UNIT-V

DIFFERENTIAL GEOMETRY OF SURFACES IN THE LARGE

Compact surfaces whose points are umblics:

Theorem: 1

The only compact surfaces of class ≥ 2 for which every point is an umblic are spheres.

Proof:

Let S be a compact surfaces of class ≥ 2 for which every point is an umblic.

Let P be any point on S, and let V be a co-ordinate neighbourhood of S containing P, in which part of S is represented parametrically by

$r = r(u, v)$

Since every point of V is an umblic, it follows that every curve lying in V must be line of curvature.

Hence from Rodrigues formula at all points of V

$dN + k\,dr = 0$ —— ①

Where k is the normal curvature of S in the direction dr.

From ① we get,

$N_1 = -k r_1 \qquad N_2 = -k r_2$

Which with the identity.

$N_{12} = N_{21}$

gives

$k_2 r_1 - k_1 r_2 = 0$

Since r_1, r_2 are linearly independent we obtain $K_1 = K_2 = 0$, so that K is constant.

Integration of ① gives, for $K \neq 0$,

$$r = a - K^{-1} N \quad \text{——— ②}$$

showing that r lies on a plane.

This completes the local part of the theorem. So far all we have proved is that in the neighbourhood of any point the surface is spherical or plane.

Associate with each point p on the surface a neighbourhood V_p having the above property.

The set of all neighbourhoods V_p covers S, and from the compactness we deduce that S is covered by a finite sub-cover formed by V_i, $i = 1, 2, \ldots N$.

Consider two over-lapping neighbourhoods V_i, V_j. From the previous local argument it follows that K is constant in V_i and also in V_j.

By considering the value of K at points in $V_i \cap V_j$ it follows that K takes the same value over the whole of the surface.

Moreover, this value cannot be zero, otherwise the surface would contain a straight line and would not be compact.

Hence the surface must be a sphere.

Hence proved.

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(5) Hilbert's lemma:

In a closed region R of a surface of constant positive Gaussian curvature without umbilics, the principal curvatures takes their extreme values at the boundary.

(or)

If at a point P_0 of any surface, the principal curvatures K_a, K_b are such that either

i) $K_a > K_b$, K_a has a maximum at P_0 and K_b has a minimum at P_0 or

ii) $K_a < K_b$, K_a has minimum at P_0 and K_b has a maximum at P_0.

then the Gaussian curvature K cannot be positive at P_0.

Proof:

Let us prove this lemma by the method of contradiction.

Suppose that the lemma is false.

Then there is a point P_0 at which the principal curvatures have distinct extreme values are maximum and the other minimum.

Taking the lines of curvature as parametric curves, the principal curvatures are

$$K_a = L/E \quad , \quad K_b = N/G \quad \text{——} (1) \qquad \text{[By Euler's formula]}$$

The codazzi equations are

$$\left.\begin{aligned} L_2 &= \frac{1}{2} E_2 \left(\frac{L}{E} + \frac{N}{G} \right) \\ N_1 &= \frac{1}{2} G_1 \left(\frac{L}{E} + \frac{N}{G} \right) \end{aligned}\right\} \text{——} (2)$$

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Differentiate ① with respect to u and v, we get

$$\frac{\partial K_a}{\partial v} = \frac{EL_2 - LE_2}{E^2}$$

from ②

$$= \frac{E\left[E_2/2\left(\frac{L}{E}+\frac{N}{G}\right)\right] - LE_2}{E^2}$$

$$= \frac{\frac{EE_2L}{2E} + \frac{EE_2N}{2G} - LE_2}{E^2}$$

$$= \frac{\frac{E_2L}{2} + \frac{EE_2N}{2G} - LE_2}{E^2}$$

$\frac{E_2L}{2} - LE_2$

$$= \frac{\frac{EE_2}{2}\left(\frac{N}{G}\right) - \frac{E_2L}{2}}{E^2}$$

$$= \frac{E_2}{2E}\left(\frac{N}{G}\right) - \frac{E_2L}{2E^2}$$

$$= \frac{E_2}{2E}\left(\frac{N}{G} - \frac{L}{E}\right)$$

$$\frac{\partial K_a}{\partial v} = \frac{1}{2}\frac{E_2}{E}(K_b - K_a)$$

$$\frac{\partial K_b}{\partial u} = \frac{GN_1 - NG_1}{G^2}$$

$$= \frac{1}{G^2}\left[\frac{GG_1}{2}\left(\frac{L}{E}+\frac{N}{G}\right) - NG_1\right]$$

