5 PLANE ELECTROMAGNETIC WAVES AND THEIR PROPAGATION

INTRODUCTION :

[In this chapter we shall show that the Maxwell's field equations, predict the existence of electromagnetic waves and discuss the propagation of these waves in free space, non-conducting, conducting and ionized media. We shall also investigate the energy flow associated with their propagation.]

§ 5.1. Electromagnetic Waves in free space.*

We know that Maxwell's equations are

$$\left.\begin{aligned} \nabla \bullet \mathbf{D} &= \rho \\ \nabla \bullet \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned}\right\} \quad \text{with} \begin{cases} \mathbf{J} = \sigma \mathbf{E} \\ \mathbf{B} = \mu \mathbf{H} \\ \mathbf{D} = \varepsilon \mathbf{E} \end{cases} \qquad \text{...(1)}$$

and in free space *i.e.* vacuum

$$\rho = 0 \qquad \varepsilon_r = 1$$
$$\sigma = 0 \qquad \mu_r = 1$$

So Maxwell's equations reduce to

$$\left.\begin{aligned} \nabla \bullet \mathbf{E} &= 0 && \text{...(a)} \\ \nabla \bullet \mathbf{H} &= 0 && \text{...(b)} \\ \nabla \times \mathbf{H} &= \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} && \text{...(c)} \\ \nabla \times \mathbf{E} &= -\mu_0 \frac{\partial \mathbf{H}}{\partial t} && \text{...(d)} \end{aligned}\right\}. \qquad \text{...(2)}$$

Now if

(I) We take the curl of equation 2 (c) then

$$\nabla \times (\nabla \times \mathbf{H}) = \varepsilon_0 \nabla \times \left(\frac{\partial \mathbf{E}}{\partial t}\right)$$

i.e.

$$\left[\nabla(\nabla \bullet \mathbf{H}) - \nabla^2 \mathbf{H}\right] = \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}). \qquad \text{...(3)}$$

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* Reader is advised to go through appendix III before starting this chapter.

But from equations 2 (b) and 2 (d)

$$\nabla \bullet \mathbf{H} = 0 \qquad \text{and} \qquad \nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}.$$

So eqn. (3) reduces to

$$\nabla^2 \mathbf{H} - \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \text{ with } \mu_0 \varepsilon_0 = \frac{1}{c^2}. \qquad \text{...(A)}$$

(II) We take the curl of equation 2 (d), then

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\mu_0 \frac{\partial \mathbf{H}}{\partial t} \right)$$

i.e.

$$\left[\nabla(\nabla \bullet \mathbf{E}) - \nabla^2 \mathbf{E}\right] = -\mu_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{H}). \qquad \text{...(4)}$$

But from equation 2 (a) and 2 (c)

$$\nabla \bullet \mathbf{E} = 0 \qquad \text{and} \qquad \nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

So equation (4) reduces to

i.e.

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \text{ with } \mu_0 \varepsilon_0 = \frac{1}{c^2}. \qquad \text{...(B)}$$

A glance at differential equations (A) and (B) reveals that these are indential in form to the equation

$$\nabla^2 \psi - \frac{1}{\upsilon^2}\frac{\partial^2 \psi}{\partial t^2} = 0. \qquad \text{...(5)}$$

However equation (5) is a standard wave equation representing unattenuated wave traveling at a speed υ*. So we conclude that *field vector E and H are propagated in free space as waves at a speed*

$$c = \frac{1}{\sqrt{(\varepsilon_0 \mu_0)}} = \sqrt{\left(\frac{4\pi}{4\pi\varepsilon_0 \mu_0}\right)} = \sqrt{(9 \times 10^9)} \times \sqrt{(10^7)}$$

$$= 3 \times 10^8 \ m/s$$

*i.e. the velocity of light.***

Further as equation (A) and (B) are vector wave equations their solution can be obtained in many forms, for instance either stationary or progressive waves or having wave fronts of particular types such as plane, cylindrical or spherical. Where no boundary conditions are imposed, as in

* For details of plane progressive wave see point (3) in appendix III.

** This result suggests that light may be electromagnetic in nature.

this chapter, plane progressive solutions are most appropriate. So as the plane progressive solution of equation (5) is

$$\psi = \psi_0 \, e^{-i(\omega t - \mathbf{k} \bullet \mathbf{r})}$$

the solutions of equations (A) and (B) will be of the form

$$\left.\begin{aligned} \mathbf{E} &= \mathbf{E}_0 \, e^{-i(\omega t - \mathbf{k} \bullet \mathbf{r})} \\ \mathbf{H} &= \mathbf{H}_0 \, e^{-i(\omega t - \mathbf{k} \bullet \mathbf{r})} \end{aligned}\right\} . \qquad \text{...(C)}$$

where **k** is the so called wave vector given by

$$\mathbf{k} = k\mathbf{n} = \frac{2\pi}{\lambda}\mathbf{n} = \frac{2\pi f}{\lambda}\mathbf{n} = \frac{\omega}{c}\mathbf{n}$$

with **n** as a unit vector in the direction of wave propagation.

The form of field vectors **E** and **H** given by eqn. (C) suggests that in case of field vectors operator ∇ is equivalent to $i\,\mathbf{k}$ while $\partial/\partial t$ is $(-i\omega)$.* So Maxwell's equations in free space *i.e.* eqn. (2) in terms of operator (i **k**) and $(-i\omega)$ can be written as

$$\left.\begin{aligned} \mathbf{k} \bullet \mathbf{E} &= 0 && \text{.....(a)} \\ \mathbf{k} \bullet \mathbf{H} &= 0 && \text{.....(b)} \\ -\mathbf{k} \times \mathbf{H} &= \omega \varepsilon_0 \mathbf{E} && \text{.....(c)} \\ \mathbf{k} \times \mathbf{E} &= \omega \mu_0 \mathbf{H} && \text{.....(d)} \end{aligned}\right\} . \qquad \text{...(4)}$$

Regarding plane electromagnetic waves in free space it is worthy to note that :

(*i*) As according eqn. 4 (a) the vector **E** is perpendicular to the direction of propagation while according to eqn. 4 (b) the vector **H** is perpendicular to the direction of propagation (*i.e.* in an electromagnetic wave both the vectors **E** and **H** are perpendicular to the direction of wave propagation), *electromagnetic waves are transverse in nature.*

Further as according to eqn. 4 (d) **H** is perpendicular to both **E** and **k** while according to eqn. 4 (a) **E** is perpendicular to **k**. This all in turn implies that *in a plane electromagnetic waves vectors* **E**, **H** *and* **k** *are orthogonal* as shown in fig. 5.1.

(*ii*) As according to equation 4 (d)

$$\mathbf{k} \times \mathbf{E} = \omega \mu_0 \mathbf{H}$$

* For details see point (3) in appendix III.

$$\mathbf{H} = \frac{k}{\omega\mu_0}(\mathbf{n} \times \mathbf{E}) \quad (\text{as } \mathbf{k} = \mathbf{n}k)$$

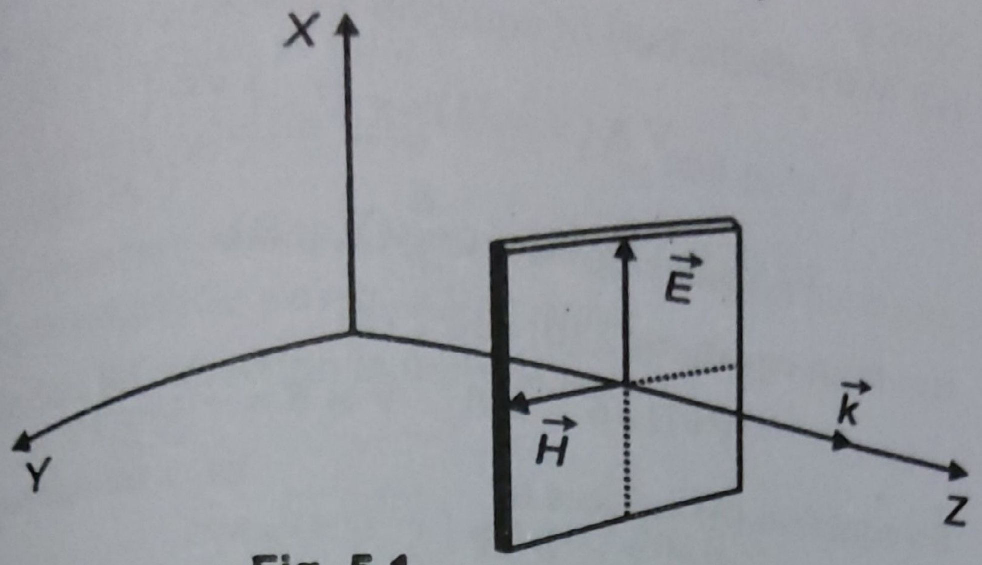

Fig. 5.1

i.e. $$\mathbf{H} = \frac{\mathbf{n} \times \mathbf{E}}{c\mu_0} = c\varepsilon_0 (\mathbf{n} \times \mathbf{E}) \qquad \left(\text{as } k = \frac{\omega}{c} \text{ and } \varepsilon_0\mu_0 = \frac{1}{c^2}\right)$$

i.e. $$\mathbf{B} = \frac{\mathbf{n} \times \mathbf{E}}{c}. \qquad ...(D)$$

and $$\left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \frac{E_0}{H_0} = c\mu_0 = \frac{1}{c\varepsilon_0} = \sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)} = Z_0 \qquad \left(\text{as } \mu_0\varepsilon_0 = \frac{1}{c^2}\right)$$

As the ratio $|\mathbf{E}/\mathbf{H}|$ is real and positive, *the vectors* **E** *and* **H** *are in phase.** *i.e.* when **E** has its maximum value **H** has also its maximum value. This is shown in fig. 5.2. From the above it is also clear that in an electromagnetic wave *the amplitude of electric vector* **E** *is* Z_0 *times that of the magnetic vector* **H**.

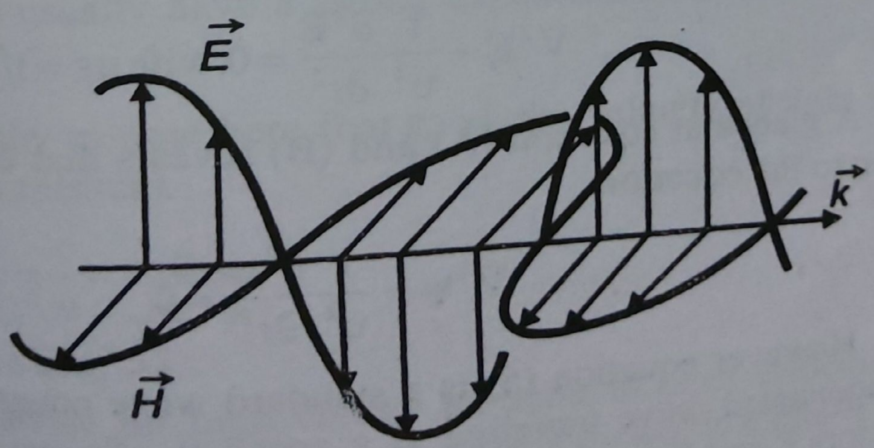

Fig. 5.2

* For details see point (4) in appendix III.

