$$y_{n+1} = y_n + \frac{1}{2} h \, [f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n))]$$

$$\therefore \quad y_1 = y_0 + \frac{1}{2} h \, [f(x_0, y_0) + f(x_1, y_0 + h f(x_0, y_0))] \quad \text{...(1)}$$

$$= 0 + \frac{0{\cdot}2}{2} [y_0 + e^{x_0} + y_0 + h\,(y_0 + e^{x_0}) + e^{x_0 + h}]$$

$$= (0{\cdot}1) \, [0 + 1 + 0 + 0{\cdot}2 \, (0 + 1) + e^{0{\cdot}2}]$$

$$y\,(0{\cdot}2) = (0{\cdot}1) \, [1 + 0{\cdot}2 + 1{\cdot}2214] = \mathbf{0{\cdot}24214}$$

$$y_2 = y_1 + \frac{1}{2} h \, [f(x_1, y_1) + f(x_1 + h, y_1 + h f(x_1, y_1))] \quad \text{...(2)}$$

Here $f(x_1, y_1) = y_1 + e^{x_1} = 0{\cdot}24214 + e^{0{\cdot}2} = 1{\cdot}46354$

$$y_1 + h f(x_1, y_1) = 0{\cdot}24214 + (0{\cdot}2)(1{\cdot}46354) = 0{\cdot}53485$$

$$f(x_1 + h, y_1 + h f(x_1, y_1)) = f(0{\cdot}4, 0{\cdot}53485)$$

$$= 0{\cdot}53485 + e^{0{\cdot}4}$$

$$= 2{\cdot}02667$$

using (2),

$$y_2 = y\,(0{\cdot}4) = 0{\cdot}24214 + (0{\cdot}1) \, [1{\cdot}46354 + 2{\cdot}02667]$$

$$= 0{\cdot}59116$$

$$y\,(0{\cdot}4) = \mathbf{0{\cdot}59116}$$

Example 4. *Compute y at $x = 0{\cdot}25$ by Modified Euler method given $y' = 2xy$, $y\,(0) = 1$.* *(BR. Nov. 1995)*

Solution. Here, $f(x, y) = 2xy$: $x_0 = 0$, $y_0 = 1$.

Take $h = 0{\cdot}25$, $x_1 = 0{\cdot}25$

By Modified Euler method,

$$y_{n+1} = y_n + h \left[f\left(x_n + \frac{1}{2} h, \; y_n + \frac{1}{2} h f(x_n, y_n) \right) \right] \quad \text{...(1)}$$

$$\therefore \quad y_1 = y_0 + h \left[f\left(x_0 + \frac{1}{2} h, \; y_0 + \frac{1}{2} h f(x_0, y_0) \right) \right]$$

$$f(x_0, y_0) = f(0, 1) = 2(0)(1) = 0.$$

$$\therefore \quad y_1 = 1 + (0{\cdot}25) \, [f(0{\cdot}125, 1)]$$

$$= 1 + (0{\cdot}25) \, [2 \times 0{\cdot}125 \times 1]$$

$$y\,(0{\cdot}25) = \mathbf{1{\cdot}0625}$$

By solving $\dfrac{dy}{dx} = 2xy$, we get $y = e^{x^2}$ using $y\,(0) = 1$,

$$y\,(0{\cdot}25) = e^{(0{\cdot}25)^2} = 1{\cdot}0645$$

Exact value of $y\,(0{\cdot}25) = \mathbf{1{\cdot}0645}$

Error is only $0{\cdot}002$.

Note: To improve the result we can take $h = 0{\cdot}125$ and get $y\,(0{\cdot}125)$ first and then get $y\,(0{\cdot}25)$. Of course, labour is more.

Example 5. *Solve the equation* $\frac{dy}{dx} = 1 - y$, *given* $y(0) = 0$ *using* ***Modified Euler's method*** *and tabulate the solutions at* $x = 0.1$, 0.2, *and* 0.3. *Compare your results with the* ***exact solutions.*** Also, get the solutions by ***Improved Euler method.*** *(Anna Ap. 2005)* *(B.R. Nov. 1991)*

Solution. Here, $x_0 = 0$, $y_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $h = 0.1$

$y' = 1 - y \quad \therefore \quad f(x, y) = 1 - y; \quad f(x_0, y_0) = 1 - y_0 = 1$