$$= \frac{G_1L}{2EG} + \frac{G_1N}{2G^2} - \frac{NG_1}{G^2}$$

$$\frac{\partial K_b}{\partial u} = \frac{G_1L}{2EG} - \frac{NG_1}{2G^2}$$

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$$= \frac{G_1}{2G}\left[\frac{L}{E} - \frac{N}{G}\right]$$

$$= \frac{G_1}{2G}(k_a - k_b)$$

$$\frac{\partial k_b}{\partial u} = \frac{1}{2}\frac{G_1}{G}(k_a - k_b)$$

therefore,

$$\left.\begin{aligned} \frac{\partial k_a}{\partial v} &= \frac{1}{2}\frac{E_2}{E}(k_b - k_a) \\ \frac{\partial k_b}{\partial u} &= \frac{1}{2}\frac{G_1}{G}(k_a - k_b) \end{aligned}\right\} \text{——— (3)}$$

Since we have assumed that the principal curvatures have extreme values.

$$\frac{\partial k_a}{\partial v} = 0 \qquad \frac{\partial k_b}{\partial u} = 0$$

$\frac{1}{2}\frac{E_2}{E}(k_b - k_a) = 0$

$E_2 = 0$

$\frac{\partial k_a}{\partial v} = \frac{1}{2}\frac{E_2}{E}(k_b - k_a)$

$\frac{\partial^2 k_a}{\partial v^2} = \frac{1}{2}(k_b - k_a)[E$

then (3) $\Rightarrow E_2 = G_1 = 0$.

Since at the point

$$\frac{\partial^2 k_a}{\partial v^2} = \frac{1}{2}\left[\frac{E E_{22} - E_2^{\,2}}{E^2}\right](k_b - k_a)$$

substitute the limit $E_2 = 0$.

$$\left.\begin{aligned} \frac{\partial^2 k_a}{\partial v^2} &= \frac{1}{2}\frac{E_{22}}{E}(k_b - k_a) \\ \frac{\partial^2 k_b}{\partial u^2} &= \frac{1}{2}\frac{G_{11}}{G}(k_a - k_b) \end{aligned}\right\} \text{——— (4)}$$

There are now two possibilities either

i) k_a has a maximum in this case,

$$k_a - k_b > 0$$

$$\frac{\partial^2 k_a}{\partial v^2} \leq 0$$

$$\frac{\partial^2 k_b}{\partial u^2} \geq 0$$

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(ii) K_a has a minimum, then,

$$K_b - K_a > 0$$

$$\frac{\partial^2 K_a}{\partial v^2} \geq 0$$

$$K = \frac{-1}{2E}\left(\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right)$$

$$\frac{\partial^2 K_b}{\partial u^2} \leq 0$$

From (4) we have in either case we see that

$E_{22} > 0 \quad G_{11} > 0$

But, we know that the Gaussian curvature,

$$K = \frac{-1}{2H}\left[\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right]$$

$G_1 = E_2 = 0$

$$K = \frac{-1}{2H}\left[\frac{H G_{11} - G_1 H_1 + H E_{22} - E_2 H_2}{H^2}\right]$$

$$K = \frac{-1}{2H}\left[\frac{G_{11} + E_{22}}{H}\right]$$

$$K = -\left[\frac{G_{11} + E_{22}}{2H^2}\right]$$

$H = \sqrt{EG - F^2}$

$-\frac{(G_{11} + E_{22})}{2G_1 E}$

$$K = -\left[\frac{G_{11} + E_{22}}{2(EG - F^2)}\right]$$

$$K = \frac{-(G_{11} + E_{22})}{2EG} \qquad [\because f = 0]$$

But $E_{22} > 0 \quad G_{11} > 0$

$\Rightarrow$ K is either zero (or) negative which a contradiction.

Hence Gaussian curvature is a positive constant.

Hence proved.

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Compact surfaces of constant Gaussian or Mean curvature :

Theorem : 3

Let S the only compact surfaces with constant Gaussian curvature are spheres.

Proof :

Let S be a compact surface with constant positive Gaussian curvature K.