The quantity Z_0 has the dimension

$$[Z_0]=\left[\sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)}\right]=\sqrt{\left(\frac{H/m}{F/m}\right)}=\sqrt{\left(\frac{\text{ohm}\times\text{sec}}{\text{coul./volt}}\right)}$$

$$=\sqrt{\left(\frac{\text{ohm}\times\text{volt}}{\text{amp}}\right)}=\text{ohm}$$

i.e. of impedance, hence it is called the *intrinsic* or *characteristic impendence* of free space. It is a constant having value

$$Z_0=\left[\sqrt{\left(\frac{\mu_0}{\varepsilon_0}\right)}\right]=\sqrt{\left[\frac{4\pi\times10^{-7}}{(1/4\pi\times9\times10^9)}\right]}=120\pi\approx377\Omega$$

(*iii*) The Poynting vector for a plane electromagnetic wave in free space will be given by

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}=\mathbf{E}\times\frac{(\mathbf{n}\times\mathbf{E})}{c\mu_0}$$

i.e. $$\mathbf{S}=\frac{(\mathbf{E}\bullet\mathbf{E})\,\mathbf{n}-(\mathbf{E}\bullet\mathbf{n})}{c\mu_0}=\frac{1}{c\mu_0}(E^2\mathbf{n})$$

(as $\mathbf{E}\bullet\mathbf{n}=0$ because $\mathbf{E}$ is $\perp$ to $\mathbf{n}$)

or $$\mathbf{S}=\varepsilon_0\,cE^2\mathbf{n}=\frac{1}{Z_0}E^2\mathbf{n}\quad\left(\text{as }\frac{1}{c\mu_0}=c\varepsilon_0=\frac{1}{Z_0}\right)$$

or $$<\mathbf{S}>=\varepsilon_0 c<E^2>\mathbf{n}=\frac{1}{Z_0}<E^2>\mathbf{n}.$$

But as

$$<E^2>=<[E_0e^{-i(\omega t-\mathbf{k}\bullet\mathbf{r})}]^2>=E_0^2<\cos^2(\omega t-\mathbf{k}\bullet\mathbf{r})>$$

i.e. $$<E^2>=\frac{E_0^2}{2}=\left(\frac{E_0}{\sqrt{2}}\right)\left(\frac{E_0}{\sqrt{2}}\right)=E_{rms}^2\;[\text{as}<\cos^2\theta>=\tfrac{1}{2}]$$

So $$<\mathbf{S}>=\varepsilon_0 cE_{rms}^2\,\mathbf{n}=\frac{1}{Z_0}E_{rms}^2\mathbf{n}\qquad\text{...(E)}$$

i.e. the flow of energy in a plane wave in free space is in the direction of wave propagation.

(iv) In case of a plane electromagnetic wave

$$\frac{u_e}{u_m}=\frac{\frac{1}{2}\varepsilon_0E^2}{\frac{1}{2}\mu_0H^2}=\frac{\varepsilon_0}{\mu_0}\left(\frac{E}{H}\right)^2=1\left(as\frac{E}{H}=\sqrt{\frac{\mu_0}{\varepsilon_0}}\right)$$

i.e. the electromagnetic energy density is equal to the magnetostatic energy density.

Further

$$\frac{<\mathbf{S}>}{<u>} = \frac{\varepsilon_0 c E_{rms}^2 \mathbf{n}}{\varepsilon_0 E_{rms}^2} = c n$$

i.e. $$\mathbf{S} = c u n$$

This implies that *electromagnetic energy in free space is transmitted with the speed of light c with which the field vectors* **E** *and* **H** *do*

In case of propagation E. M. W. in free space.

(*i*) *The wave propagates with a speed equal to that of light in free space.*

(*ii*) *The electromagnetic waves are transverse in nature.*

(*iii*) *The wave vectors E and H are mutually perpendicular.*

(*iv*) *The vector E and H are in phase.*

(*v*) *The electrostatic energy density is equal to the magnetostatic energy density.*

(*vi*) *The electromagnetic energy is transmitted in the direction of wave propagation at speed c.*

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§ 5.2. Propagation of E. M. W. in Isotropic Dielectrics.*

We know that Maxwell's field equations are

$$\left.\begin{aligned} &\nabla \bullet \mathbf{D} = \rho \\ &\nabla \bullet \mathbf{B} = 0 \\ &\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ &\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \end{aligned}\right\} \quad \text{with} \begin{cases} \mathbf{J} = \sigma \mathbf{E} \\ \mathbf{B} = \mu \mathbf{H} \\ \mathbf{D} = \varepsilon \mathbf{E} \end{cases} \qquad \text{...(1)}$$

and in isotropic dielectrics

$$\sigma = 0 \text{ and } \rho = 0.$$

So Maxwell's equations reduce to

$$\left.\begin{aligned} &\nabla \bullet \mathbf{E} = 0 && \text{...(a)} \\ &\nabla \bullet \mathbf{H} = 0 && \text{...(b)} \\ &\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} && \text{...(c)} \\ &\nabla \Delta \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} && \text{...(d)} \end{aligned}\right\} \qquad \text{...(2)}$$

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* A non-conducting medium whose properties are same in all directions is called isotropic dielectric.

Now if

(I) We take the curl of equation 2 (c) then

$$\nabla \times (\nabla \times \mathbf{H}) = \varepsilon\, \nabla \times \left(\frac{\partial \mathbf{E}}{\partial t}\right)$$

or
$$\nabla(\nabla \bullet \mathbf{H}) - \nabla^2 \mathbf{H} = \varepsilon \frac{\partial}{\partial t}(\nabla \times \mathbf{E}). \qquad \text{...(3)}$$

But from equations 2 (b) and 2 (d)

$$\nabla \bullet \mathbf{H} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}.$$

So equation (3) reduces to

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

i.e.
$$\nabla^2 \mathbf{H} - \frac{1}{\upsilon^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \text{ with } \mu\varepsilon = 1/\upsilon^2 \qquad \text{...(A)}$$

(II) We take the curl of eqn. 2 (d) then

$$\nabla \times (\nabla \bullet \mathbf{E}) = \nabla \times \left(-\mu \frac{\partial \mathbf{H}}{\partial t}\right)$$

i.e.
$$\nabla(\nabla \bullet \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) \qquad \text{...(4)}$$

But from equations 2 (a) and 2 (c)

$$\nabla \bullet \mathbf{E} = 0 \quad \text{and} \quad \nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}.$$

So equation (4) reduces to

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

i.e.
$$\nabla^2 \mathbf{E} - \frac{1}{\upsilon^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \text{ with } \mu\varepsilon = 1/\upsilon^2 \qquad \text{...(B)}$$

A glance at equation (A) and (B) reveals that these are identical in form to the equation

$$\nabla^2 \psi - \frac{1}{\upsilon^2} \frac{\partial^2 \psi}{\partial t^2} = 0. \qquad \text{...(5)}$$

However equation (5) is a standard wave equation representing an unattenuated wave traveling at a speed υ. So we conclude that field vectors **E** and **H** propagate in isotropic dielectric as waves given by

$$\begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} = \begin{Bmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{Bmatrix} e^{-i(\omega t - \mathbf{k} \bullet \mathbf{r})} \qquad \text{...(C)}$$

$$\upsilon = \frac{1}{\sqrt{(\varepsilon\mu)}} = \frac{1}{\sqrt{(\varepsilon_r \mu_r \varepsilon_0 \mu_0)}} \qquad (\text{as } \varepsilon = \varepsilon_r \varepsilon_0 \text{ and } \mu = \mu_r \mu_0)$$

$$\upsilon = \frac{c}{\sqrt{(\varepsilon_r \mu_r)}} < c \; [\text{as } \varepsilon_0 \mu_0 = 1/c^2 ; \varepsilon_r \text{ and } \mu_r > 1]. \qquad ...(6)$$

The speed of electromagnetic wave in isotropic dielectrics is less than speed of electromagnetic waves in free space.

Further as index of refraction is defined as

$$n = (c/\upsilon)$$

So in this particular case

$$n = \sqrt{(\varepsilon_r \mu_r)} \; [\text{as } \upsilon = c/\sqrt{(\varepsilon_r \mu_r)}]$$

and as in a non-magnetic medium $\mu_r = 1$

$$n = \sqrt{(\varepsilon_r)} \; i.e. \; n = \varepsilon_r \qquad ...(7)$$

Equation (7) is called Maxwell's relation and has been actually confirmed by experiments for long waves *i.e.* radio frequency and slow infrared oscillations. In visible region of the spectrum this relation is also fairly well satisfied for some substances such as H_2, CO_2, N_2 and O_2. But for many other substances it fails, when as a rule the substance shows infrared selective absorption. With water the failure is especially marked. For water $\mu_r \approx 1$, $\varepsilon_r \approx 81$ so that $n \approx 9$. But it is well known that the index of refraction of water for light is very closely given by 4/3 *i.e.* 1.33. The solution of this appearent contradiction lies in the fact that our macroscopic formulation of electromagnetic theory gives no indication of the values to be expected for ε_r and μ_r and we must rely on experiment to obtain them. It turns out that these quantities are not really constant for a given material but usually have a strong dependence on frequency due to dispersion*.

It is also worthy to note here that $\varepsilon_r > 1$ the velocity of light in an isotropic dielectric medium.

$$\upsilon = \frac{c}{n} = \frac{c}{\sqrt{(\varepsilon_r)}}$$

is always less then c as $\varepsilon_r > 1$. ...(8)

It is therefore possible for high energy particles to have velocities in excess of υ. When such particles pass through a dielectric a bluish light known as *Cerenkov-radition* is emitted due to the interaction of uniformly moving charged particles with the medium.

* For details see § 7.6 and 7.7.

Further as the form of field vector **E** and **H** given by equation (C) suggests that

$$\nabla \to i\mathbf{k} \text{ and } \frac{\partial}{\partial t} \to -i\omega$$

So in terms of these operators eqn. (2) reduces to

$$\left.\begin{aligned} \mathbf{k} \bullet \mathbf{E} &= 0 \quad &\ldots(a)\\ \mathbf{k} \bullet \mathbf{H} &= 0 \quad &\ldots(b)\\ -\mathbf{k} \times \mathbf{H} &= \omega\varepsilon\mathbf{E} \quad &\ldots(c)\\ \mathbf{k} \times \mathbf{E} &= \omega\mu\mathbf{H} \quad &\ldots(d)\end{aligned}\right\} \quad \ldots(9)^*$$

From this form of Maxwell's equation it is self evident that in a plane electromagnetic wave propagating through isotopic dielectric—

(*i*) *The vectors* **E**, **H** *and* **k** *are orthogonal i.e. the electromagnetic wave is transverse in nature and in it the electric and magnetic vectors are also mutually orthogonal.* This is because

	according to 9 (a)	**E** is ⊥ to **k**
	according to 9 (b)	**H** is ⊥ to **k**
	according to 9 (c)	**E** is ⊥ to both **k** and **H**
and	according to 9 (d)	**H** is ⊥ to both **k** and **E**

(*ii*) *The vectors* **E** *and* **H** *are in phase and their magnitudes are related to each other by the relation.*

$$\left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \frac{E_0}{H_0} = \sqrt{\left(\frac{\mu_r}{\varepsilon_r}\right)} Z_0 = Z$$

where Z is called the impendence of the medium.

