THE STRUCTURE OF COMMUTATIVE BANACH ALGEBRAS

Applications of the formulas

$$r(x) = \lim \|x^n\|^{1/n}.$$

ug ug.

Theorem: 1

The following conditions on A are all equivalent to one another.

i) $\|x^2\| = \|x\|^2$ for every x.

(ii) $r(x) = \|x\|$. " "

(iii) $\|\hat{x}\| = \|x\|$.

proof:-

(i) $\Rightarrow$ (ii)

First we assume that $\|x^2\| = \|x\|^2$.

$$\Rightarrow \|x^4\| = \|(x^2)^2\|.$$

$$= \|x^2\|^2$$

$$= \|x\|^4$$

In general, $\|x^{2^k}\| = \|x\|^{2^k}$ for every positive integer k.

By the spectral radius formula

$$r(x) = \lim \|x^n\|^{1/n}.$$

$$= \lim \|x^{2^k}\|^{\frac{1}{2^k}}$$

$$= \lim \|x\|$$

$$r(x) = \|x\|.$$

(ii) $\Rightarrow$ (iii)

Assume that $r(x) = \|x\| \longrightarrow$ (1)

To prove that $\|\hat{x}\| = \|x\|$.

It is enough to show that

$r(x) = \|\hat{x}\|$

If x is an element of A, then $\hat{x}$ the function defined on $\mathcal{M}$ by $\hat{x}(M) = x(M)$ $\hookrightarrow$ (2)

$\therefore$ From (1)

$r(x) = \|x\| = \|\hat{x}\|$ by (2)

Hence $\|\hat{x}\| = \|x\|$.

(iii) $\Rightarrow$ (i)

Consider $\|\hat{x}\| = \|x\| \longrightarrow$ (1) for every x

$r(x) = \|\hat{x}\|$.

We know that $r(x) = \lim \|x^n\|^{1/n}$.

$= \|x^n\|^{1/n}$.

From (1), $\|\hat{x}\| = \|x\|$.

$\|x^n\|^{1/n} = \|x\|$

$\|x^n\| = \|x\|^n$.

put $n=2$.

$\|x^2\| = \|x\|^2$ for all x.

Hence the proof.

Involutions in Banach algebras

A Banach algebra A is called a Banach*_algebra, if it has an involution. If there exists a mapping $x \to x^*$ of A into itself with the following properties.

i) $(x+y)^* = x^* + y^*$

ii) $(\alpha x)^* = \bar{\alpha} x^*$

iii) $(xy)^* = y^* x^*$

iv) $x^{**} = x$.

Theorem:

The involution is continuous.

(or)

If $x_n \to x$ then $x_n^* \to x_n^*$.

Proof:-

Let us assume that $x_n \to x$.

(i.e) $\|x_n\| \to \|x\|$

$\|x_n - x\| \to 0 \to$ ①

To show that $x_n^* \to x^*$

$\|x_n^* - x^*\| = \|(x_n - x)^*\|$.

$= \|x_n - x\|^*$

$\|x_n^* - x^*\| \to 0$ by ①

Hence $x_n^* \to x^*$

ug (X.)

Theorem 3

If x is a normal element in a B^*-algebra, then $\|x^2\| = \|x\|^2$.

Proof:-

To prove that $\|x^2\| = \|x\|^2$.

Clearly, $\|x^2\| \leq \|x\|^2 \longrightarrow (1)$.

$$\|x^*\|^2 \|x\|^2 = (\|x^*\| \cdot \|x\|)^2.$$

$$= \|x^* \cdot x\|^2.$$

$$= \|(x^* x)^* \; x^* x\|.$$

$$= \|x^* \cdot x \; x^* x\|$$

$$= \|x^* x^* \cdot x x\|$$

$$= \|(x^*)^2 \cdot x^2\|.$$

$$= \|\cancel{(x^2)^*} \; \cancel{x^2}\|.$$

$$= \|(x^2)^* \cdot x^2\|.$$

$$= \|(x^2)^*\| \cdot \|x^2\|.$$

$$\|x^*\|^2 \cdot \|x\|^2 \leq \|(x^*)^2\| \cdot \|x^2\|.$$

$$\|x\|^2 \leq \|x^2\| \longrightarrow (2)$$

From (1) & (2)

$$\|x^2\| = \|x\|^2.$$

ug (X.) (X.)