By *Modified Euler method,*

$$y_{n+1} = y_n + hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right) \qquad ...(1)$$

$$\therefore \quad y_1 = y_0 + hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hf(x_0, y_0)\right) \qquad ...(2)$$

$$x_0 + \frac{1}{2}h = \frac{0.1}{2} = 0.05$$

$$y_0 + \frac{1}{2}hf(x_0, y_0) = 0 + \frac{0.1}{2}[1] = 0.05$$

using (2),

$$\therefore \quad y_1 = 0 + 0.1\,[f(0.05, 0.05)] = (0.1)(1 - 0.05)$$

$$y_1 = y(0.1) = \mathbf{0.095}$$

$$f(x_1, y_1) = 1 - y_1 = 0.905;$$

$$\therefore \quad y_2 = y_1 + hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}hf(x_1, y_1)\right)$$

$$= 0.095 + (0.1)\,[f(0.15, 0.14025)]$$

$$= 0.095 + (0.1)\,[1 - 0.14025]$$

$$y(0.2) = \mathbf{0.18098}$$

$$y_3 = y_2 + hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}hf(x_2, y_2)\right) \qquad ...(3)$$

$$x_2 + \frac{1}{2}h = 0.25$$

$$y_2 + \frac{1}{2}hf(x_2, y_2) = 0.18098 + (0.05)\,[1 - 0.18098]$$

$$= 0.22193$$

using (3), we get

$$y(0.3) = y_3 = 0.18098 + (0.1)\,[1 - 0.22193] = \mathbf{0.258787}.$$

Exact solution: $\frac{dy}{dx} = 1 - y$ gives $\frac{dy}{1 - y} = dx$

$$\therefore \quad -\log(1 - y) = x + c$$

$$\log(1 - y) = -x - c$$

$\therefore \quad 1 - y = e^{-x} A$

At $x = 0, y = 0 \quad \therefore \quad A = 1 \quad \therefore \quad y = 1 - e^{-x}$

using this exact solution,

$$y(0.1) = 1 - e^{-0.1} = 0.09516258$$
$$y(0.2) = 1 - e^{-0.2} = 0.181269247 \quad \ldots(4)$$
$$y(0.3) = 1 - e^{-0.3} = 0.259181779$$

By Improved Euler method,

$$y_{n+1} = y_n + \frac{1}{2} h [f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n))]$$

$$\therefore \quad y_1 = y_0 + \frac{1}{2} h [f(x_0, y_0) + f(x_1, y_0 + h f(x_0, y_0))] \quad \ldots(5)$$

$$f(x_0, y_0) = 1 - y = 1 - 0 = 1$$
$$f(x_1, y_0 + h f(x_0, y_0)) = f(0.1, 0.1) = 1 - 0.1 = 0.9 \quad \ldots(6)$$

using in (6),

$$y_1 = y(0.1) = 0 + \frac{0.1}{2} [1 + 0.9] = \frac{0.19}{2} = \mathbf{0.095}$$

$$y_2 = y_1 + \frac{1}{2} h [f(x_1, y_1) + f(x_2, y_1 + h f(x_1, y_1))]$$

$$f(x_1, y_1) = 1 - y_1 = 1 - 0.095 = 0.905 \quad \ldots(7)$$
$$f(x_2, y_1 + h f(x_1, y_1)) = f(0.2, 0.095 + (0.1)(0.905)) = 0.8145$$

using in (7), we get

$$y_2 = y(0.2) = 0.095 + \frac{0.1}{2} [0.905 + 0.8145]$$

$$y(0.2) = \mathbf{0.18098}$$

$$y_3 = y_2 + \frac{1}{2} h [f(x_2, y_2) + f(x_3, y_2 + h f(x_2, y_2))]$$

$$f(x_2, y_2) = 1 - y_2 = 1 - 0.18098 = 0.81902 \quad \ldots(8)$$

$$y_2 + h f(x_2, y_2) = 0.18098 + (0.1)(0.81902) = 0.26288$$

using in (8),

$$y_3 = y(0.3) = 0.18098 + \frac{0.1}{2} [0.81902 + 1 - 0.26288]$$

$$y(0.3) = \mathbf{0.258787}$$

The values are tabulated.

x	*Modified Euler*	*Improved Euler*	*Exact solution*
0.1	0.095	0.095	0.09516
0.2	0.18098	0.18098	0.18127
0.3	0.258787	0.258787	0.25918

Modified Euler and Improved Euler methods give the same values correct to six decimal places.

