The fourth term is a similar relativistic correction to the potential energy, classical analogue and the last term is the **spin orbit coupling energy** which appears as an automatic consequence of the Dirac equation.

Thus the spin-orbit coupling energy is

$$U_{s-0} = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\,\mathbf{S}\cdot\mathbf{L}, \qquad \text{...(13)}$$

where **S** is spin angular momentum and **L** is orbital angular momentum.

14.10 ZITTERBEWEGUNG

Let us consider the motion of an electron according to Dirac equation in Heisenberg representation where the Hamiltonian becomes time dependent.

The relativistic linearised Hamiltonian in electromagnetic field described by **vector and scalar potentials ϕ and A** is

$$H = \vec{\alpha} \cdot (c\mathbf{p} - e\mathbf{A}) + \beta mc^2 + e\phi \quad \text{...(1)}$$

Now according to Heisenberg representation, the equation of motion for operator x

$$\dot{x} = \frac{1}{i\hbar}[x, H] = \frac{1}{i\hbar}[x, \vec{\alpha} \cdot (c\,\mathbf{p} - e\mathbf{A}) + \beta mc^2 + e\phi] \quad \text{...(2)}$$

Omitting the terms which commute with x, we get

$$\dot{x} = \frac{1}{i\hbar}[x, \alpha_x cp_x] = \frac{c}{i\hbar}\alpha_x [x, p_x] = \frac{c}{i\hbar}\alpha_x i\hbar = c\alpha_x. \quad \text{...(3)}$$

In general the velocity operator $\dot{\mathbf{r}} = \mathbf{v}$ is given by

$$\mathbf{v} = c\vec{\alpha} \quad \text{...(4)}$$

The probability density function according to Dirac equation is $\psi^\dagger \psi$; thereby giving momentum density for Dirac particle as $\psi^\dagger \mathbf{p}\, \psi$ and the *velocity density* for Dirac particle appears to be

$$\psi^\dagger \mathbf{v}\, \psi = \psi^\dagger (c\vec{\alpha})\, \psi = c\,\psi^\dagger \cdot \vec{\alpha}\psi$$

The eigen-values of each α are ± 1 therefore the observed value of any component of velocity is $\pm c$ this is peculiar result, since according to relativistic mechanics speed of light is the upper limit for the speed of material particle.

To find the significance of the this result let us investigate the motion of an electron under no field (*i.e.* $\mathbf{A} = 0$ and $\phi = 0$). The Hamiltonian then is expressed as

$$H = c\vec{\alpha} \cdot \mathbf{p} + \beta mc^2. \quad \text{...(5)}$$

Writting the equation of motion for operator α_x, we have

$$\dot{\alpha}_x = \frac{1}{i\hbar}[\alpha_x, H] = \frac{1}{i\hbar}(\alpha_x H - H\alpha_x) \quad \text{...(6)}$$

But
$$\alpha_x H + H\alpha_x = \alpha_x (c\,\vec{\alpha} \cdot \mathbf{p} + \beta mc^2) + (c\,\vec{\alpha} \cdot \mathbf{p} + \beta mc^2)\alpha_x$$
$$= c\alpha_x (\alpha_x p_x + \alpha_y p_z + \alpha_z p_z + \beta mc) + c(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta mc)\alpha_x = 2cp_x.$$

$\therefore$
$$H\alpha_x = 2cp_x - \alpha_x H. \quad \text{...(7)}$$

In view of this equation (6) gives

$$\dot{\alpha}_x = \frac{1}{i\hbar}(2\alpha_x H - 2cp_x) \quad \text{...(8)}$$

As for a free particle the energy and momentum are conserved, we have

$$i\hbar \dot{p}_x = [p_x, H] = 0 \text{ and } i\hbar \dot{H} = [H, H] = 0$$

therefore H and p_x are independent of time. Keeping this in mind, the differential of equation (8) with respect to time gives

$$\ddot{\alpha}_x = \frac{1}{i\hbar}(2\,\dot{\alpha}_x H) \quad \text{...}$$

This may be expressed as

$$\frac{\ddot{\alpha}_x}{\dot{\alpha}_x} = -\frac{2i}{\hbar}H$$

Integrating w.r. to time t, we get

$$\log(\dot{\alpha}_x) = -\frac{2i}{\hbar}Ht + K \qquad \text{...(10)}$$

K being a constant of integration.