Since S is compact, there is an point P_0 at which the maximum value of the principal curvature is attained.

Since the product of the principal curvatures (ie, the Gaussian curvature) is constant.

Follows that the principal curvatures have respect a maximum and minimum value at P_0, with the maximum not less than the minimum.

From Hilbert's lemma it follows that the two principal curvatures must be equal.

ie, no point does either principal curvature exceed $\sqrt{K}$.

Hence every point of S is am umblic, and the theorem now follows from theorem : 1

"The only constant compact surfaces of class2 for which every point is an umblic are spheres".

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Theorem: 4

The only compact surfaces whose Gaussian curvature is positive and mean curvature constant are spheres.

Proof:

Let S be a compact surface of positive Gaussian curvature and constant mean curvature and denote by K_a, K_b respectively the larger and smaller principal curvatures.

Since K_a is continuous and S compact, there is a point P_0 at which K_a attains the maximum value.

Since the mean curvature is constant it follows that K_b attains its minimum value at P_0.

Now we have the relation $K_a \geq K_b$ everywhere if $K_a > K_b$ at P_0, then the Hilbert lemma would apply and we could conclude that

$K \leq c$ contrary to hypothesis.

Hence we must have $K_a = K_b = \mu$ at P_0, and hence everywhere on S.

This completes the proof of the theorem.

Metric space:

There is a real valued function

$e : S \times S \longrightarrow R$,

with the properties

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i) $\rho(A,B) = 0$ iff $A = B$,

ii) $\rho(A,B) = \rho(B,A)$

iii) $\rho(A,C) \leq \rho(A,B) + \rho(B,C)$

For all points A, B, C of S

⑧ Cauchy sequence:

A sequence of points $\{x_n\}$ on the surface is said to form a cauchy sequence when, given a positive real number ε, an integer n_0 can be found such that $\rho(x_p, x_q) < \varepsilon$ provide p and q both exceed n_0. Evidently, if $\{x_n\}$ converges to a limit x, then the sequence $\{x_n\}$ is a cauchy sequence.

⑨ Complete:

If the surface is such that every cauchy sequence converges, then the metric is said to be complete.

⑩ Characterization of complete surfaces:

State any two prop of complete surface.

a) Every cauchy sequence of points of S is convergent

b) Every geodesic can be prolonged indefinitely in either direction or else it forms a closed curve.

c) Every bounded set of points of S is relatively compact.

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Hilberts theorem:

A complete analytic surface, free from singularities, with constant negative Gaussian curvature cannot exist in three-dimensional Euclidean space.

Proof:

Let P be a part on the surface S, and let Q be the set of all paths of S which begin at P.

We divide the set Q into classes, putting into each class the totality of paths that are homotopically equivalent.

We denote the set of these classes by S'. So that a point of S is an equivalent class of paths on S.

There is a natural mapping ϕ of the set S' on the space S, for if A is a points or S'. Then all the equivalent paths in S belonging to A must end in the same point a, and we write $a = \phi(A)$.

It is shown that the set of points S can be considered as forming a surface called the universal covering surface which has the following important properties

1. The natural mapping of S' on S is a continuous open mapping moreover, ϕ is a locally homeomorphic mapping.

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2. The universal covering surface S' of a surface S is always simply connected.

We assume that a surface S exists having the required properties and we obtain a proof by contradiction.

Consider an arbitrary geodesic line on the surface S and take an arbitrary points O on this geodesic as origin.

If s denotes the arc length of this geodesic measured from O. The completeness of S ensure that the geodesic can be continued in both directions form $-\infty$ to $+\infty$.

It is possible that the geodesic will ultimately cross itself so that the same point on S will ~~be~~ have two different s-values.

However, if we considered instead of S its universal covering surface S', then different values of s will correspond to different points

This follows because on a surface of negative Gaussian curvature two geodesic arcs cannot enclose a simply connected.