Example 6. *Given $y' = x^2 - y$, $y(0) = 1$, find correct to four decimal places the value of $y(0{\cdot}1)$, by using Improved Euler method.*

Solution. By improved Euler method,

$$y_{n+1} = y_n + \frac{1}{2} h\,[f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n))]$$

$$\therefore \quad y_1 = y_0 + \frac{1}{2} h\,[f(x_0, y_0) + f(x_1, y_0 + h f(x_0, y_0))]$$

$$x_0 = 0,\ y_0 = 1,\ x_1 = 0{\cdot}1$$

$$f(x_0, y_0) = x_0^2 - y_0 = 0 - 1 = -1$$

$$y_0 + h f(x_0, y_0) = 1 + (0{\cdot}1)(-1) = 1 - 0{\cdot}1 = 0{\cdot}9$$

$$f(x_1, y_0 + hf(x_0, y_0)) = x_1^2 - 0{\cdot}9 = (0{\cdot}1)^2 - (0{\cdot}9) = -0{\cdot}89$$

$$\therefore \quad y_1 = 1 + \frac{0{\cdot}1}{2}[-1 + (-0{\cdot}89)]$$

$$y(0{\cdot}1) = 1 - \frac{0{\cdot}1}{2} \times 1{\cdot}89 = \mathbf{0{\cdot}9055}.$$

Example 7. *Using improved Euler method find y at $x = 0{\cdot}1$ and y at $x = 0{\cdot}2$ given*

$$\frac{dy}{dx} = y - \frac{2x}{y},\quad y(0) = 1. \qquad \textit{(MS. Ap. 1991)}$$

Solution. By Improved Euler method,

$$y_{n+1} = y_n + \frac{1}{2} h\,[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))] \qquad \text{...(1)}$$

$$\therefore \quad y_1 = y_0 + \frac{1}{2} h\,[f(x_0, y_0) + f(x_1, y_0 + h f(x_0, y_0))] \qquad \text{...(2)}$$

$$f(x_0, y_0) = y_0 - \frac{2x_0}{y_0} = 1 - 0 = 1$$

$$f(x_1, y_0 + hf(x_0, y_0)) = f(0{\cdot}1, 1{\cdot}1) = 1{\cdot}1 - \frac{2 \times (0{\cdot}1)}{1{\cdot}1} = 0{\cdot}91818$$

$$y(0{\cdot}1) = y_1 = 1 + \frac{0{\cdot}1}{2}[1 + 0{\cdot}91818] = \mathbf{1{\cdot}095909}$$

$$y_2 = y(0{\cdot}2) = y_1 + \frac{1}{2} h\,[f(x_1, y_1) + f(x_2, y_1 + h f(x_1, y_1))] \qquad \text{...(3)}$$

$$f(x_1, y_1) = y_1 - \frac{2x_1}{y_1} = 1{\cdot}095909 - \frac{2 \times 0{\cdot}1}{1{\cdot}095909}$$

$$= 0{\cdot}913412$$

$$f(x_2, y_1 + h f(x_1, y_1)) = f(0{\cdot}2, 1{\cdot}095909 + (0{\cdot}1)(0{\cdot}913412))$$

$$= f(0.2, 1.18732) = 1.18732 - \frac{2 \times 0.2}{1.18732} = 0.8504268$$

using in (3),

$$y_2 = 1.095909 + \frac{0.1}{2}[0.913412 + 0.850427]$$

$$= \mathbf{1.1841009}$$

x	0	0·1	0·2
y	1	1·095907	1·1841009

Example 8. *Using Modified Euler method, find y (0·2), y (0·1) given*

$$\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1. \qquad (MS.\ Ap.\ '92)$$