If at $t = 0$, $\dot{\alpha}_x = (\dot{\alpha}_x)_{t=0}$; we have

$$K = \log(\dot{\alpha}_x)_{t=0}$$

$\therefore$ Equation (10) gives

$$\log \dot{\alpha}_x = -\frac{2i}{\hbar}Ht + \log(\dot{\alpha}_x)_{t=0}$$

i.e.

$$\log\left\{\frac{\dot{\alpha}_x}{(\dot{\alpha}_x)_{t=0}}\right\} = -\frac{2i}{\hbar}Ht$$

or

$$\dot{\alpha}_x = (\dot{\alpha}_x)_{t=0}\exp\left\{\left(-\frac{2i}{\hbar}\right)Ht\right\} \qquad \text{...(11}$$

Substituting this in (8), we get

$$(\dot{\alpha}_x)_{t=0}\exp\left(-\frac{2i}{\hbar}Ht\right) = \frac{1}{i\hbar}(2\alpha_x H - 2cp_x) \qquad \text{...(12}$$

We have $H^2 = p^2c^2 + m^2c^4 = E^2$, therefore

$$H = (p^2c^2 + m^2c^4)H^{-1}$$

This implies that H is the reciprocal of H^{-1} with eigen value E^{-1}

From equation (12), we have

$$\alpha_x = cp_x H^{-1} + \tfrac{1}{2}i\hbar(\dot{\alpha}_x)_{t=0}\exp\left(-\frac{2i}{\hbar}Ht\right)H^{-1} \qquad \text{...(1}$$

Therefore

$$\dot{x} = c\,\alpha_x = c^2 p_x H^{-1} + \tfrac{1}{2}i\hbar c(\dot{\alpha}_x)_{t=0}\exp\left(-\frac{2i}{\hbar}Ht\right)H^{-1} \qquad \text{...(}$$

Integration of above equation gives

$$x = c^2 p_x H^{-1}t - \tfrac{1}{4}c\hbar^2(\dot{\alpha}_x)_{t=0}\exp\left(-\frac{2i}{\hbar}Ht\right)H^{-2} + x_0 \qquad \text{...(}$$

when x_0 is constant of integration

Since

$$\dot{\alpha}_x^{\dagger} = \left\{\frac{1}{i\hbar}[\alpha_x, H]\right\}^{\dagger} = -\frac{1}{i\hbar}[H, \alpha_x] = \frac{1}{i\hbar}[\alpha_x, H] = \dot{\alpha}_x$$

Also p_x and H are Hermitian, therefore right hand side of (15) is Hermitian if we ignore the con of integration which would obviously represent the initial position of particle. Then x would b observable quantity and its expectation value is given by

$$\langle x \rangle = \frac{c^2 p_x t}{E} - \frac{1}{4}\frac{c\hbar^2}{E^2}\langle(\dot{\alpha}_x)_{t=0}\rangle\exp\left(-\frac{2i}{\hbar}Et\right)$$

where $\langle(\dot{\alpha}_x)_{t=0}\rangle$ is the eigen value of α_x at $t = 0$.

Now we have

$$E = (p^2c^2 + m^2c^4)^{1/2} \approx mc^2$$

and writing $\frac{2E}{\hbar} = \omega$, equation (16) may be expressed as

$$<x> = \frac{c^2 p_x t}{E} - \frac{1}{4}\frac{c\hbar^2}{E^2} < (\dot{\alpha}_x)_{t=0} > e^{i\omega t} \quad ...(17)$$

In this equation the first term represents usual term

$$\left(\frac{c^2 p_x t}{E} \approx \frac{c^2 p_x t}{mc^2} = \frac{mv_x t}{m} = \right) v_x t$$

of classical mechanics and the second term because of the exponential factor represents the motion of particle, oscillating with angular frequency ω. This trembling motion of the electron was first observed by Schroedinger and is called the *Zitterbewgung* and it imparts the value to the velocity of electron. That is the electron's motion is something like the superposition of classical motion and electromagnetic wave motion.