This is because according to equation 9 (d).

$$\mathbf{H} = \frac{k}{\omega\mu}(\mathbf{n} \times \mathbf{E}) = \frac{1}{\mu\upsilon}(\mathbf{n} \times \mathbf{E}) \qquad \left(\text{as } k = \frac{\omega}{\upsilon}\right)$$

i.e.
$$\mathbf{H} = \sqrt{\left(\frac{\varepsilon}{\mu}\right)}(\mathbf{n} \times \mathbf{E}) = \frac{(\mathbf{n} \times \mathbf{E})}{\mathbf{Z}} \qquad \left(\text{as } \upsilon = \frac{1}{\sqrt{(\mu\varepsilon)}}\right)$$

with
$$\mathbf{Z} = \sqrt{\left(\frac{\varepsilon}{\mu}\right)} = \sqrt{\left(\frac{\mu_r}{\varepsilon_r}\frac{\mu_0}{\varepsilon_0}\right)} = \frac{\mu r \mathbf{Z}_0}{n} \qquad \left(n = \sqrt{(\mu_r \varepsilon_r)}\right)$$

or
$$\left|\frac{\mathbf{E}}{\mathbf{H}}\right| = \frac{\mathbf{E}_0}{\mathbf{H}_0} = Z = \text{real quantity.} \qquad \ldots(10)$$

* In this case

$$\mathbf{k} = k\mathbf{n} = \frac{2\pi}{\lambda}\mathbf{n} = \frac{2\pi f}{\upsilon}\mathbf{n} = \frac{\omega}{\upsilon}\mathbf{n}$$

(*iii*) *The direction of flow of energy is the direction in which the wave propagates and the Poynting vector is* (n/μ_r) *times of the Poynting vector if the same wave propagates, through free space.*

It is because

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \mathbf{E} \times \frac{(\mathbf{n} \times \mathbf{E})}{Z}$$

i.e. $$\mathbf{S} = \frac{1}{Z}[(\mathbf{E} \bullet \mathbf{E})\,\mathbf{n} - (\mathbf{E} \bullet \mathbf{n})\,\mathbf{E}]$$

i.e. $$\mathbf{S} = \frac{1}{Z} E^2 \mathbf{n} \; [\text{as } \mathbf{E} \bullet \mathbf{n} = 0 \text{ because } \mathbf{E} \text{ is} \perp \text{to } \mathbf{n}]$$

i.e. $$\mathbf{S} = \frac{1}{Z} E^2 \mathbf{n} = \frac{n}{\mu_r}[\varepsilon_0 c E^2]\,\mathbf{n} \qquad \left(as\ \frac{1}{Z} = \frac{n}{\mu_r}\frac{1}{Z_0} = \frac{n}{\mu_r}\varepsilon_0 c \right)$$

i.e. $$<\mathbf{S}> = \frac{1}{Z} E_{rms}^2\ \mathbf{n} = \frac{n}{\mu_r}[\varepsilon_0 c E_{rms}^2]\,\mathbf{n} \qquad ...(11)$$

(*iv*) *The electromagnetic energy density is equal to the magneto-static energy density and the total energy density is* ε_r *times of the energy density if the same wave propagates through free space.*

This is because

$$\frac{u_e}{u_m} = \frac{\frac{1}{2}\varepsilon E^2}{\frac{1}{2}\mu H^2} = \frac{\varepsilon}{\mu}\left(\frac{E^2}{H^2}\right) = \frac{\varepsilon}{\mu}(Z^2) = \frac{\varepsilon}{\mu} \times \frac{\mu}{\varepsilon} = 1 \quad \left(\text{as } |\mathbf{H}| = \frac{|\mathbf{E}|}{Z} \right)$$

and $$u = u_e + u_m = \varepsilon E^2 = \varepsilon_r\,(\varepsilon_0 E^2)$$

Further $$\frac{<\mathbf{S}>}{<u>} = \frac{\frac{n}{\mu_r}[\varepsilon_0 c E_{rms}^2]}{[\varepsilon_r \varepsilon_0 E_{rms}^2]} = \frac{nc}{\mu_r \varepsilon_r}\mathbf{n}$$

i.e. $$<\mathbf{S}> = \frac{nc}{n^2} <u>\mathbf{n} \qquad [(\text{as } n = \sqrt{(\mu_r \varepsilon_r)}\,]$$

i.e. $$<\mathbf{S}> = \upsilon <u>\mathbf{n} \qquad (\text{as } c/n = \upsilon)$$

i.e. electromagnetic energy is transmitted with the same velocity with which the fields do.

§ 5.3. Propagation of E.M.W. in Anisotropic Dielectric*.

In anisotropic medium the relative permitivity is no longer a scalar and to deal with wave propagation we refer all fields to the principal axes so that

$$D_x = \varepsilon_x \varepsilon_0 E_x \,;\ D_x = E_y \varepsilon_0 E_y \text{ and } D_z = \varepsilon_z \varepsilon_0 E_z \qquad ...(1)$$

Further since the medium is non-conducting *i.e.*

$$\mathbf{J} = 0 \,;\ \rho = 0 \text{ and } \mu_r = 1$$

* A non-conducting medium whose properties depend on direction is called anisotropic dielectric.

So Maxwell's equation in an anisotropic dielectric medium reduce to

$$\left.\begin{aligned} &\text{div } \mathbf{D} = 0 && (a)\\ &\text{div } \mathbf{H} = 0 && (b)\\ &\text{curl } \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} && (c)\\ &\text{curl } \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} && (d) \end{aligned}\right\} \quad ...(2)$$

It is important to note that in this case though

$$\text{div } \mathbf{D} = 0, \ \text{div } \mathbf{E} \neq 0$$

because **D** in general is not in the direction of **E**.

Now consider a plane wave advancing with phase velocity v along the direction of wave normal **n** (*i.e.* wave vector **k**). Let it be

$$\begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} = \begin{Bmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{Bmatrix} e^{-i(\omega t - \mathbf{k} \cdot \mathbf{r})} \quad ...(3)$$

So the operator ∇ and $\frac{\partial}{\partial t}$ will be

$$\nabla \rightarrow i\mathbf{k} \text{ and } \frac{\partial}{\partial t} \rightarrow (-i\omega).$$

And in terms of these operations equations (2) can be written as

$$\left.\begin{aligned} &\mathbf{k} \cdot \mathbf{D} = 0 && ...(a)\\ &\mathbf{k} \cdot \mathbf{H} = 0 && ...(b)\\ &-\mathbf{k} \times \mathbf{H} = \omega \mathbf{D} && ...(c)\\ &\mathbf{k} \times \mathbf{E} = \mu_0 \omega \mathbf{H} && ...(d) \end{aligned}\right\} \quad ...(4)$$

From this form of Maxwell's eqns. it is clear that

(*i*) *The E. M. W. are transverse in nature w.r.t.* **D** *and* **H** (and not w. r. t. **E** and **H** as in a isotropic media). It is because according to 4 (a) **k** is ⊥ to **D** while according to 4 (b) **k** is ⊥ to **H** *i.e.* **k** is ⊥ to both **H** and **D** as shown in fig. 5.3.

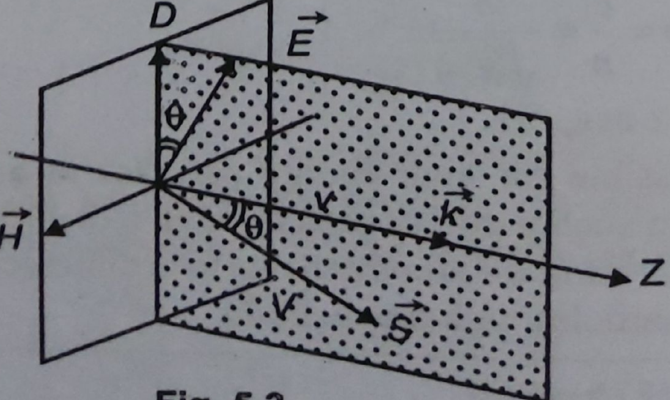


Fig. 5.3

(ii) *The vectors* **D**, **H** *and* **k** *are orthogonal* because according to eqn. 4 (b) **k** is ⊥ to **H** while according to eqn. 4 (c) **D** is ⊥ to both **k** and **H**.

(iii) *The vectors* **D**, **E** *and* **k** *are co-planer.* This is because according to equation 4 (c)

$$\mathbf{D} = -(\mathbf{k} \times \mathbf{H}) / \omega \qquad ...(5)$$

while according to 4 (d)

$$\mathbf{H} = (\mathbf{k} \times \mathbf{E})/\mu_0 \omega \qquad ...(6)$$

So from equations (5) and (6)

$$\mathbf{D} = -[\mathbf{k} \times \mathbf{k} \times \mathbf{E}]/\mu_0\omega^2$$

i.e.

$$\mathbf{D} = -[\mathbf{k} \bullet \mathbf{E})\, \mathbf{k} - k^2 \mathbf{E}]/\mu_0 \omega^2 \qquad ...(7)$$

(iv) *In an anisotropic medium energy is not propagated in general in the direction of wave propagation (i.e. the direction of k and S are not same) and the Poynting vector is coplaner with* **D**, **E** *and* **k**. *This is* because the Poynting vector is given by

$$\mathrm{S} = \mathrm{E} \times \mathrm{H}$$

i.e. **S** is normal to the plane of **E** and **H** and not to the plane of **D** and **H** (which is the direction of **k**).

Example 1. *Show that in case of propagation of plane electromagnetic waves through an anisotropic dielectric*

$$\frac{\cos^2\alpha}{v^2 - v_x^2} + \frac{\cos^2\beta}{v^2 - v_y^2} + \frac{\cos^2\gamma}{v^2 - v_z^2} = 0$$

where v *is the phase velocity of the wave,* α, β, *and* γ *are the angles which the wave vector makes with the principal axes and* $v_x = c/\sqrt{(\varepsilon_x)}$, $v_y = c/\sqrt{\varepsilon_y}$ and $v_z = c/\sqrt{\varepsilon_z}$.