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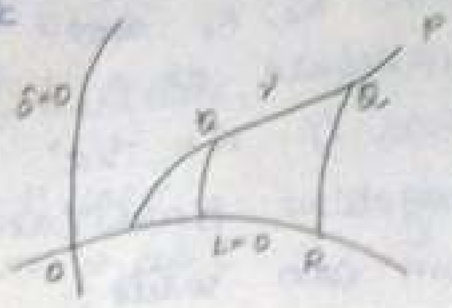
Now consider the asymptotic lines on the surface S. These are given by the difference equation.



$$L\,ds^2 + 2M\,ds\,dt + N\,dt^2 = 0$$

Since, $K<0$ we conclude that $LN - M^2 < 0$ and hence that at each point of the $(s-t)$ plane. Since the $(s-t)$ plane is simply connected it follow that the differential eqn gives rise to two vectors fields which can continued over the whole plane.

From the known theorem, we observes that each asymptotic line can be prolonged to an arbitrary extent in both directions and it t^r denote the arc length then

$$\lim_{t\to -\infty} (s^2+t^2) = \infty, \quad \lim_{t\to \infty} (s^2+t^2) = \infty$$

We next prove that each asymptotic line of the system cuts each asymptotic line of the other system is exactly one points.

Suppose that T is a neighbourhood of S in which the lines of curvature are choosen as parametric lines.

If K_a, K_b denote the principal curvature at a point P on N and if $K = -1/a^2$ is the constant negative Gaussian curvature, we can write,

$$K_a = a^{-1} \cot \rho,$$
$$K_b = -a^{-1} \tan \rho$$

where, $0 < \rho < \frac{1}{2}\pi$

An argument leads to the equations

$$\left.\begin{aligned} \frac{\partial K_a}{\partial v} &= \frac{1}{2} \frac{E_2}{E} (K_b - K_a) \\ \frac{\partial K_b}{\partial u} &= \frac{1}{2} \frac{G_1}{G} (K_a - K_b) \end{aligned}\right\} \text{——} (1)$$

Substitute for K_a, K_b to get.

$$\left.\begin{aligned} \frac{E_2}{E} &= 2\rho_2 \cot \rho \\ \frac{G_1}{G} &= -2\rho_1 \tan \rho \end{aligned}\right\} \text{——} (2)$$

These equations integrate to given

$$\left.\begin{aligned} E &= U(u) \sin^2 \rho \\ G &= V(v) \cos^2 \rho \end{aligned}\right\} \text{——} (3)$$

where, $U(u)$ and $V(v)$ are certain functions of u and v respectively.

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The first fundamental form then becomes

$$\sin^2\rho \, du^2 + \cos^2\rho \, dv^2$$

In terms of the new parameters,

$L = K_a$, $\quad F = a^{-1} \sin\rho \cos\rho$

$N = K_b$, $\quad G = -a^{-1} \sin\rho \cos\rho$

$M = 0$

and the asymptotic lines are given by

$$du^2 - dv^2 = 0$$

Choose new parameters σ, τ where,

$$\sigma = \frac{1}{2}(v+u)$$

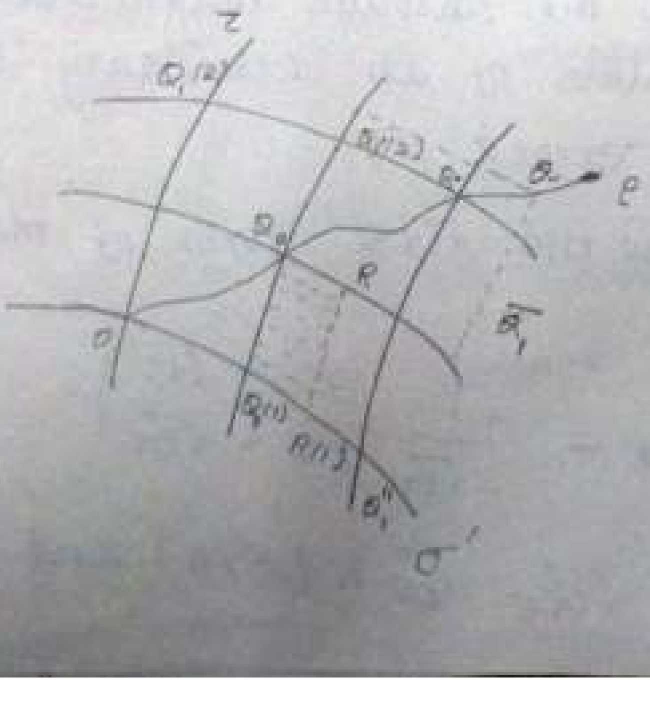
$$\tau = \frac{1}{2}(v-u)$$

Then the parametric curves $\sigma =$ constant, $\tau =$ constant are asymptotic lines.