THE GELFAND NEUMARK THEOREM:

If A is a commutative B^*-algebra then the Gelfand mapping $x \to \hat{x}$ is an isometric $*$-isomorphism of A onto the commutative B^*- algebra $\mathcal{C}(\mathcal{M})$.

<u>proof</u>:-

Since A is Commutative, each of its elements is normal.

We know that $\|x^2\| = \|x\|^2$ for all x.

It is enough to show that,

$$\hat{x}^*(M) = \overline{\hat{x}(M)} \text{ for each } x \in A \;\&\; M$$

and

i) $\hat{x}(M)$ is real

If x is self - adjoint, then $\hat{x}(M)$ is real for every M.

Suppose, $\hat{x}(M)$ is not real. Then $\exists$ an M such that $\hat{x}(M) = \alpha + i\beta$

Since x is self adjoint.

$y = (x - \alpha I)/\beta$ is also self adjoint.

If $\hat{y}(M) = i$, then $y - i1 \in M$.

From the properties of involution in A,

$M^* = \{m^*, m \in M\}$ is a maximal ideal

Since $(y - i1)^* = y + i1$

and $\hat{y}(M^*) = -i$

If K is any +ve number, there.

$\widehat{(y - iK1)}(M) = i(1+K)$

Hence, $1+K \leq \|\widehat{y - iK1}\| \leq \|y - iK1\| \rightarrow (1)$

Similarly $1+K \leq \|y + iK1\| \longrightarrow (2)$.

multiplying (1) & (2),

$$(1+k)^2 \le \|y-ik1\| \cdot \|y+ik1\|.$$
$$= \|(y+ik1)^*\| \cdot \|y+ik1\|.$$
$$= \|(y+ik1)^*(y+ik1)\|.$$
$$= \|(y-ik1)(y+ik1)\|.$$
$$= \|y^2+k^2\|.$$
$$\le \|y^2\| + k^2.$$

So $1+2k \le \|y^2\|$.

$\because$ k is arbitrary, this is impossible

~~Since~~ Hence (i) is proved.

(ii) $\hat{x^*}(M) = \overline{\hat{x}(M)}$ ~~for every~~.

x is any element of A, then

$\hat{x}(M) = \overline{\hat{x}(M)}$ for every M.

clearly, $y = (x+x^*)/2$ and

$z = (x-x^*)/2i$

are self-adjoint and $x = y+iz$.

$$\hat{x^*}(M) = \widehat{(y-iz)}(M)$$
$$= \hat{y}(M) - i\hat{z}(M)$$
$$= \overline{\hat{y}(M)} - \overline{i\hat{z}(M)}$$
$$= \overline{\hat{y}(M) + i\hat{z}(M)}$$
$$= \overline{\hat{x}(M)}$$

Hence the proof.

U4 U4 What is the Gelfand mapping 2M

Gelfand Mapping theorem:-

The natural homomorphism $x \to x+M$ of A onto $A/M = C$ assigns to each element $x \in A$ a complex number $x(M)$ defined by.

$x(M) = x+M$, and the mapping $x \to x(M)$ has the following properties.

i) $(x+y)(M) = x(M) + y(M)$

ii) $(\alpha x)(M) = \alpha x(M)$

iii) $(xy)(M) = x(M) \cdot y(M)$.

iv) $x(M) = 0 \Leftrightarrow x \in M$.

v) $1(M) = 1$

(vi) $|x(M)| \leq \|x\|$.

CHAPTER FOURTEEN

The Structure of Commutative Banach Algebras

The set $\mathcal{C}(X)$ of all bounded continuous complex functions defined on a topological space X is the simplest of the really interesting Banach algebras. Our purpose in this chapter is to prove the famous Gelfand-Neumark theorem, which says that every commutative Banach algebra A of a certain type is essentially identical with $\mathcal{C}(X)$ for a suitable compact Hausdorff space X. More precisely, we shall prove that a compact Hausdorff space X can be built out of the inner structure of A, that X is accompanied by a natural mapping of A into $\mathcal{C}(X)$, and that this mapping is one-to-one onto and preserves all the structure assumed to be present in A.

70. THE GELFAND MAPPING

Let A be an arbitrary commutative Banach algebra. Our first theorem below is the principal source of the structure theory of A, and the remainder of the chapter will be devoted entirely to shaping its consequences into the elegant form of the Gelfand-Neumark theorem.