However the frequency

$$\omega = \frac{2E}{\hbar} \geq \frac{2mc^2}{\hbar}$$

is so high that the departure from the classical mechanics term $v_x t$ is undetectable. The Zitterbewgung did not appear in our nonrelativistic theory, the reason that this phenomenon is due to the rest energy of the electron which remains unaccounted in classical mechanics.

14.11 DIRAC'S EQUATION OF A CENTRAL FIELD FORCE (H-ATOM)

Hence Dirac's theory automatically endows the electron ... phenomenon previously ascribed to a hypothetically spinning motion of the electron.

14.9 SPIN-ORBIT ENERGY

The spin orbit coupling energy follows as a result to Dirac equation in a central field. The term is however of order v^2/c^2 and in order to obtain a consistent approximation we proceed by two-component reduction of Dirac equation in the central field $V(r)$.

The Dirac equation for the central field is

$$[c\,\vec{\alpha}\cdot\mathbf{p} + \beta mc^2 + V(r)]\,\psi = E\,\psi.$$

Writing $\psi = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}$ which represent the first and the last two components of ψ respectively.

i.e.
$$\left\{c\begin{bmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{bmatrix}\cdot\mathbf{p}+\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}mc^2+V(r)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}=E\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}$$

i.e.
$$\begin{bmatrix}c & \vec{\sigma}\cdot\mathbf{p} & \psi_2\\ c & \vec{\sigma}\cdot\mathbf{p} & \psi_1\end{bmatrix}+\begin{bmatrix}mc^2\,\psi_1\\ -mc^2\,\psi_2\end{bmatrix}+\begin{bmatrix}V\psi_1\\ V\psi_2\end{bmatrix}=\begin{bmatrix}E\psi_1\\ E\psi_2\end{bmatrix}$$

This equation is equivalent to following two equations

$$c\,\vec{\sigma}\cdot\mathbf{p}\,\psi_2+mc^2\,\psi_1+V\psi_1=E\,\psi_1$$

and
$$c\,\vec{\sigma}\cdot\mathbf{p}\,\psi_1-mc^2\,\psi_2+V\psi_2=E\,\psi_2$$

or
$$\left.\begin{aligned}(E-V-mc^2)\,\psi_1-c\,\vec{\sigma}\cdot\mathbf{p}\,\psi_2&=0\\ (E-V+mc^2)\,\psi_2-c\,\vec{\sigma}\cdot\mathbf{p}\,\psi_1&=0\end{aligned}\right\}\qquad ...(1)$$

Assuming that ψ_1 and ψ_2 together constitute a non-relativistic energy eigen-function, which means that

$$E = E' + mc^2$$

is regarded as a number rather than an operator; the non-relativistic energy E' and V are assumed to be smaller in comparison with mc^2.

The wave equations (1) then become

$$\left.\begin{aligned}\{E'-V(r)\}\,\psi_1-c\,\vec{\sigma}\cdot\mathbf{p}\,\psi_2&=0 \quad ...(a)\\ \{E'+2mc^2-V(r)\}\,\psi_2-c\,\vec{\sigma}\cdot\mathbf{p}\,\psi_1&=0 \quad ...(b)\end{aligned}\right\}\qquad ...(3)$$

Using the identity

we have

Also if u is any functi

i.e.

or

∴

∴

∴ $(\vec{\sigma}\cdot\mathbf{p})\frac{(E'-V}{4m^2c^2}$

Using (6)

(E

From (3b), we have

$$\psi_2 = \frac{c\,\vec{\sigma}\cdot\mathbf{p}}{E' + 2mc^2 - V(r)}\,\psi_1 \qquad \text{...(4)}$$

Substituting this value of ψ_2 in (3a), we get

$$[E' - V(r)]\,\psi_1 = c^2(\vec{\sigma}\cdot\mathbf{p})\,[E' + 2mc^2 - V(r)]^{-1}\,\vec{\sigma}\cdot\mathbf{p}\,\psi_1$$

$$= \frac{\vec{\sigma}\cdot\mathbf{p}}{2m}\left[1 + \frac{E' - V(r)}{2mc^2}\right]^{-1}\vec{\sigma}\cdot\mathbf{p}\,\psi_1$$

$$= \frac{\vec{\sigma}\cdot\mathbf{p}}{2m}\left[1 - \frac{E' - V}{2mc^2}\right]\vec{\sigma}\cdot\mathbf{p}\,\psi_1$$

$$= \frac{1}{2m}(\vec{\sigma}\cdot\mathbf{p})(\vec{\sigma}\cdot\mathbf{p})\,\psi_1 - (\vec{\sigma}\cdot\mathbf{p})\frac{(E' - V)}{4m^2c^2}\,\vec{\sigma}\cdot\mathbf{p}\,\psi_1 \qquad \text{...(5)}$$