Solution. Here, $x_0 = 0, y_0 = 1, h = 0.1, x_1 = 0.1, f(x, y) = x^2 + y^2$

By Modified Euler method,

$$y_1 = y_0 + hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}hf(x_0, y_0)\right) \qquad ...(1)$$

$$y_0 + \frac{1}{2}hf(x_0, y_0) = y_0 + \frac{1}{2}h\left(x_0^2 + y_0^2\right)$$

$$= 1 + \frac{0.1}{2}(0 + 1) = 1.05$$

using in (1)

$$y_1 = 1 + (0.1)[f(0.05, 1.05)]$$

$$y(0.1) = 1 + (0.1)[(0.05)^2 + (1.05)^2] = \mathbf{1.1105}$$

$$y_2 = y_1 + hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}hf(x_1, y_1)\right)$$

$$f(x_1, y_1) = f(0.1, 1.1105) = (0.1)^2 + (1.1105)^2 = 1.24321$$

$$y_1 + \frac{1}{2}hf(x_1, y_1) = 1.1105 + (0.05)(1.24321) = 1.172660$$

$$\therefore\ y_2 = 1.1105 + (0.1)[f(0.15, 1.172660)]$$

$$= 1.1105 + (0.1)[(0.15)^2 + (1.17266)^2]$$

$y(0.2) = \mathbf{1.25026}$.

EXERCISE 11.3

1. Compute y (0·3) taking $h = 0.1$ given $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$ using improved Euler method.

2. Find $y(0{\cdot}6)$, $y(0{\cdot}8)$, $y(1)$ given $\frac{dy}{dx} = x + y$, $y(0) = 0$ taking $h = 0{\cdot}2$ by improved Euler method.
3. Using Improved Euler method find
$$y(0{\cdot}2),\ y(0{\cdot}4) \text{ given } \frac{dy}{dx} = y + x^2,\ y(0) = 1.$$
4. Use Euler's method to find $y(0{\cdot}4)$ given $y' = xy$, $y(0) = 1$.
5. Use Improved Euler method to find $y(0{\cdot}1)$ given $y' = \frac{y-x}{y+x}$, $y(0) = 1$.
6. Use Modified Euler method and obtain $y(0{\cdot}2)$ given
$\frac{dy}{dx} = \log(x + y)$, $y(0) = 1$, $h = 0{\cdot}2$.
7. Using Modified Euler method, get $y(0{\cdot}2)$, $y(0{\cdot}4)$, $y(0{\cdot}6)$ given
$\frac{dy}{dx} = y - x^2$, $y(0) = 1$.
8. Using Euler's Improved method, find $y(0{\cdot}2)$, $y(0{\cdot}4)$ given
$$\frac{dy}{dx} = x + |\sqrt{y}|,\ y(0) = 1.$$
9. Find $y(0{\cdot}1)$ given $y' = x^2 + y$, $y(0) = 1$ using Improved Euler method.

Using Euler's method do the problems (10-11):

10. Find $y(1{\cdot}5)$ taking $h = 0{\cdot}5$ given
$$y' = y - 1,\ y(0) = 1{\cdot}1.$$
11. If $y' = 1 + y^2$, $y(0) = 1$, $h = 0{\cdot}1$, find $y(0{\cdot}4)$.
12. Use Euler's improved method to calculate $y(0{\cdot}5)$, taking $h = 0{\cdot}1$, and $y' = y + \sin x$, $y(0) = 2$.
13. Find $y(1{\cdot}6)$ if $y' = x\log y - y\log x$, $y(1) = 1$ if $h = 0{\cdot}1$.
14. Find by Improved Euler to get $y(1{\cdot}2)$, $y(1{\cdot}4)$ given $\frac{dy}{dx} = \frac{2y}{x} + x^3$ if $y(1) = 0{\cdot}5$.
15. Use Improved Euler and Modified Euler method, to get $y(1{\cdot}6)$ if
$$\frac{dy}{dx} = y^2 - \frac{y}{x},\ \text{ if } y(1) = 1.$$
16. Solve $y' = 3x^2 + y$ given $y(0) = 4$, if $h = 0{\cdot}25$ to obtain $y(0{\cdot}25)$, $y(0{\cdot}5)$.
17. Given $y' = \frac{y}{x} - \frac{5}{2}x^2 y^3$; $y(1) = \frac{1}{\sqrt{2}}$ find $y(2)$ if $h = 0{\cdot}125$.
18. Find $y(0{\cdot}2)$ by Improved Euler method, given $y' = -xy^2$, $y(0) = 2$ if $h = 0{\cdot}1$.

11·12. Runge-Kutta Method

The use of the previous methods to solve the differential equation numerically is restricted due to either slow convergence or due to labour involved, especially in Taylor-series method. But, in Runge-Kutta methods, the derivatives of higher order are not required and we require only the given function values at different points. Since the derivation of fourth order Runge-Kutta method is tedious, we will derive Runge-Kutta method of second order.