Moreover, the metric assumes the form

$$d\sigma^2 + 2\cos^2\rho \, d\sigma \, d\tau + d\tau^2 \quad \text{(4)}$$

and σ, τ measure the arc lengths of the asymptotic lines.



Let ω be the angle between the parametric curves. Then from known formula,

$$\cos\omega = \frac{r_1 \cdot r_2}{|r_1||r_2|} = \frac{F}{\sqrt{EG}}$$

$$\sin\omega = \frac{|r_1 \times r_2|}{|r_1||r_2|} = \frac{H}{\sqrt{EG}}$$

We see that $\omega = 2\rho$ and hence $0 < \omega < \pi$.

Now using equation for the Gaussian curvature $K = -a^{-2}$.

we find.

$$\frac{\partial^2 \omega}{\partial\sigma\,\partial\tau} = -K \sin\omega \longrightarrow (5)$$

Consider now the quadrilateral formed by the asymptotic lines $\sigma = \pm 2$ $\tau = \pm\alpha$.

(see above figure).

we have for the total curvature

$$\iint K\,ds = \iint K \sin\omega \, d\sigma\, d\tau$$

$$= \omega_1 - \omega_2 + \omega_3 - \omega_4$$

$$= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2\pi$$

It follows that the absolute magnitude of the total curvature of an arbitrary large region cannot exceed 2π.

Consider now the first form of metric $ds^2 + G(s)\,dt^2$.

we have, $K = \frac{-1}{2\sqrt{G}} \frac{\partial}{\partial s}\left(\frac{G_s}{\sqrt{G}}\right)$

The total curvature over a region bounded by parametric lines $s = \pm l$, $t = \pm l$ is

$$\iint K \cdot dS = \iint K\sqrt{G}\, ds\, dt$$

$$= -\frac{1}{2}\iint \frac{\partial}{\partial s}\left(\frac{G_s}{\sqrt{G}}\right) ds \cdot dt$$

$$= -\frac{4l}{a} \sin h\, l/a$$

But in magnitude this tends to infinitely as $l \to \infty$ which contradicts the previous assertation that the absolute magnitude of the total curvature cannot conjugate points on geodesics.

Theorem :

If P and Q are two points of a geodesic which can be embeded in a field of geodesic then the arc PQ of the geodesic is shorter than any other arc which joints P to Q and lies entirely in that region of the surface covered by the field.

Proof :

To prove the theorem choose parameters so that the geodesic of the family are the curves v = constants with $v = v_0$ as the given geodesic and let the curves u = constant be geodesic parallel orthogonal to them. so choosen that the metric reduces to the form

$ds^2 = du^2 + \lambda^2 dv^2$

If the co-ordinates of P and Q are (u_1, v_0), (u_2, v_0) with $u_2 > u_1$, the length of the geodesic arc PQ is $(u_2 - u_1)$

Let C be an arbitrary curve passing through P and Q given by the equation

$v = \phi(u)$

Where, $\phi(u_1) = v_0$, $\phi(u_2) = v_0$. Then the arc length of C is

$$l = \int_{u_1}^{u_2} \{1 + \lambda^2 (d\phi/du)^2\}^{1/2} du$$

Evidently l exceeds $u_2 - u_1$ unless $d\phi/du = 0$. When, C is the given geodesic

Theorem:

When the surface S has negative curvature everywhere, the length of a geodesic which joins any two points A, B is always less than the lengths of neighbouring curves through A and B.

Proof:

Let one system of parametric curves be the geodesic normal to the given geodesic AB and the other system be the orthogonal trajectory

Let u denote the length of the geodesic on normal PQ from P to AB, and let v denote the length PQ.

The line element of the surface becomes

$$ds^2 = du^2 + \lambda^2 dv^2$$

where ; $\lambda(0,v) = 1$, $\lambda(0,v) = 0$

Interms of these parameters the Gaussian curvature is given by.

$$K = \frac{-\lambda_{11}}{\lambda}, \text{ so that } \lambda_{11} = -\lambda K$$

The function λ may thus be expanded as a power series in u in the form

$$\lambda = 1 - K\frac{u^2}{2} - K_1\frac{u^3}{6} + O(u^4) \ldots$$

where,

K and K_1 are evaluated with $u = 0$.