Using the identity

$$(\vec{\sigma}\cdot\mathbf{B})(\vec{\sigma}\cdot\mathbf{C}) = \mathbf{B}\cdot\mathbf{C} + i\,\vec{\sigma}\cdot\mathbf{B}\times\mathbf{C}$$

we have

$$(\vec{\sigma}\cdot\mathbf{p})(\vec{\sigma}\cdot\mathbf{p}) = \mathbf{p}\cdot\mathbf{p} + i\,\vec{\sigma}\cdot\mathbf{p}\times\mathbf{p} = \mathrm{p}^2 \qquad \text{...(6)}$$

Also if u is any function

$$[\mathbf{p}, V]\,u = (\mathbf{p}\,V - V\,\mathbf{p})\,u = \mathbf{p}\,V u - V\,\mathbf{p}\,u$$

$$= \frac{\hbar}{i}\nabla(Vu) - V\frac{\hbar}{i}\nabla u$$

$$= \frac{\hbar}{i}[V\nabla u + u\nabla V] - V\frac{\hbar}{i}\nabla u$$

$$= \frac{\hbar}{i}u\nabla V = \left(\frac{\hbar}{i}\nabla V\right)u$$

i.e. $$(\mathbf{p}\,V - V\,\mathbf{p})\,u = (-i\hbar\nabla V)\,u$$

or $$(\mathbf{p}V - V\mathbf{p}) = -i\hbar\nabla V \qquad \text{...(7)}$$

$\therefore$ $$\mathbf{p}\,V = V\,\mathbf{p} - i\hbar\nabla V \qquad \text{...(8)}$$

$\therefore$ $$(\vec{\sigma}\cdot\mathbf{p})\,V = V(\vec{\sigma}\cdot\mathbf{p}) - \vec{\sigma}\cdot i\hbar\nabla V$$

$\therefore$ $$(\vec{\sigma}\cdot\mathbf{p})\frac{(E' - V)}{4m^2c^2}\,\vec{\sigma}\cdot\mathbf{p} = \frac{E'}{4m^2c^2}(\vec{\sigma}\cdot\mathbf{p})(\vec{\sigma}\cdot\mathbf{p}) - \frac{1}{4m^2c^2}(\vec{\sigma}\cdot\mathbf{p})\,V(\vec{\sigma}\cdot\mathbf{p})$$

$$= \frac{E'}{4m^2c^2}(\vec{\sigma}\cdot\mathbf{p})(\vec{\sigma}\cdot\mathbf{p}) - \frac{1}{4m^2c^2}\{V(\vec{\sigma}\cdot\mathbf{p})(\vec{\sigma}\cdot\mathbf{p}) - \vec{\sigma}\cdot(i\hbar\nabla V)(\vec{\sigma}\cdot\mathbf{p})\}$$

$$= \frac{E'}{4m^2c^2}p^2 - \frac{1}{4m^2c^2}Vp^2 + \frac{i\hbar}{4m^2c^2}(\vec{\sigma}\cdot\nabla V)(\vec{\sigma}\cdot\mathbf{p})$$

$$= \left(\frac{E' - V}{4m^2c^2}\right)p^2 + \frac{i\hbar}{4m^2c^2}\{\nabla V\cdot\mathbf{p} + i\,\vec{\sigma}\cdot\nabla V\times\mathbf{p}\} \qquad \text{...(}$$

Using (6) and (9), equation (5) gives

$$(E' - V)\,\psi_1 = \frac{1}{2m}p^2\,\psi_1 - \left\{\left(\frac{E' - V}{4m^2c^2}\right)p^2 + \frac{i\hbar}{4m^2c^2}(\nabla V\cdot\mathbf{p} + i\,\vec{\sigma}\cdot\nabla V\times\mathbf{p})\right\}\psi_1.$$