11·13. Second order Runge-Kutta method (for first order O.D.E.)

AIM. To solve $\dfrac{dy}{dx} = f(x, y)$ given $y(x_0) = y_0$. ...(1)

Proof. By Taylor series, we have,

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + O(h^3) \qquad ...(2)$$

Differentiating the equation (1) w.r.t. x,

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = f_x + y' f_y = f_x + ff_y \qquad ...(3)$$

Using the values of y' and y'' got from (1) and (3), in (2), we get,

$$y(x+h) - y(x) = hf + \frac{1}{2} h^2 [f_x + ff_y] + O(h^3)$$

$$\therefore \qquad \Delta y = hf + \frac{1}{2} h^2 (f_x + ff_y) + O(h^3) \qquad ...(4)$$

Let $\Delta_1 y = k_1 = f(x, y) . \Delta x = hf(x, y)$ ...(5)

$\Delta_2 y = k_2 = hf(x + mh, y + mk_1)$ ...(6)

and le $\Delta y = ak_1 + bk_2$ ...(7)

where a, b and m are constants to be determined to get the better accuracy of Δy.

Expand k_2 and Δy in powers of h.

Expanding k_2, by Taylor series for two variables, we have

$$k_2 = hf(x + mh, y + mk_1)$$

$$= h\left[f(x, y) + \left(mh\frac{\partial}{\partial x} + mk_1\frac{\partial}{\partial y}\right)f + \frac{\left(mh\frac{\partial}{\partial x} + mk_1\frac{\partial}{\partial y}\right)^2}{2!} f + \cdots\right]$$

$$= h\left[f + mhf_x + mhff_y + \frac{\left(mh\frac{\partial}{\partial x} + mk_1\frac{\partial}{\partial y}\right)^2 f}{2!} + \cdots\right] \qquad ...(8)$$

since $k_1 = hf$

$$= hf + mh^2 (f_x + ff_y) + \cdots \text{ higher powers of } h \qquad ...(9)$$

Substituting k_1, k_2 in (7),

$$\Delta y = ahf + b\left[hf + mh^2 (f_x + ff_y) + O(h^3)\right]$$

$$= (a + b) hf + bmh^2 (f_x + ff_y) + O(h^3) \qquad ...(10)$$

Equating Δy from (4) and (10), we get

$$= hf + mh^2 (f_x + ff_y) + \cdots \text{ higher powers of } h \quad ...(9)$$

Substituting k_1, k_2 in (7),

$$\Delta y = ahf + b\left[hf + mh^2 (f_x + ff_y) + O(h^3)\right]$$

$$= (a+b)\, hf + bmh^2 (f_x + ff_y) + O(h^3) \quad ...(10)$$

Equating Δy from (4) and (10), we get

$$a + b = 1 \quad \text{and} \quad bm = \frac{1}{2} \quad ...(11)$$

Now we have only two equations given by (1) to solve for three unknowns a, b, m.

From $a + b = 1, a = 1 - b$ and also $m = \frac{1}{2b}$ using (7),

$$\Delta y = (1-b)\, k_1 + bk_2$$

where $k_1 = hf(x, y)$

$$k_2 = hf\left(x + \frac{h}{2b}, y + \frac{hf}{2b}\right)$$

Now $\Delta y = y(x+h) - y(x)$

$$\therefore y(x+h) = y(x) + (1-b)\, hf + bhf\left(x + \frac{h}{2b}, y + \frac{hf}{2b}\right)$$

i.e.,

$$y_{n+1} = y_n + (1-b)\, hf(x_n, y_n) + bhf\left(x_n + \frac{h}{2b}, y_n + \frac{h}{2b} f(x_n, y_n)\right) + O(h^3)$$

From this general second order Runge-Kutta formula, setting $a = 0, b = 1, m = \frac{1}{2}$, we get the second order Runge-Kutta algorithm as

$$k_1 = hf(x, y)$$

$$k_2 = hf\left(x + \frac{1}{2}h, y + \frac{1}{2}k_1\right)$$

and $\Delta y = K_2$ where $h = \Delta x$.

Second order R.K. algorithm

Since the derivations of third and fourth order Runge-Kutta algorithms are tedious, we state them below for use.