A neighbouring curve APB which differs very little from AB will have an equation of the form $u = \phi(v)$ where, u will be small.

The length of this curve will be

$$l = \int_A^B \{\phi'^2 + \lambda^2\}^{1/2} dv$$

$$= \int_A^B \{1 + \phi'^2 - K\phi^2 - \tfrac{1}{3}\phi^3\}^{1/2} dv.$$

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where, terms of the fourth order are neglected.

We now assume that ϕ' never becomes infinite and is thus of the same order of smallness as u.

With this assumption the difference between l and the geodesic arc length s may be written

$$l - s = \frac{1}{2}\int_A^B \{\phi'^2 - K\phi^2 - \frac{1}{3}K_1\phi^3\}\, dv$$

Now the sign of the variation of the arc length will be given by the second order term provided that these do not vanish identically.

If only these terms retained the equations becomes.

$$l - s = \frac{1}{2}\int_A^B (\phi'^2 - K\phi^2)\, dv.$$

Now, if K is always negative, the integrated is always positive and so $l > s$

This proves the required result.

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Lemma: (Erdmann's lemma)

In order that the geodesic distance AB should be the shortest distance it is necessary and sufficient that B lies between A and its conjugate point A_1.

Proof:

consider the problem of finding a curve

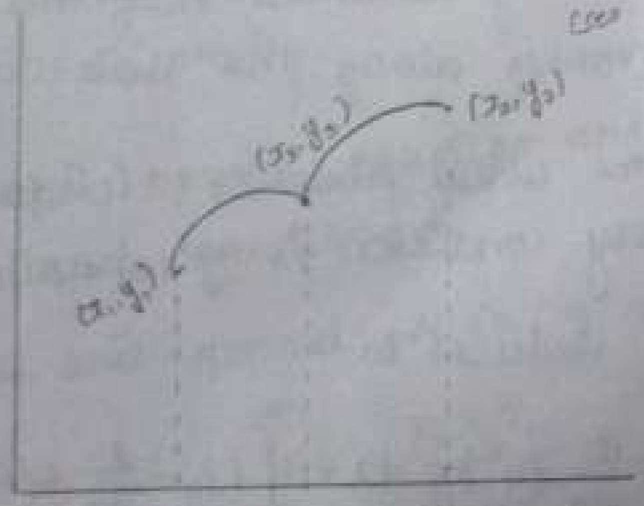
$$y = y(x).$$

which passes through two points (x_1, y_1), (x_2, y_2) has a discontinuity of slope on the line $x = x_3$ and is such that the integral

$$J = \int_{x_1}^{x_2} f(x, y, y') \, dx$$

assumes an extreme value.

(see figure)



Let $y'_+ = \lim_{\delta \to 0} y'(x_3 + \delta)$

$$y'_- = \lim_{\delta \to 0} y'(x_3 - \delta)$$

Where δ is positive

Then Erdamann's lemma states that for an extreme value. In addition to the equation of Euler it is necessary that

$$f_{+y'} = f_{y'}(x_3, y_3, y'_+).$$

$$f_{-y'} = f_{y'}(x_3, y_3, y'_-)$$

To prove the lemma, we note that the variation of the integral over the curves $y(x)$ and $y + \epsilon\eta(x)$, where $\eta(x_1) = 0$, $\eta(x) = 0$ is given by.

$$J(\epsilon) = \int_{x_1}^{x_2} f(x, y+\epsilon\eta, y'+\epsilon\eta')\,dx + \int_{x_3}^{x_2} f(x, y+\epsilon\eta, y'+\epsilon\eta')\,dx$$

it being assumed that the "corner" still moves along the line $x = x_3$

In the usual manner, it follows that a necessary condition is $J'(0) = 0$.

This reduces to

$$\int_{x_1}^{x_3} \left(f_y - \frac{d}{dx} f_{y'}\right)\eta\,dx + \int_{x_3}^{x_2} \left(f_y - \frac{d}{dx} f_{y'}\right)\eta\,dx + \eta_3(f_{-y'} - f_{+y'}) = 0$$

From this it follows that in addition to Eulers equation.

$$f_y - \frac{d\, f_{y'}}{dx} = 0.$$

It is necessary to have $f_{+y'} = f_{-y'}$ and the lemma is proved.

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