $p_i(T)^{r_i}$ is 0 on the subspace W_i. This shows that the minimal polynomial for T_i divides $p_i^{r_i}$. Conversely, let g be any polynomial such that $g(T_i) = 0$. Then $g(T) f_i(T) = 0$. Thus $g f_i$ is divisible by the minimal polynomial p of T. ie) $p_i^{r_i} f_i$ divides $g f_i$. It is easily seen that $p_i^{r_i}$ divides g.

Hence the minimal polynomial for T_i is $p_i^{r_i}$.

Corollary:

If $E_1, \dots, E_k$ are the projections associated with the primary decomposition of T, then each E_i is a polynomial in T, and accordingly if a linear operator U commutes with T then U commutes with each of the E_i. ie) each subspace W_i is invariant under U.

~~Definition~~ **Theorem:**

Let T be a linear operator on the finite-dimensional vector space V over the field F. Suppose that the minimal polynomial for T decomposes over F into a product of linear polynomials. Then there is a diagonalizable operator D on V and a nilpotent operator N on V such that

(i) $T = D + N$,

(ii) $DN = ND$

The diagonalizable operator D and the nilpotent operator N are uniquely determined by (i) and (ii) and each of them is a polynomial in T.

Proof:

We have just observed that we can write

$T = D + N$ where D is diagonalizable and N is nilpotent, and where D and N not only commute but are polynomials in T. Now suppose that we also have $T = D' + N'$ where D' is diagonalizable, N' is nilpotent, and $D'N' = N'D'$. We shall prove prove that $D = D'$ and $N = N'$.

Since D' and N' commute with one another and $T = D' + N'$, we see that D' and N' commute with T. Thus D' and N' commute with any polynomial in T; hence they commute with D and with N. Now we have

$$D + N = D' + N'$$

(or)

$$D - D' = N' - N$$

and all four of these operators commute with one another. Since D and D' are both diagonalizable and they commute, they are simultaneously diagonalizable and $D - D'$ is diagonalizable.

Since N and N' are both nilpotent and they commute, the operator $(N' - N)$ is nilpotent, for using the fact that N' and N' commute

$$(N' - N)^r = \sum_{j=0}^{r} \binom{r}{j} (N')^{r-j} (-N)^j$$

and so when r is sufficiently large every term in this expression for $(N'-N)^r$ will be 0.

25 [Actually, a nilpotent operator on an n-dimensional space must have its nth power 0; if we take $r = 2n$ above, that will be large enough. It then follows that $r = n$ is large enough, but this is not obvious from the above expression].

Now $D - D'$ is a diagonalizable operator which is also nilpotent. Such an operator is obviously the zero operator; for since it is nilpotent, the minimal polynomial for this operator is of the form x^r for some $r \leq m$; but then since the operator is diagonalizable, the minimal polynomial cannot have a repeated root; hence $r = 1$ and the minimal polynomial is simply x, which says the operator is 0. Thus we see that $D = D'$ and $N = N'$.

Definition:

Let N be a linear operator on the vector space V. We say that N is nilpotent if there is some positive integer r such that $N^r = 0$.

Corollary :

Let V be a finite-dimensional vector space
26 over an algebraically closed field F , eg the field of complex numbers . Then every linear operator T on V can be written as the sum of a diagonalizable operator D and a nilpotent operator N which commute . These operators D and N are unique and each is a polynomial in T .

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