The third order Runge-Kutta method algorithm is given below:

$$k_1 = hf(x, y)$$

$$k_2 = hf\left(x + \frac{1}{2}h, y + \frac{1}{2}k_1\right)$$

$$k_3 = hf(x + h, y + 2k_2 - k_1)$$

and $\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$

Third order R.K. algorithm

The fourth order Runge-Kutta method algorithm is mostly used in problems unless otherwise mentioned. It is

$$k_1 = hf(x, y)$$

$$k_2 = hf\left(x + \frac{1}{2}h, y + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x + \frac{1}{2}h, y + \frac{1}{2}k_2\right)$$

$$k_4 = hf(x + h, y + k_3)$$

and $$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y(x + h) = y(x) + \Delta y$$

Fourth order R.K. algorithm

Working Rule: To solve $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$.

Calculate $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

and $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where $\Delta x = h$.

Now $y_1 = y_0 + \Delta y$.

Now starting from (x_1, y_1) and repeating the process, we get (x_2, y_2) etc.

Note 1. In second order Runge-Kutta method,

$$\Delta y_0 = k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right)$$

$$\Delta y_0 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}hf(x_0, y_0)\right)$$

$$\therefore \quad y_1 = y_0 + \Delta y_0 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}hf(x_0, y_0)\right)$$

This is exactly the *Modified Euler method.*

So, the Runge-Kutta method of second order is nothing but the Modified Euler method.

Note 2. If $f(x, y) = f(x)$, *i.e.*, only a function x alone, then the fourth order Runge-Kutta method reduces to

$$k_1 = hf(x_0)$$

$$\Delta y = \frac{1}{6}h\left[f(x_0) + 4f\left(x_0 + \frac{h}{2}\right) + f(x_0 + h)\right]$$

$$= \frac{\left(\frac{h}{2}\right)}{3}\left[f(x_0) + 4f\left(x_0 + \frac{h}{2}\right) + f(x_0 + h)\right]$$

= the area of $y = f(x)$ between $x = x_0$ and $x = x_0 + h$ with 2 equal intervals of length $\frac{h}{2}$ by Simpson's one-third rule.

i.e., Δy reduces to the area by Simpson's one-third rule.

Note 3. In all the three methods. (2nd order, 3rd order and 4th order) the values of k_1, k_2 are same. Therefore, one need not repeat the work while doing by all the three methods.

Example 1. *Apply the fourth order Runge-Kutta method to find $y(0{\cdot}2)$ given that $y' = x + y$, $y(0) = 1$.* *(Ap. 1992)*

Solution. Since h is not mentioned in the question, we take $h = 0{\cdot}1$

$y' = x + y;\ y(0) = 1 \quad \therefore \quad f(x, y) = x + y,\ x_0 = 0,\ y_0 = 1$

$x_1 = 0{\cdot}1,\ x_2 = 0{\cdot}2$

By fourth order Runge-Kutta method, for the first interval,

$k_1 = hf(x_0, y_0) = (0{\cdot}1)(x_0 + y_0) = (0{\cdot}1)(0 + 1) = 0{\cdot}1$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0{\cdot}1)f(0{\cdot}05, 1{\cdot}05)$$

$$= (0{\cdot}1)(0{\cdot}05 + 1{\cdot}05) = 0{\cdot}11$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$= (0{\cdot}1)f(0{\cdot}05, 1{\cdot}055) = (0{\cdot}1)(0{\cdot}05 + 1{\cdot}055) = 0{\cdot}1105$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= (0{\cdot}1)f(0{\cdot}1, 1{\cdot}1105) = (0{\cdot}1)(0{\cdot}1 + 1{\cdot}1105) = 0{\cdot}12105$$

$$\therefore \quad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0{\cdot}1 + 0{\cdot}22 + 0{\cdot}2210 + 0{\cdot}12105) = 0{\cdot}110341667.$$

$$y(0{\cdot}1) = y_1 = y_0 + \Delta y = 1{\cdot}110341667 \approx \mathbf{1{\cdot}110342.}$$

Now starting from (x_1, y_1) we get (x_2, y_2). Again apply Runge-Kutta algorithm replacing (x_0, y_0) by (x_1, y_1).