Solution. In case of propagation of plane **E.M.W.** in an anisotropic dielectric we know that

$$\mathbf{D} = -\frac{1}{\mu_0\omega^2}[(\mathbf{k}\bullet\mathbf{E})\,\mathbf{k} - k^2\mathbf{E}] \quad \text{(See eqn. 7 in § 5.3)}$$

i.e.

$$\mathbf{D} = \frac{k^2}{\mu_0\omega^2}[\mathbf{E} - (\mathbf{n}\bullet\mathbf{E})\,\mathbf{n}] \qquad (\text{as } \mathbf{k} = \mathbf{n}\cdot k)$$

$$\mathbf{D} = \frac{1}{\mu_0 v^2}[\mathbf{E} - (\mathbf{n}\bullet\mathbf{E})\,\mathbf{n}] \qquad (\text{as } k = \omega/v) \qquad ...(1)$$

Equation (1) in terms of components can be written as

$$D_x = \frac{1}{\mu_0 v^2}[\mathbf{E} - (\mathbf{n}\bullet\mathbf{E})\,\mathbf{n}]_x$$

i.e. $$D_x = \frac{1}{\mu_0 \nu^2}[\mathbf{E}_x - (\mathbf{n} \bullet \mathbf{E}) \cos\alpha] \qquad ...(2)$$

Similarly

$$D_y = \frac{1}{\mu_0 \nu^2}[\mathbf{E}_y - (\mathbf{n} \bullet \mathbf{E}) \cos\beta] \qquad ...(3)$$

And $$D_z = \frac{1}{\mu_0 \nu^2}[\mathbf{E}_z - (\mathbf{n} \bullet \mathbf{E}) \cos\gamma] \qquad ...(4)$$

Now as in an anisotropic medium

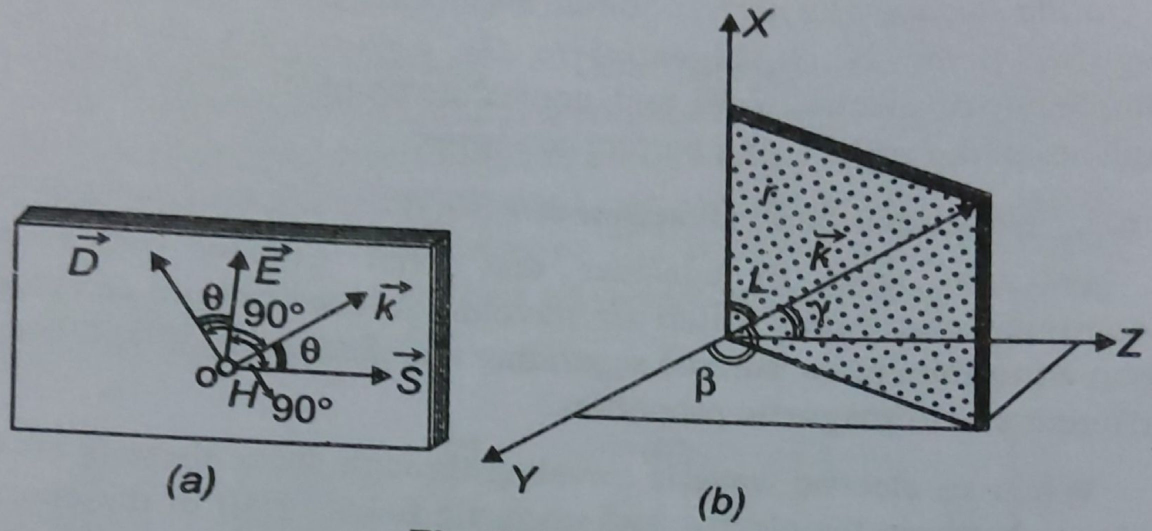


Fig. 5.4

Vector **H** is normal to the plane of the paper and outward.

$$k_x = k\cos\alpha$$
$$k_y = k\cos\beta$$
$$k_z = k\cos\gamma$$

$$D_x = \varepsilon_x \varepsilon_0 E_x \qquad D_y = \varepsilon_y \varepsilon_0 E_y \qquad \text{and} \qquad D_z = \varepsilon_z \varepsilon_0 E_z$$

i.e. $$E_x = \frac{D_x}{\varepsilon_x \varepsilon_0} \qquad D_y = \frac{D_y}{\varepsilon_y \varepsilon_0} \qquad \text{and} \qquad E_z = \frac{D_z}{\varepsilon_z \varepsilon_0}$$

or $$E_x = \frac{c^2 \mu_0}{\varepsilon_x \varepsilon_0} D_x \qquad E_y = \frac{c^2 \mu_0}{\varepsilon_y \varepsilon_0} D_y \qquad \text{and} \qquad E_z = \frac{c^2 \mu_0}{\varepsilon_z \varepsilon_0} D_z$$

(as $1/\varepsilon_0 = \mu_0 c^2$)

or $$E_x = \mu_0 v_x^2 D_x \qquad E_y = \mu_0 v_y^2 D_y \qquad \text{and} \qquad E_z = \mu_0 v_z^2 D_z \qquad ...(5)$$

[as $c/\sqrt{(\varepsilon_x)} = v_x$, $c/\sqrt{(\varepsilon_y)} = v_y$, and $c/\sqrt{(\varepsilon_z)} = v_z$,]

So on substituting the values of E_x, E_y and E_z from equation (5) in 2, 3 and 4 respectively we get

$$D_x = \frac{1}{\mu_0 \nu^2}\left[\mu_0 v_x^2 D_x - (\mathbf{n} \bullet \mathbf{E})\cos\alpha\right]$$

So boundary conditions become

(i) $D_{1n} = \sigma$

(ii) $B_{1n} = 0$

(iii) $H_{1t} = J_s$

(iv) $E_{1t} = 0$

at the surface of a *perfect conductor electric field* **E** *is normal while magnetic fields* **H** *is tangential to the surface. i.e.* the tangential component of electric field and normal component of magnetic field vanishes at the surface of a perfect conductor.

§ 6.2. Reflection and refraction of E.M.W.

we now need to consider that what happens when plane electromagnetic waves which are traveling in one medium are incident upon an infinite plane surface separating this medium from another with different electromagnetic properties.

When an electric wave is traveling through space there is an exact balance between the electric and magnetic fields. Half of the energy of wave as a matter of fact is in electric field and half in the magnetic.* If the wave enters some different medium, there must be a new distribution of energy (due to the change in field vectors). Whether the new medium is a dielectric, a magnetic, a conducting or an ionised region, there will have to be a readjustment of energy relations as the wave reaches its surface. Since no energy can be added to the wave as it passes through the boundary surface, the only way that a new balance can be achieved is for some of the incident energy to be reflected. This is what actually happens. The transmitted energy constitutes the refracted wave and the reflected one the reflected wave.

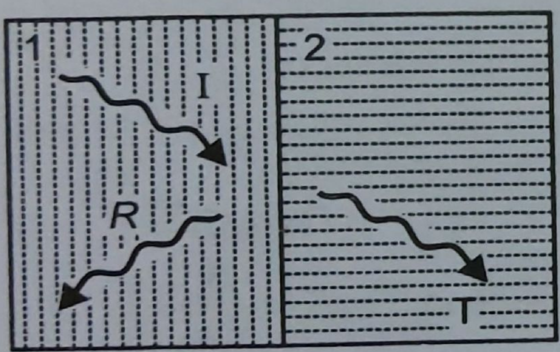


Fig. 6.3

The reflection and refraction of light at a plane surface between two media of different dielectric properties is a familiar example of reflection and refraction of electromagnetic waves. The various aspects of the phenomenon divide themselves into two classes :

* See Art § 5.2.

(A) Kinematic Properties :

Following are the kinematic properties of reflection and refraction :

(*i*) **Law of Frequency :** *The frequency of the wave remains unchanged by reflection or refraction.*

(*ii*) *The reflected and refracted waves are in the same plane as the incident wave and the normal to the boundary surface.*

(*iii*) **Law of Reflection :** *In case of reflection the angle of reflection is equal to the angle of incidence i.e.*

$$\theta_i = \theta_R$$

(*iv*) **Snell's Law :** *In case of refraction the ratio of the sin of the angle of refraction to the sin of angle of incidence is equal to the ratio of the refractive indices of the two media i.e.*

$$n_1 \sin\theta_i = n_2 \sin\theta_\tau$$

(B) Dynamic Properties :

These properties are concerned with the :—

(*i*) *Intensities of reflected and refracted waves.*

(*ii*) *Phase changes and polarisation of waves.*

The kinematic properties follow immediately from the wave nature of phenomenon and the fact that there are boundary condition to be satisfied. But they do not depend on the nature of the waves or the boundary conditions. On the other hand the dynamic properties depend entirely on the specific nature of electromagnetic fields and the boundary conditions. Kinematic properties are proved in example—1 while dynamic properties are discussed in details in forth-coming articles.

Example 1. *Assuming that the electric vector of an electromagnetic wave is given by*

$$\mathbf{E} = \mathbf{E}_0 e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}$$

and in crossing a boundary the tangential component of electric intensity is continuous prove the various laws of reflection and refraction.

Solution. Let the medium below the plane $z = 0$ (*i.e.* *x-y*. plane) have permitivity and permeability ε_1 and μ_1 respectively while above it ε_2 and μ_2. If the plane wave with vector k_i in the *x-z* plane and frequency ω_i is incident from medium - 1 while the waves with wave vector $\mathbf{k}_R$ and $\mathbf{k}_T$ and frequencies ω_R and ω_T are the reflected and transmitted wave, given boundary condition

(ii) *The normal component of* **B** *at the surface of the conducting ne is zero*, since it must be zero within the plane and the normal nponent of **B** is continuous.

In the light of above boundary conditions consider the propagation of ctromagnetic waves in the space between two parallel perfectly nducting planes separated by a distance d. Let the planes be in the x-z ane situated at $y=0$ at $y=d$ as shown in fig. 6.17. If an electromagnetic ve is incident on any plane it will be reflected from the wall whenever it ikes the wall and hence by a process of multiple reflections the wave ll propagate in a direction parallel to planes. Since in fig. 6.17 the x and directions are physically indistinguishable, no generality is lost if we onsider only waves with wave vector $\mathbf{k_0}$ in y-z plane making an angle θ ith y-axis. Such wave will strike on perfectly conducting surface at $=d$ and will be reflected as waves whose propagation vector $\mathbf{k'}$ makes he angle θ with the minus y-axis (as $\theta_i = \theta_r$). When these waves are eflected again by the surface at $y=0$. They become waves of initial type gain and have propagation vector $\mathbf{k_0}$. Thus the propagation between two conducting planes can be described in terms of exponential factors.