Theorem A. *If M is a maximal ideal in A, then the Banach algebra A/M is a division algebra, and therefore equals the Banach algebra C of complex numbers. The natural homomorphism $x \to x + M$ of A onto $A/M = C$ assigns to each element x in A a complex number $x(M)$ defined by*

$$x(M) = x + M,$$

and the mapping $x \to x(M)$ has the following properties:

(1) $(x + y)(M) = x(M) + y(M)$;
(2) $(\alpha x)(M) = \alpha x(M)$;
(3) $(xy)(M) = x(M)y(M)$;
(4) $x(M) = 0 \Leftrightarrow x \in M$;
(5) $1(M) = 1$;
(6) $|x(M)| \leq \|x\|$.

PROOF. Theorems 69-B and 66-D tell us that A/M is indeed a Banach algebra. Since A contains an identity, M is maximal as a ring ideal (see the comments on this matter in Sec. 61), and therefore, by Theorem 11-C, A/M is a division algebra. We now appeal to Theorem 67-B to conclude that A/M equals C. (Actually, of course, A/M equals the set of all scalar multiples of its own identity, but we identify this set with C in accordance with the remarks following Theorem 67-B.) Finally, properties (1) to (5) are obvious consequences of the nature of the homomorphism under discussion, and (6) follows from

$$|x(M)| = \|x + M\| = \|x + M\| = \inf \{\|x + m\| : m \in M\} \leq \|x\|.$$

It is interesting to observe that this proof depends, either directly or indirectly, on virtually every major theorem in the previous chapter. We also note that the ultimate reason for assuming that A is commutative lies in Theorem 11-A, which is definitely not true in the non-commutative case (see Problem 11-1).

The language of Theorem A is oriented toward the idea that $x(M)$ is a function of x for each fixed M. The notation, however, suggests that we reverse this point of view and that for each fixed x we regard $x(M)$ as a complex function defined on the set $\mathfrak{M}$ of all maximal ideals in A. This is the direction in which we now proceed.

If x is a given element of A, we denote by $\hat{x}$ the function defined on $\mathfrak{M}$ by $\hat{x}(M) = x(M)$, and we put $\hat{A} = \{\hat{x} : x \in A\}$. Our next step is to define a topology for $\mathfrak{M}$ in such a manner that every function in $\hat{A}$ is continuous. The most natural way of doing this is to introduce the weak topology generated by $\hat{A}$. It will be recalled that this is the weakest topology on $\mathfrak{M}$ relative to which every function $\hat{x}$ is continuous and that a typical subbasic open set has the form

$$S(\hat{x}, M_0, \epsilon) = \{M : M \in \mathfrak{M} \text{ and } |\hat{x}(M) - \hat{x}(M_0)| < \epsilon\}.$$

We call the topological space $\mathfrak{M}$ the *space of maximal ideals*, or the *maximal ideal space*, and the mapping $x \to \hat{x}$ of A onto $\hat{A}$ will be referred to as the *Gelfand mapping*.

We are now in a position to reformulate Theorem A, and to extend it, in such a way that the Gelfand mapping is displayed as the object of central importance.

Theorem B. *The Gelfand mapping $x \to \hat{x}$ is a norm-decreasing (and therefore continuous) homomorphism of A into $\mathscr{C}(\mathfrak{M})$ with the following properties:*

(1) the image $\hat{A}$ of A is a subalgebra of $\mathscr{C}(\mathfrak{M})$ which separates the points of $\mathfrak{M}$ and contains the identity of $\mathscr{C}(\mathfrak{M})$;

(2) the radical R of A equals the set of all elements x for which $\hat{x} = 0$, so $x \to \hat{x}$ is an isomorphism $\Leftrightarrow$ A is semi-simple;

(3) an element x in A is regular $\Leftrightarrow$ it does not belong to any maximal ideal $\Leftrightarrow$ $\hat{x}(M) \neq 0$ for every M;

(4) if x is an element of A, then its spectrum equals the range of the function $\hat{x}$ and its spectral radius equals the norm of $\hat{x}$, that is, $\sigma(x) = \hat{x}(\mathfrak{M})$ and $r(x) = \sup |\hat{x}(M)| = \|\hat{x}\|$.