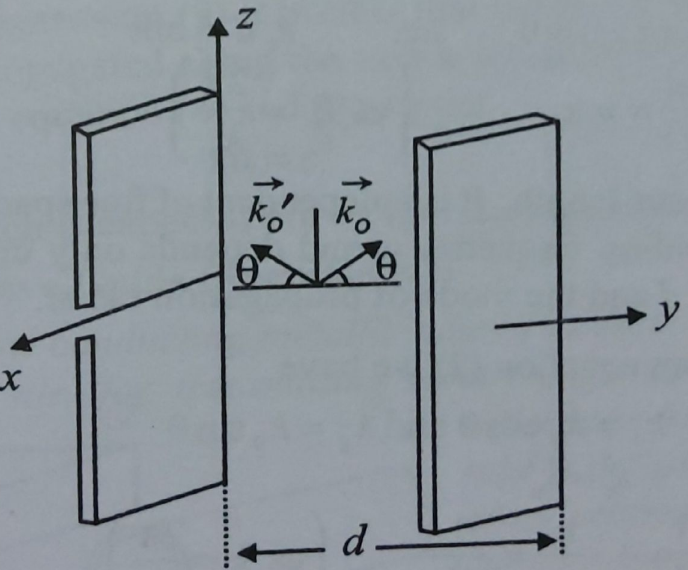


Fig. 6.17

$$e^{-i(\omega t - \mathbf{k}_0 \cdot \mathbf{r})} = e^{-i\omega} e^{-ik_0 (y\cos\theta + z\sin\theta)}$$

[for incident wave]

and
$$e^{-i(\omega t - \mathbf{k'}_0 \cdot \mathbf{r})} = e^{-i\omega} e^{-ik_0 (-y\cos\theta + z\sin\theta)}$$

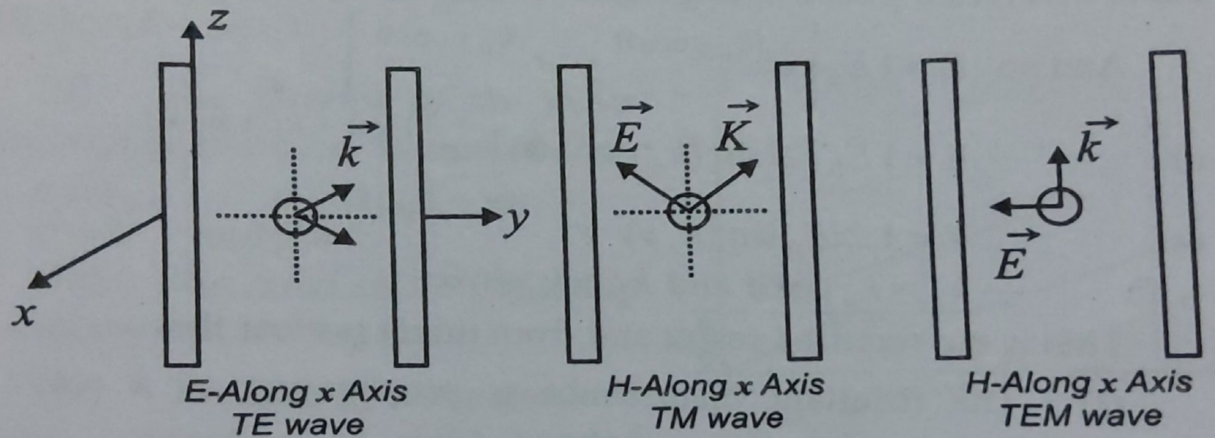
[for reflected wave]

where $|\mathbf{k}_0| = |\mathbf{k'}_0| = |(2\pi/\lambda_0) = (\omega/c)|$ and λ_0 is called the *free space wavelength*.

Now since in an **E.M.W. E, H** and **k** are orthogonal, in general there are three possible modes of propagation *viz.*

(A) TE Waves (or Mode) : This is characterised by an **E.M.W.** having an electric field **E** which is entirely in a plane transverse to the assumed axis of propagation (which is z-axis here). Only the magnetic field **H** has a component along the assumed axis of propagation and hence this type of wave is also known as *H-wave*. This is shown in fig. 6.18 (a). For *TE* wave it is possible to express all field components in terms of the axial magnetic field component H_z.

(B) TM wave (or Mode) : This is characterised by an **E.M.W.** having magnetic field **H** which is entirely in a plane transverse to the assumed axis of propagation (which is z-axis here). Only the electric field **E** has a component along the assumed axis of propagation and hence this type of wave is also known as *E*-wave. This is shown in fig. 6.18 (b). For **TM** wave it is possible to express all field components in terms of axial electric field components E_z.



(a) (b) (c)

Fig. 6.18

(C) TEM wave (or Mode) : It is characterial by an **E.M.W.** having both the electric and magnetic fields entirely in a plane transverse to the assumed axis of propagation *i.e.* it is an electromagnetic wave in which the direction of wave motion is along the assumed axis of propagation. This is shown in fig. 6.18 (c) [In coaxial cables usually **EMW** are propagated in this mode].

As an example here *we shall discuss only TE wave*. The electric fields for incident and reflected waves in *TE* case will be

$$E_i = \mathbf{i}\ E_0 e^{-i\omega t}\ e^{-ik_0\,(y\cos\theta + z\sin\theta)}$$

$$\mathbf{E}_R = \mathbf{i}\ \mathbf{E}'_0\, e^{-i\omega t}\, e^{-i\mathbf{k}_0(-y\cos\theta + z\sin\theta)}$$

So by principle of super position the resultant electric field in the region between the planes in **TE** will be

$$\mathbf{E} = \mathbf{E}_I + \mathbf{E}_R = \mathbf{i}e^{i\omega t}\left[E_0\, e^{ik_0(y\cos\theta + z\sin\theta)} + E'_0\, e^{ik_0(-y\cos\theta + z\sin\theta)}\right]$$

Now as by boundary condition that tangential component of E must vanishes at the surface of the conducting plane *i.e.*

$$\mathbf{E} = 0 \quad \text{at } y = 0$$

We get $\mathbf{i}e^{i\omega t}\left[E_0\, e^{ik_0 z\sin\theta} + E'_0\, e^{ik_0 z\sin\theta)}\right] = 0$

i.e. $\quad E_0 + E' = 0 \quad$ or $\quad E' = -E_0$

This condition simply indicates that *the reflection at the conducting plane involves a phase change of* π *and no change in amplitude*

And so $\quad \mathbf{E} = \mathbf{i}\, E_0\left[e^{ik_0 y\cos\theta} - e^{ik_0 y\cos\theta}\right] e^{-i(\omega t - k_0 z\sin\theta)}$

or $\quad \mathbf{E} = \mathbf{i}\, E_0\left[2i\sin(k_0 y\cos\theta)\right] e^{-i(\omega t - k_0 z\sin\theta)}$

or $\quad \mathbf{E} = \mathbf{i}\ 2iE_0\sin(k_c y)\ e^{-i(\omega t - k_g z)}$ ...(1)

with $\quad k_c = k_0\cos\theta$ and $k_g = k_0\sin\theta$ ...(2)

This is the required result and from this it is clear that

(I) The resultant disturbance is propagating as a wave along z-axis with a wave length

$$\lambda_g = \frac{2\pi}{k_g} = \frac{2\pi}{k_0\sin\theta} \qquad (\text{as } k_g = k_0\sin\theta)$$

$$\lambda_g = \frac{\lambda_0}{\sin\theta} \qquad \left(\text{as } k_0 = \frac{2\pi}{\lambda_0}\right) \qquad ...(3)$$

λ_g is called the guide wavelength and is $> \lambda_0$ as $\sin\theta$ is < 1.

And so the velocity of the wave will be

$$v = \frac{\omega}{k_g} = \frac{\omega}{k_0\sin\theta} \qquad (\text{as } k_g = k_0\sin\theta)$$

$$v = \frac{c}{\sin\theta} \qquad \left(\text{as } k_0 = \frac{\omega}{c}\right) \qquad ...(4)$$

This velocity is called *phase velocity* and is greater than c as $\sin\theta < 1$. At first glance this appears to be in direct conflict with special theory of relatively according to which no signal can be propagated with a speed greater then c. The solution of this apparent difficulty lies in the fact that energy (signal) along the axis is propagated with group velocity υ_z given by eqn. (9) which is lesser than c and not with *phase velocity* $(> c)$.

(II) The amplitude of resultant disturbance

i.e. $2iE_0 \sin(k_c y)$

shows that it varies sinusoidally with y *i.e.*, along y-axis. So the resultant wave given by equation (1) is a stationary wave along y-axis with effective wavelength

$$\lambda_c = \frac{2\pi}{k_c} = \frac{2\pi}{k_0 \cos\theta} \qquad (\text{as } k_c = k_0 \cos\theta)$$

or $$\lambda_c = \frac{\lambda_0}{\cos\theta} \qquad \left(\text{as } k_0 = \frac{2\pi}{\lambda_0}\right) \qquad ...(5)$$

These standing waves were experimentally observed in the experiment of *Lippmann*.

Further the boundary condition $E = 0$ at $y = d$ requires

$$\sin(k_c d) = 0 \qquad i.e. \qquad k_c d = m\pi$$

or $$\frac{2\pi d}{\lambda_c} = m\pi \qquad \left(\text{as } k_c = \frac{2\pi}{\lambda_c}\right) \qquad ...(6)$$

λ_c is called cut of wave length. It is independent of free space wavelength λ_0 or the corresponding frequency ω and depends only on the distance between the planes d and the mode of propagation *i.e.* m.

(III) From equation (2) we have

$$k_c = k_0 \cos\theta \text{ and } k_g = k_0 \sin\theta$$

so $$k_c^2 + k_g^2 = k_0^2 . \qquad ...(7)$$

or $$\frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2} \qquad \left(\text{as } k = \frac{2\pi}{\lambda}\right) \qquad ...(8)$$

Equation (8) is called the guide equation and from this *i.e.*

$$k_g^2 = k_0^2 - k_c^2$$

it is clear that for $k_0 < k_c$, k_g will be imaginary which in turn results in attenuation* of **E**. This in turn means that we cannot propagate wave for which

$$k_0 < k_c \quad i.e. \quad \lambda_0 > \lambda_c$$

* See Appendix III.

i.e. only those waves are propagated for which

$$\lambda_0 < \lambda_c. \quad \text{or} \quad \omega > \omega_c$$

i.e. λ_c is the largest wavelength or ω_c is the lowest frequency which can be propagated. This is why λ_c is called cut of wavelength and the given problem acts as high pass filter.

(IV) The velocity with which energy is propagated along the axis is called group velocity and is given by

$$\upsilon_z = \frac{\partial \omega \dagger}{\partial k_g}$$

But from equation (7)

$$k_0 = \sqrt{(k_c^2 + k_g^2)} \quad \text{or} \quad \omega = c\sqrt{(k_c^2 + k_g^2)} \qquad [\text{as } k_0 = (\omega/c)]$$

or

$$\upsilon_z = \frac{\partial \omega}{\partial k_g} = c\tfrac{1}{2}(k_c^2 + k_g^2)^{-1/2} \times 2k_g$$

i.e.

$$\upsilon_z = c\frac{k_g}{k_0} = c\tfrac{1}{2}\ [\text{as } k_0 = (k_c^2 + k_g^2)^{1/2}]$$

i.e.

$$\upsilon_z = c\sin\theta \qquad [\text{as } k_z = k_0 \sin\theta] \qquad ...(9)$$

From expression (9) it is clear that the group velocity v_z with which energy is propagated along the axis is lesser than c as $\sin\theta < 1$. Further multiplying equation (4) and (9) we get

$$\upsilon\,\upsilon_z = c^2$$

a result which is expected but by no means apparent.

§ 6.8. Wave Guide (Rectangular)

A hollow conducting metallic tube of uniform cross section usually filled with air, for transmitting electromagnetic wave by successive reflections from inner walls of the tube is called a wave guide. If the cross section is rectangular it is called rectangular wave guide and if the cross section is circular it is called cylindrical wave guide.

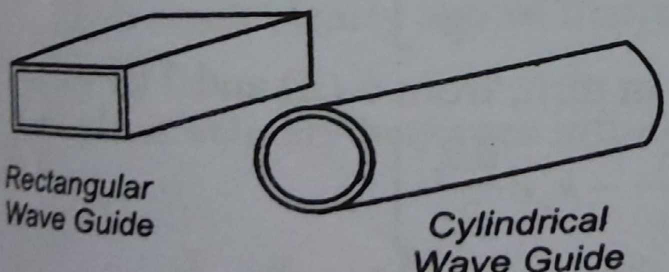


Fig. 6.19

It is used in U.H.F. and microwave region such as radar ($f > 3000$ MHz or $\lambda < 10$cm) as an alternative to transmission lines as at these frequencies it can handle more power with lesser losses as compared to transmission lines.

Propagation of **E.M.W.** in wave guides can be considered as a phenomenon in which either *TE* or *TM* waves are reflected from wall to wall and hence pass down the wave guide in zig-zag fashion. [in transmission lines **E.M.W.** are usually propagated along the axis of cable as *TEM* waves.]

As essential feature of wave guide propagation is that it exhibits a cut off characteristic frequency similar to that of a high pass filter. At frequencies below the cut off value, the wave is simply reflected backwards and forwards across the wave guide and makes no forward progress. [Transmission line do not have any cut off frequency and are broad band devies.]

Theory :

For making the treatment simple we assume that

(i) *The walls of the guide are perfectly conducting* so that tangential component of **E** and normal component of **B** vanishes at its surface.

(ii) *The interior of the wave guide is free space i.e.* vacuum so that

$\varepsilon = \varepsilon_0$, $\mu = \mu_0$,
$\sigma = 0$ and $\rho = 0$.

(iii) *The cross section of guide is uniform* and rectangular.

(iv) *The axis of wave guide is along z*-direction of right handed co-ordinate system.

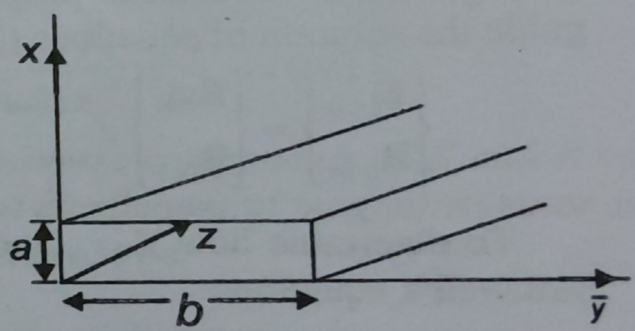


Fig. 6.20

In the light of above assumptions to discuss the propagation of **E.M.W.** in the guide consider Maxwell's eqns. in free space viz.

$$\left.\begin{array}{llll} \text{Div } \mathbf{E} = 0 & \text{.....(a)} & \text{Div } \mathbf{B} = 0 & \text{.....(b)} \\ \text{Curl } \mathbf{B} = \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t} & \text{.....(c)} & \text{Curl } \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \text{.....(d)} \end{array}\right] \quad \text{...(1)}$$

Taking the curl of eqn. 1 (d) we get

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \text{ curl } (\mathbf{B})$$

or
$$\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B})$$

$$[\text{as } \nabla \times \nabla \times \mathbf{V} = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}]$$

which in the light of equations 1 (a) and (c) reduces to

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad ...(2)$$

Similarly taking curl of eqn. 1 (c) and using 1 (b) and (d) we get

$$\nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \quad ...(3)$$

As equations (2) and (3) are of the form

$$\nabla^2 \psi - \frac{1}{\upsilon^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

We come to the conclusion that *fields* **E** *and* **B** *are propagated as waves in the guide at a speed c.*

Now as the solution of above wave equation when it is propagating along z-axis is

$$\psi = \psi_0 e^{-i\,(\omega t - kz)}$$

so if k_g is the *wave vector or propagation constant* along z-axis *i.e.* axis of guide the solution of equations (2) and (3) will be

$$\begin{Bmatrix} \mathbf{E}_{(r,t)} \\ \mathbf{B}_{(r,t)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{E}_{(xy)} \\ \mathbf{B}_{(xy)} \end{Bmatrix} e^{-i\,(\omega t - k_g z)} \quad ...(4)$$

To determine how $\mathbf{E}_{(x,\,y)}$ and $\mathbf{B}_{(x,\,y)}$ vary with x and y we start with Maxwell's equations

$$\text{curl } \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \quad \text{and} \quad \text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

which in terms of components can be written as

$$\begin{aligned} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} &= \frac{1}{c^2}\frac{\partial E_x}{\partial t} & \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{\partial B_x}{\partial t} \\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} &= \frac{1}{c^2}\frac{\partial E_y}{\partial t} \quad \text{and} & \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{\partial B_y}{\partial t} \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= \frac{1}{c^2}\frac{\partial E_z}{\partial t} & \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{\partial B_z}{\partial t} \end{aligned} \quad ...(5)$$

But from equation (4) it is apparent that

$$\frac{\partial}{\partial z} \to ik_g \quad \text{and} \quad \frac{\partial}{\partial t} \to -i\omega \to -ik_0 c \left[\text{as } k_0 = \frac{\omega}{c}\right]$$

So equation (5), reduces to

$$\left.\begin{aligned} &\frac{\partial B_z}{\partial y} - ik_g B_y = -\frac{ik_0}{c} E_x \quad \ldots\text{(i)} \\ &ik_g B_x - \frac{\partial B_z}{\partial x} = -\frac{ik_0}{c} E_y \quad \ldots\text{(ii)} \\ &\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{ik_0}{c} E_z \quad \ldots\text{(iii)} \end{aligned}\right\} \quad \ldots(6)$$

and

$$\left.\begin{aligned} &\frac{\partial E_z}{\partial y} - ik_g E_y = -ik_0 c B_x \quad \ldots\text{(i)} \\ &ik_g E_x - \frac{\partial E_z}{\partial x} = -ik_0 c B_y \quad \ldots\text{(ii)} \\ &\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -ik_0 c B_z \quad \ldots\text{(iii)} \end{aligned}\right\} \quad \ldots(7)$$

If we substitute the value of B_y from equation 7 (ii) in 6 (i), we get

$$\frac{\partial B_z}{\partial y} - ik_g \left(\frac{k_g}{k_0 c} E_x - \frac{1}{ik_0 c} \frac{\partial E_z}{\partial x} \right) = -\frac{ik_0}{c} E_x$$

i.e.,

$$\frac{\partial B_z}{\partial y} + \frac{k_g}{k_0 c} \frac{\partial E_z}{\partial x} = \left(\frac{ik_g^2}{ck_0} - \frac{ik_0}{c} \right) E_x$$

or

$$E_x = \frac{i}{[k_0^2 - k_g^2]} \left[k_g \frac{\partial E_z}{\partial x} + k_0 c \frac{\partial B_z}{\partial y} \right] \quad \ldots\text{(A)}$$

And if we substitute the value of E_x from 6 (i) 7 (ii), we get

$$B_y = \frac{i}{[k_0^2 - k_g^2]} \left[\frac{k_0}{c} \frac{\partial E_z}{\partial x} + k_g \frac{\partial B_z}{\partial y} \right] \quad \ldots\text{(B)}$$

Similarly eliminating B_x and E_y in turn, from 6 (ii) and 7 (i) we get

$$E_y = \frac{i}{[k_0^2 - k_g^2]} \left[k_g \frac{\partial E_z}{\partial y} - k_0 c \frac{\partial B_z}{\partial x} \right] \quad \ldots\text{(C)}$$

and

$$B_x = \frac{i}{[k_0^2 - k_g^2]} \left[-\frac{k_0}{c} \frac{\partial E_z}{\partial y} + k_g \frac{\partial B_z}{\partial x} \right] \quad \ldots\text{(D)}$$

Examination of equations (A), (B), (C) and (D) shows that :

(*i*) If a electromagnetic wave is to be propagated along z axis then as $E_z = B_z = 0$, the equations (A), (B), (C) and (D) vanish. Therefore there is no non-zero component of **E** or **B**. This in turn implies that *TEM waves cannot be propagated along the axis of a wave guide.*

(ii) If we set $k_0^2 - k_g^2 = k_c^2$ *i.e.*, $k_g{}^2 = k_0^2 - k_c^2 = k_g^2$ we find that for $k_0 < k_c, k_g$ is imaginary which in turn results in the attenuation of **E** and **H** given by eqn. (4). This in turn means that we cannot propagate waves for which $k_0 < k_c$ (or $f_0 < f_c$) *i.e. a guide acts as a short of high pass filter in the sense that one can propagate waves along it whose frequencies are greater than cut off frequency.*

The equation

$$k_0^2 - k_g^2 = k_c^2 \quad i.e. \quad k_0^2 = k_g^2 + k_c^2$$

i.e.

$$\frac{1}{\lambda_0^2} = \frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2} \qquad (\text{as } k = 2\pi/\lambda)$$

is called guide equation. It relates the free space wavelength λ_0 to cut off wavelength λ_c and guide wavelength λ_g. According to it

$$\lambda_g = \frac{\lambda_0}{\sqrt{\left\{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2\right\}}} \qquad \text{...(E)}$$

(iii) The phase velocity in the guide will be given by

$$\upsilon = \frac{\omega}{k_g} = c\frac{k_0}{k_g} \qquad \left[\text{as } k_0 = \frac{\omega}{c}\right]$$

or

$$\upsilon = \frac{c\, k_0}{\sqrt{(k_0^2 - k_c^2)}} = \frac{c}{\sqrt{[1 - (k_c/k_0)^2]}} \qquad [\text{as } k_g^2 = k_0^2 - k_c^2]$$

or

$$\upsilon = \frac{c}{\sqrt{[1 - (\lambda_0/\lambda_c)^2]}} \qquad \left[\text{as } k = \frac{2\pi}{\lambda}\right] \qquad \text{...(F)}$$

This result clearly shows that $\upsilon > c$ and for $\lambda_0 = \lambda_c$

$$\upsilon = \infty$$

i.e. phase velocity becomes infinite exactly at cut off.

(iv) As

$$k_0^2 - k_g^2 = k_c^2 \quad i.e. \quad \omega = c\,(k_g^2 + k_c^2)^{1/2} \qquad [\text{as } k_0 = \omega/c]$$

The group velocity with which energy is propagated along the axis of the guide will be given by

$$\upsilon_z = \frac{\partial w}{\partial k_g} = \frac{\partial}{\partial k_g}[c\,(k_g^2 + k_c^2)^{1/2}]$$

i.e.

$$\upsilon_z = c\tfrac{1}{2}(k_g^2 + k_c^2)^{-1/2}\, 2\, k_g$$

i.e. $$\upsilon_z = c\frac{k_g}{k_0} = c\sqrt{[1-(k_c/k_0)^2]}$$ [as $k_0^2 = k_g^2 + k_c^2$] ...(G)

or $$\upsilon_z = c\sqrt{[1-(\lambda_0/\lambda_c)^2]}$$ [as $k = 2\pi/\lambda$]

From this equation it is clear that $\upsilon_z < c$ and $\upsilon\upsilon_z = c^2$.