PROOF. The definition of the topology on $\mathfrak{M}$ guarantees that each function $\hat{x}$ is continuous, and part (5) of Theorem A shows that $\hat{x}$ is bounded and that $\|\hat{x}\| = \sup |\hat{x}(M)| \leq \|x\|$, so $x \to \hat{x}$ is a norm-decreasing mapping of A into $\mathscr{C}(\mathfrak{M})$. The fact that this mapping is a homomorphism is immediate from parts (1), (2), and (3) of Theorem A.

Since $x \to \hat{x}$ is a homomorphism, $\hat{A}$ is obviously a subalgebra of $\mathscr{C}(\mathfrak{M})$. The stated properties of $\hat{A}$ follow readily from parts (4) and (5) of Theorem A: if $M_1 \neq M_2$ and if (say) x is in M_1 but not in M_2, then $\hat{x}(M_1) = 0$ and $\hat{x}(M_2) \neq 0$; and $\hat{1}(M) = 1$ for every M.

If we recall that R is the intersection of all the M's, then the proof of (2) is easy: we have only to notice that part (4) of Theorem A tells us that $\hat{x}(M) = 0$ for every M $\Leftrightarrow$ x is in every M.

To prove (3), it suffices—in view of part (4) of Theorem A—to show that x is regular $\Leftrightarrow$ it does not belong to any M. It is elementary that a regular element cannot lie in any proper ideal, so we confine our attention to showing that if x is singular, then it does belong to some M. We prove this by observing that the singularity of x implies that $Ax = \{yx : y \in A\}$ is a proper ideal which contains x and can therefore be imbedded in a maximal ideal M which also contains x.

Finally, we use (3) to prove (4). By the definition of the spectrum of x, we have $\lambda \in \sigma(x) \Leftrightarrow x - \lambda 1$ is singular $\Leftrightarrow (\widehat{x - \lambda 1})(M) = 0$ for at least one $M \Leftrightarrow (\hat{x} - \lambda 1)(M) = 0$ for at least one $M \Leftrightarrow \hat{x}(M) = \lambda$ for at least one M, so $\sigma(x)$ equals the range of $\hat{x}$. The rest of (4) follows from this statement and the definition of the spectral radius.

We add the final touch to this portion of the theory by showing that $\mathfrak{M}$ is a compact Hausdorff space. The reader will recall that if A^* is the conjugate space of A, then its closed unit sphere

$$S^* = \{f : f \in A^* \text{ and } \|f\| \leq 1\}$$

is a compact Hausdorff space in the weak* topology (see Theorem 49-A).

Our strategy is to identify $\mathfrak{M}$, both as a set and as a topological space, with a closed subspace of A^*.

A *multiplicative functional* on A is a functional f in the ordinary sense—that is, an element of the conjugate space A^*—which is non-zero and satisfies the additional condition $f(xy) = f(x)f(y)$. Theorem A shows that to each M in $\mathfrak{M}$ there corresponds a multiplicative functional f_M defined by $f_M(x) = x(M)$. It is important for us to know that $M \to f_M$ is a one-to-one mapping of $\mathfrak{M}$ onto the set of all multiplicative functionals. It will facilitate our work if we begin by proving the

Lemma. *If f_1 and f_2 are multiplicative functionals on A with the same null space M, then $f_1 = f_2$.*

PROOF. We first show that $f_1 = \alpha f_2$ for some scalar α. Let x_0 be an element of A which is not in M. If x is an arbitrary element of A, it is easy to see that x can be expressed uniquely in the form $x = m + \beta x_0$ with m in M (set $\beta = f_1(x)/f_1(x_0)$, put $m = x - \beta x_0$, and observe that $f_1(m) = 0$). It now follows that

$$f_1(x) = f_1(m) + \beta f_1(x_0) = \beta f_1(x_0) = [f_1(x_0)/f_2(x_0)]f_2(x),$$

so $f_1 = \alpha f_2$ with $\alpha = f_1(x_0)/f_2(x_0)$. We complete the proof by showing that α equals 1. Let x be an element not in M, so that $f_2(x) \neq 0$. Then $\alpha f_2(x)^2 = \alpha f_2(x^2) = f_1(x^2) = f_1(x)^2 = [\alpha f_2(x)]^2 = \alpha^2 f_2(x)^2$ implies that

$$\alpha^2 = \alpha,$$

so $\alpha = 0$ or $\alpha = 1$. Since $f_1 \neq 0$, we conclude that $\alpha = 1$.