(v) Transverse components of the fields *i.e.* E_x, E_y, B_x, and B_y of a guided wave are independent of one another and depend only on the values of the longitudinal components E_z or B_z of the guided wave, so it is possible to express them in terms of a linear superposition of two independent solutions, one for which E_z=0 (*TE*) and one for which $B_z = 0$ (*TE*). Transverse electric waves are sometimes known as *H* wave and transverse magnetic waves as *E*-waves.

TE Waves :

For these as E_z=0 and $k_c^2 = k_0^2 - k_g^2$ equations (A), (B), (C) and (D) reduce to

$$E_x = \frac{ik_0 c}{k_c^2}\frac{\partial B_z}{\partial y} \quad ...(i) \qquad B_x = \frac{ik_g}{k_c^2}\frac{\partial B_z}{\partial x} \quad ...(iii)$$

$$E_y = \frac{ik_0 c}{k_c^2}\frac{\partial B_z}{\partial x} \quad ...(ii) \qquad B_y = \frac{ik_g}{k_c^2}\frac{\partial B_z}{\partial y} \quad ...(iv)$$ ...(8)

Thus in **TE** *mode all the transverse components of E and B can be expressed in terms of longitudinal component of magnetic vector B_z.* In order to compute B_z we use equation (3) *i.e.*

$$\nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = 0$$

which in the light of eqn. (4) *i.e.*

$$\boldsymbol{B}_{(r,t)} = \boldsymbol{B}_{(x,\,y)}\, e^{-i(\omega t - k_g z)}$$

i.e. with $\frac{\partial}{\partial z} \to (ik_g)$ and $\frac{\partial}{\partial t} \to (-i\omega)$ becomes

$$\frac{\partial^2\mathbf{B}}{\partial x^2} + \frac{\partial^2\mathbf{B}}{\partial y^2} + (ik_g)^2\,\mathbf{B} - \frac{1}{c^2}(-i\omega)^2\,\mathbf{B} = 0$$

i.e. $$\frac{\partial^2\mathbf{B}}{\partial y^2} + \frac{\partial^2\mathbf{B}}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k_g^2\right)\mathbf{B} = 0$$

i.e. $$\frac{\partial^2\mathbf{B}}{\partial x^2} + \frac{\partial^2\mathbf{B}}{\partial y^2} + k_c^2\,\mathbf{B} = 0$$ [as $k_0 = \omega/c$ and $k_0^2 = k_g^2 + k_c^2$]

As above equations is a vector equation so must be satisfied for each component of **B**. For z-component of **B** it reduces to

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + k_c^2 B_z = 0 \quad \text{...(a)}$$

with boundary condition $|\partial B_z/\partial n|s = 0$ *i.e.*

$$\frac{\partial B_z}{\partial x} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a.$$

and
$$\frac{\partial B_z}{\partial y} = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b.$$

Such a solution is

$$B_z = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \quad \text{...(H)}$$

with
$$k_c^2 = \pi^2\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right] \quad \text{...(I)}$$

where the indices m and n specify the mode. The cut of wavelength is given by

$$\left(\frac{1}{\lambda_c}\right)_{mn} = \frac{1}{2}\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^{1/2} \qquad \left(\text{as } k = \frac{2\pi}{\lambda}\right)$$

i.e.
$$(\lambda_c)_{mn} = \frac{2}{\sqrt{\left[\frac{m}{a}\right]^2 + \left[\frac{n}{a}\right]^2}} \quad \text{...(J)}$$

while cut off frequency will be

$$\omega_{mn} = \pi c\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^{1/2} \qquad \left[\text{as } \omega \frac{2\pi c}{\lambda}\right] \quad \text{...(K)}$$

The modes corresponding to m and n are designated as TE_{mn} mode. The case $m = n = 0$ gives a static field which do not represent a wave propagation. So TE_{00} mode does not exist. If $a < b$ the lowest cut off frequency result for $m = 0$ and $n = 1$ *i.e.*

$$(\omega)_{01} = \frac{\pi c}{b} \text{ or } k_c = \frac{\pi}{b}$$

The TE_{01} mode is called the principal or dominant mode.

The fields in the guide for TE mode will be obtained from eqn. (8) by substituting the solution for B_z, which is

$$B_{z(r,t)} = B_{z(x,y)}\, e^{-i(\omega t - k_g z)}$$

i.e. $B_{z(r,t)} = B_0 \cos\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{a}\right] e^{-i(\omega t - k_g z)}$

Thus we have

$$E_x = -\frac{in\pi c k_0}{k_c^2 b} B_0 \cos\left[\frac{m\pi x}{a}\right] \sin\frac{n\pi y}{b} e^{-i(\omega t - k_g z)}$$

$$E_y = -\frac{in\pi c k_0}{k_c^2 a} B_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-i(\omega t - k_g z)}$$

$$B_x = -\frac{im\pi k_g}{k_c^2 a} B_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-i(\omega t - k_g z)}$$

$$B_y = -\frac{in\pi k_g}{k_c^2 b} B_0 \cos\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{b}\right] e^{-i(\omega t - k_g z)}$$

TM Waves :

For there as $B_z = 0$ and as $k_0^2 - k_g^2 = k_c^2$ equations A, B, C and D reduce to

$$\left.\begin{array}{llll} E_x = \frac{ik_g}{k_c^2}\frac{\partial E_z}{\partial x} & \ldots(\text{i}) & B_x = -\frac{ik_0}{ck_c^2}\frac{\partial E_z}{\partial y} & \ldots(\text{iii}) \\ E_y = \frac{ik_g}{k_c^2}\frac{\partial E_z}{\partial y} & \ldots(\text{ii}) & B_y = \frac{ik_0}{ck_c^2}\frac{\partial E_z}{\partial x} & \ldots(\text{iv}) \end{array}\right\} \quad \ldots(10)$$

Thus in *TM mode, all the transverse components of* **E** *and* **B** *can be expressed in terms of longitudinal component of the electric field* E_z, E_z may be computed by using the eqn. (4) for z-component

i.e. $$E_{z(r,t)} = E_{z(x,y)} e^{-i(\omega t - k_g z)}$$

so that it satisfies eqn. (2) (for z component) *i.e.*

$$\nabla^2 E_z - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 0$$

i.e. $$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + (ik_g)^2 E_z \frac{(-i\omega)^2}{c^2} E_z = 0$$

or $$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_c^2 E_z = 0 \qquad \left(\text{as } \frac{\omega^2}{c^2} - k_g^2 = k_0^2 - k_g^2 = k_c^2\right)$$

with boundary condition $E_z/s = 0$ *i.e.*

	$E_z = 0$	at	$x = 0$	and	$x = a$
and	$E_z = 0$	at	$y = 0$	and	$y = b$

Such a solution is

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad ...(L)$$

with

$$k_c^2 = \pi^2\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right] \qquad ...(M)$$

which corresponds to a cut off wavelength

$$\left(\frac{1}{\lambda_c}\right)_{mn} = \frac{1}{2}\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]^{1/2} \qquad [\text{as } k = 2\pi/\lambda]$$

and a cut off frequency

$$\omega_{mn} = \pi c\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]^{1/2} \qquad [\text{as } k = \omega/c]$$

Comparing eqn. (M) with (I) we find that in a rectangular waveguide *TE* and *TM* modes have the same set of cut off frequencies. However the cases $m=0$ and $n=1$ or $m=1$ and $n=0$ which were dominant in *TE* mode do not exist for *TM* wave because the field vanishes or m or $n=0$.

The value of the fields for *TM* mode will be obtained from eqn. (10) by substituting the solution for E_z, which is

$$E_{z(r,t)} = E_{z(x,y)}\, e^{-i(\omega t - k_g z)}$$

i.e.

$$E_{z(r,t)} = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) e^{-i(\omega t - k_g z)}$$

Thus we have

$$E_x = \frac{im\pi k_g}{k_c^2 a} E_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) e^{-i(\omega t - k_g z)}$$

$$E_y = \frac{in\pi k_g}{k_c^2 b} E_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) e^{-i(\omega t - k_g z)}$$

$$B_x = \frac{in\pi k_0}{bck_c^2} E_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) e^{-i(\omega t - k_g z)}$$

and

$$B_y = \frac{im\pi k_0}{ack_c^2} \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) e^{-i(\omega t - k_g z)}$$

Note : In solving numericals related to wave guides keep in mind that

(*a*) *The cut off wavelength* λ_c for a given mode and *free space wavelength* λ_0 *are given by*

$$\lambda_c = \frac{2}{\sqrt{\left\{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right\}}} \qquad \text{and} \qquad \lambda_0 = \frac{c}{f}$$

(*b*) *The guide wavelength*

$$\lambda_g = \frac{\lambda_0}{\sqrt{\left\{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2\right\}}}$$

(*c*) *The propagation constant*

$$k_g = \frac{2\pi}{\lambda_g}$$

(*d*) *The phase velocity*

$$\upsilon = \frac{c}{\sqrt{\left\{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2\right\}}}$$

(*e*) *The group velocity*

$$\upsilon_z = c\sqrt{\left\{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2\right\}}$$

(*f*) *The guide impedence for TE mode*

$$Z_E = \frac{Z_0}{\sqrt{\left\{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2\right\}}}$$

while for **TM** mode

$$Z_M = Z_0\sqrt{\left\{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2\right\}}$$

where $Z_0 = 377$ ohm≡free space impedance.

Example 6. *What must be the width of a rectangular guide such that the energy of electromagnetic radiations whose free space wavelength is 3 cm. travels down the guide at 95% of the speed of light, in principal mode.*

Solution. We know that in a wave guide energy travels with group velocity

$$v_z = c\sqrt{[1-(\lambda_0/\lambda_c)^2]}$$

but here

$$v_z = \cdot 9\ c$$

and for principal mode

$$(\lambda_c)_{01} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}} = \frac{2}{\sqrt{\left(\frac{0}{a}\right)^2+\left(\frac{1}{b}\right)^2}} = 2b$$

so $\cdot 95\ c = c\sqrt{[1-(3/2b)^2]}$

or $b = 4.8$ cm. **Ans.**

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Example 7. *Find (a) what transmission modes are possible at an operating frequency of 3 GH_z in a hollow rectangular wave guide of inner dimension 3.44×7.22 cm and (b) the corresponding values of phase constant, phase velocity and group velocity.*

Solution. (a) In this problem

$$\lambda_0 = (c/f) = (3\times10^{10}/3\times10^9) = 10 \text{ cm}$$

Now as

$$[\lambda_c]_{mn} = \frac{2}{\sqrt{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}}$$

a particular mode can be transmitted if its corresponding λ_c is greater than λ_0 *i.e.* 10 cm.

(1) For TE_{00} mode $m=0$, $n=0$ and there will be no propagation.

(2) For $TE_{1,0}$ mode $m=1$, $n=0$ and so

$$\lambda_c = 2a = 2\times3.44 = 6.88 \text{ cms.}$$

Thus $(\lambda_c)_{10} < \lambda$, hence $TE_{1,0}$ mode does not exist.

(3) For TE_{01} mode $m=0$, $n=1$ and so

$$\lambda_c = 2b = 2\times7.22 = 14.44 \text{ cm.}$$

Thus $(\lambda_c)_{01} < \lambda_0$, hence T_{01} mode will propagate.

Now as we will see higher TE modes does not propagate because in every case $(\lambda_c)_{mn} > \lambda_0$.