We now use this to prove

Theorem C. *$M \to f_M$ is a one-to-one mapping of the set $\mathfrak{M}$ of all maximal ideals in A onto the set of all its multiplicative functionals.*

PROOF. The mapping is easily seen to be one-to-one, for if $M_1 \neq M_2$, and if (say) x is in M_1 and not in M_2, then $f_{M_1}(x) = 0$ and $f_{M_2}(x) \neq 0$. To prove that it is onto, let f be an arbitrary multiplicative functional and consider its null space $M = \{x : f(x) = 0\}$. It is clear by the assumed properties of f that M is a proper closed ideal in A. Furthermore, M is maximal, for if it were properly contained in a proper ideal I, then $f(I)$ would be a non-trivial ideal in C, contrary to Theorem 41-A. Since f and f_M are multiplicative functionals with the same null space, the lemma just proved implies that $f = f_M$, and our proof is complete.

In some of its more concrete applications, this theorem is used to replace the algebraic problem of determining the maximal ideals in A by the analytic problem of finding its multiplicative functionals. Its importance for our current task of showing that $\mathfrak{M}$ is a compact Hausdorff again is that it enables us to regard $\mathfrak{M}$ as a subset of A^*. We can say even more than this, for parts (1) and (2) of Theorem A tell us that every multiplicative functional f_M has norm 1, so $\mathfrak{M}$ is a subset of the closed unit sphere S^*. We recall earlier that S^* is a compact Hausdorff space with respect to the weak* topology, which is (see Sec. 49) the weak topology generated by all the functions F_x defined on A^* by $F_x(f) = f(x)$. We now observe that when F_x is restricted to $\mathfrak{M}$, it is precisely $\hat{x}$, for

$$F_x(f_M) = f_M(x) = x(M) = \hat{x}(M).$$

Therefore, by Problem 19-1, the topology which $\mathfrak{M}$ has as a subspace of S^* is exactly its topology as the space of maximal ideals. These considerations permit us to regard $\mathfrak{M}$ as a subspace of S^*.

Theorem B. *The maximal ideal space $\mathfrak{M}$ is a compact Hausdorff space.*

PROOF. In view of the above discussion, it suffices to show that $\mathfrak{M}$ is a closed subspace of S^*. We accomplish this by forming the subspace $\mathfrak{L}$ of S^* defined by

$$\mathfrak{L} = \cap_{x,y \in A} \{f : f \in S^* \text{ and } f(xy) = f(x)f(y)\}.$$

It is evident that $\mathfrak{L}$ is simply $\mathfrak{M}$ together with the zero functional, and since we have

$$\begin{aligned} \mathfrak{L} &= \cap_{x,y \in A} \{f : f \in S^* \text{ and } f(xy) - f(x)f(y) = 0\} \\ &= \cap_{x,y \in A} \{f : f \in S^* \text{ and } F_{xy}(f) - F_x(f)F_y(f) = 0\} \\ &= \cap_{x,y \in A} \{f : f \in S^* \text{ and } (F_{xy} - F_xF_y)(f) = 0\}, \end{aligned}$$

it is easy to see that $\mathfrak{L}$ is closed in S^* (note that each of the sets last written has this property). We next remark that F_1 is continuous on $\mathfrak{L}$ and equals 1 on $\mathfrak{M}$ and 0 at the zero functional. It follows from this that $\mathfrak{M}$ is closed in $\mathfrak{L}$ and is therefore closed in S^*.

It is worthy of notice that the topology we imposed on $\mathfrak{M}$ is the only one which makes it into a compact Hausdorff space on which all the functions $\hat{x}$ are continuous, for by Theorem 26-E, any stronger compact Hausdorff topology must equal the given one.

When Theorems B and [illegible] are taken together, the result is often called the Gelfand representation theorem. In essence, this tells us that every commutative semi-simple Banach algebra is isomorphic to an algebra of continuous complex functions on a suitable compact Hausdorff space. In general, the norm is not preserved by this isomorphism and the representing algebra does not exhaust the continuous functions on the underlying space. We shall remove these deficiencies in the following sections by assuming that additional structure is present in the Banach algebra under discussion.