Also since for a TM wave to propagate the lowest mode is TM_{11} mode for which $\lambda_c < \lambda_0$ *i.e.* TM mode will not exist in this guide at all.

From above it is clear that at 3 GH_z frequency given guide can propagate only T_{01} mode.

(b) As guide wavelength

$$\lambda_g = \frac{\lambda_0}{\sqrt{[1-(\lambda_0/\lambda_c)^2]}} = \frac{10}{\sqrt{[1-(10/14\bullet4)^2]}} = \frac{10}{.7} = 14\bullet3 \text{ cm}$$

phase constant

$$k_g = \frac{2\pi}{\lambda_g} = \frac{2\times3\cdot14}{14\cdot3} = 0\cdot45 \text{ cm}^{-1}$$

phase velocity

$$v = \frac{c}{\sqrt{[1-(\lambda_0/\lambda_c)^2]}} = \frac{3\times10^8}{\cdot7} = 43\times10^8 \text{ m/s}$$

group velocity

$$v_z = c\sqrt{[1-(10/14.4)^2]} = 3\times10^8\times\cdot7 = 2.1\times10^8 \text{ m/s}.$$

§ 6.9. Cavity Resonator :

A cavity resonator is an energy storing device, similar to a resonant circuit at low frequencies. Virtually any metallic enclosure, when properly excited will function as a cavity resonator or electromagnetic cavity. For certain specific frequencies electromagnetic field oscillations can be sustained within the enclosure with a very small expenditure of power loss in the cavity walls. Cavity resonators have the advantages of reasonable dimensions, simplicity, remarkable high Q and very high impendence.

A cavity resonator is usually superior to conventional **L-C** circuit by a factor of about 20. *i.e* the fraction of the stored energy dissipated per cycle in a cavity resonator is about (1/20) the fraction dissipated per cycle in an L-C circuit. An additional advantage is that cavity resonators of practical size have resonant frequencies which range upward from a few hundred maga cycles just the region where it is almost impossible to construct a L-C circuit.

Cavity resonators are used as resonant circuit in high frequency tubes such as Klystron, for band pass filters and for wave meters to measure frequency.

Theory : Consider a rectangular cavity as shown in fig. 6.21, with the assumptions.

(i) The walls are perfectly conducting.
(ii) The interior of cavity is free-space.
(iii) The cavity is rectangular.
(iv) The wave is advancing along z-axis.

As there are two possible modes of propagation *TE* or *TM* in the cavity, we shall deal them separately.

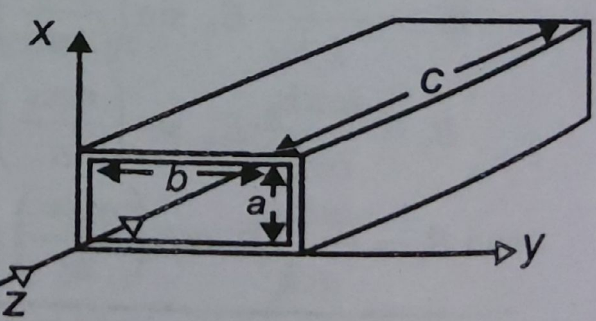

Fig. 6.21

Case I. TE Mode. In this mode $E_z = 0$ so that the electric field propagating along +ive z-direction may be expressed as

$$\mathbf{E}_{i(r,t)} = \mathbf{E}_{(x,y)} \; e^{-i(\omega t - k_g z)}$$

The electric field of reflected wave propagating along z-axis will therefore be

$$\mathbf{E}_{r(r,t)} = \mathbf{E}_{(x,y)} \; e^{-i(\omega t - k_g z)}$$

So the resultant electric field

$$\mathbf{E}_{(r,t)} = \mathbf{E}'_{(x,\, y)}\, e^{-i(\omega t - k_g z)} + \mathbf{E}'_{(x,\, y)}\, e^{-i(\omega t + k_g z)}$$

The boundary condition that tangential component of E is zero at the boundary $z = 0$ (for all values of x, y and t) requires

$$\mathbf{E} + \mathbf{E}' = 0 \qquad i.e. \qquad \mathbf{E}' = -\mathbf{E}$$

so that

$$\mathbf{E}_{(r,t)} = \boldsymbol{E}_{(x,\, y)}\, e^{-i\omega t}\, [e^{ik_g z} - e^{-ik_g z}]$$

$$\mathbf{E}_{(r,t)} = 2i\, \mathbf{E}_{(x,\, y)} \sin k_g z\; e^{-i\omega t}$$

i.e.

the boundary condition $E_{(r,t)} = 0$ at $z = d$ implies that

$$\sin k_g d = 0 \qquad \text{or} \qquad k_g d = p\pi$$

$$k_g = p\pi/d$$

i.e. ...(1)

so that

$$E_{(r,t)} = 2i\, E_{(x,\, y)} \sin\left(\frac{p\pi z}{d}\right) e^{-i\omega t}$$

which in terms of components will be

$$\left.\begin{aligned} E_{x(r,t)} &= 2i\, E_{x(x,\, y)} \sin\left(\frac{p\pi z}{d}\right) e^{-i\omega t} \\ \text{and} \quad E_{y(r,t)} &= 2i\, E_{y(x,\, y)} \sin\left(\frac{p\pi z}{d}\right) e^{-i\omega t} \end{aligned}\right\} \qquad \text{...(2)}$$

In order to calculate E_x and E_y we write Maxwell's equations curl $\mathbf{B} = (1/c^2)\, (\partial \mathbf{E}/\partial t)$ and curl $\mathbf{E} = -(\partial \mathbf{B}/\partial t)$ in terms of components and solve to get

$$E_x = \frac{ik_0 c}{kc^2}\frac{\partial B_z}{\partial y} \quad \text{and} \quad E_y = -\frac{ik_0 c}{kc^2}\frac{\partial B_z}{\partial x}^{*} \qquad \text{...(3)}$$

Now B_z will be obtained by solving the z-component of wave equation for $\mathbf{B}$ *i.e.*

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) B_z = 0$$

i.e.

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \frac{\partial^2 B_z}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2} = 0$$

But as for a wave propagating along z-axis

$$(\partial/\partial z) \to ik_g \qquad \text{and} \qquad (\partial/\partial t) \to (-i\omega)$$

* Equations 8 (i) and 8 (ii) of 6.1.

so $$\frac{\partial^2 B_z}{\partial x^2}+\frac{\partial^2 B_z}{\partial y^2}+\left[\frac{\omega^2}{c^2}-k_g{}^2\right]B_z=0$$

or $$\frac{\partial^2 B_z}{\partial x^2}+\frac{\partial^2 B_z}{\partial y^2}+k_c{}^2 B_z=0$$

[with $\omega/c=k_0$ and $k_0{}^2-k_g{}^2=k_c{}^2$] ...(4)

The boundary condition $|\partial B/\partial n|s=0$ *i.e.*

$$\frac{\partial B_z}{\partial x}=0 \quad \text{at} \quad x=0 \quad \text{and} \quad x=a$$

and $$\frac{\partial B_z}{\partial y}=0 \quad \text{at} \quad y=0 \quad \text{and} \quad y=b$$

when applied to equation (4) yields

$$B_z=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \quad ...(5)$$

with $$k_c^2=\pi^2\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right] \quad ...(6)$$

So substituting the value of B_z from (5) in (3) we get

$$E_{x(x,\,y)}=-\frac{ik_0c}{k_c{}^2}\left(\frac{n\pi}{b}\right)B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$E_{y(x,\,y)}=\frac{ik_0c}{k_c{}^2}\left(\frac{m\pi}{a}\right)B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

The above equation when substituted in eqns. (2) results

$$E_{x(r,\,t)}=\frac{2k_0c}{k_c{}^2}\left(\frac{n\pi}{b}\right)B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \quad ...(A)$$

$$E_{y(r,\,t)}=\frac{2k_0c}{k_c{}^2}\left(\frac{m\pi}{a}\right)B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \quad ...(B)$$

with $E_{z(r,\,t)}=0$ as wave is *TE* ...(C)

The components of magnetic field in this will be obtained by using the Maxwell's curl $\mathbf{E}=(-\partial\mathbf{B}/\partial t)$ in terms of components *i.e.*

$$\left.\begin{aligned}
&\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{\partial B_x}{\partial t} && \frac{-\partial E_y}{\partial z}=i\omega\, B_x\\
&\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\frac{\partial B_y}{\partial t} \quad \text{or} && \frac{\partial E_x}{\partial z}=i\omega\, B_y\\
\text{and}\ &\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\frac{\partial B_z}{\partial t} && \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega\, B_z
\end{aligned}\right] \quad ...(7)$$

[as $E_z=0$ *and* $(\partial/\partial t)\rightarrow -i\omega$]

So equation (11) in the light of **(H)** and **(I)** and with $\pi p/d = k_g$ yields

$$B_x = \frac{2i\omega}{k_c^2}\frac{E_0}{c^2}\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \quad \text{...(J)}$$

$$B_y = -\frac{2i\omega}{k_c^2}\frac{E_0}{c^2}\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \quad \text{.......(K)}$$

and $$E_z = 2E_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \quad \text{...(L)}$$

Equations **(H)** to **(L)** represents the components of field vectors and from these it is evident that modes TM_{000}, TM_{001}, TM_{100}, TM_{010}, TM_{011}, TM_{101} do not exist. The physically possible lowest mode is TM_{110}.

The resonant frequency will be given by the condition

$$k_0^2 = k_g^2 + k_c^2$$

i.e. $$\omega = \pi c\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2\right]^{\frac{1}{2}}$$

QUESTIONS AND PROBLEMS

Fresnel Formulae :

1. Determine the boundary conditions satisfied by electromagnetic fields at the interface between two media of different permeabilities and permitivities.

[**Hint :** See § 6.1].

2. (a) A plane electromagnetic wave is incident normally at the boundary of two non-conducting media. Discuss the phenomenon of reflection and refraction.

[**Hint :** See § 6.6 to prove equation's (4) and (5). Note also that same results are obtained by using equation's (A′) and (B′) or (C′) and (D′) with $\theta_i = \theta_R = \theta_T = 0$ in § 6.3].

(b) Prove that for glass-air interface ($n_2 = 1{\cdot}5$ and $n_1 = 1{\cdot}0$) for normal incidence the reflection and transmission coefficient are $R_n = 0{\cdot}04$ and $T_n = 0{\cdot}96$.

[**Hint :** $R = \dfrac{S_R \cos\theta_R}{S_i \cos\theta_i} = \dfrac{S_R}{S_T}$ (as $\theta_i = \theta_r = 0$)

i.e. $$R = \left(\frac{E_R}{E_i}\right)^2 \qquad \left(\text{as } S = \frac{n}{\mu_r Z_0}E^2 = \frac{n}{Z_0}E^2\right)$$

$$\therefore\ R = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 = \left(\frac{1{\cdot}5 - 1}{1{\cdot}5 + 1}\right)^2 = {\cdot}04 \left(\text{as } \frac{E_R}{E_i} = \frac{n_2 - n_1}{n_2 + n_1}\right)$$