$k_1 = hf(x_1, y_1) = (0{\cdot}1)(x_1 + y_1) = (0{\cdot}1)(0{\cdot}1 + 1{\cdot}110342) = 0{\cdot}1210342$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_1\right) = (0{\cdot}1)f(0{\cdot}15, 1{\cdot}170859)$$

$$= (0{\cdot}1)(0{\cdot}15 + 1{\cdot}170859) = 0{\cdot}1320859$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_2\right) = (0\cdot1)\,f(0\cdot15, 1\cdot1763848)$$

$$= (0\cdot1)(0\cdot15 + 1\cdot1763848) = 0\cdot13263848$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0\cdot1)\,f(0\cdot2, 1\cdot24298048)$$

$$= 0\cdot144298048$$

$$y(0\cdot2) = y(0\cdot1) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1\cdot110342 + \frac{1}{6}(0\cdot794781008)$$

$y(0\cdot2) = \mathbf{1\cdot2428055}$

Correct to four decimal places, $y(0\cdot2) = \mathbf{1\cdot2428}$.

Example 2. *Obtain the values of y at x = 0·1, 0·2 using R.K. method of (i) second order (ii) third order and (iii) fourth order for the differential equation $y' = -y$, given $y(0) = 1$.* (*MKU 1971*)

Solution. Here, $f(x, y) = -y$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0\cdot1$, $x_2 = 0\cdot2$.

(*i*) Second order:

$$k_1 = hf(x_0, y_0) = (0\cdot1)(-y_0) = -0\cdot1$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0\cdot1)\,f(0\cdot05, 0\cdot95)$$

$$= -0\cdot1 \times 0\cdot95 = -0\cdot095 = \Delta y$$

$$y_1 = y_0 + \Delta y = 1 - 0\cdot095 = 0\cdot905$$

$$y_1 = y(0\cdot1) = \mathbf{0\cdot905}$$

Again starting from (0·1, 0·905) replacing (x_0, y_0) by (x_1, y_1) we get

$$k_1 = (0\cdot1)\,f(x_1, y_1) = (0\cdot1)(-0\cdot905) = -0\cdot0905$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)$$

$$= (0\cdot1)\,[f(0\cdot15, 0\cdot85975)] = (0\cdot1)(-0\cdot85975) = -0\cdot085975$$

$$\Delta y = k_2$$

$$\therefore\ y_2 = y(0\cdot2) = y_1 + \Delta y = \mathbf{0\cdot819025}$$

(*ii*) Third order:

$$k_1 = hf(x_0, y_0) = -0\cdot1$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = -0\cdot095$$

$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$

$$= (0\cdot1)\,f(0\cdot1, 0\cdot9) = (0\cdot1)(-0\cdot9) = -0\cdot09$$

$$\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$y(0\cdot1) = y_1 = y_0 + \Delta y = 1 - 0\cdot09 = \mathbf{0\cdot91}$$

Again taking (x_1, y_1) as (x_0, y_0) repeat the process.

$$k_1 = hf(x_1, y_1) = (0.1)(-0.91) = -0.091$$

$$\therefore \quad k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)$$

$$= (0.1)f(0.15, 0.865) = (0.1)(-0.865) = -0.0865$$

$$k_3 = hf(x_1 + h, y_1 + 2k_2 - k_1)$$

$$= (0.1)f(0.2, 0.828) = -0.0828$$

$$y_2 = y_1 + \Delta y = 0.91 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$= 0.91 + \frac{1}{6}(-0.091 - 0.3460 - 0.0828)$$

$$y(0.2) = \mathbf{0.823366}$$

(iii) Fourth order:

$$k_1 = hf(x_0, y_0) = (0.1)f(0, 1) = -0.1$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1)f(0.05, 0.95) = -0.095$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)f(0.05, 0.9525)$$

$$= -0.09525$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)f(0.1, 0.90475)$$

$$= -\mathbf{0.090475}$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = y_0 + \Delta y = 1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = y(0.1) = \mathbf{0.9048375}$$

Again start from this (x_1, y_1) and replace (x_0, y_0) and repeat

$$k_1 = hf(x_1, y_1) = (0.1)(-y_1) = -0.09048375$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)$$

$$= (0.1)f(0.15, 0.8595956) = -0 \text{ cdot } 08595956$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right)$$

$$= (0.1)f(0.15, 0.8618577) = -0 \text{ cdot } 08618577$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= (0.1)f(0.2, 0.8186517) = -0.08186517$$

$$\Delta y = \frac{1}{6}(-0.09048375 - 2 \times 0.08595956$$

$$- 2 \times 0.08618577 - 0.08186517)$$

$= -0.0861066067$

$y_2 = y(0.2) = y_1 + \Delta y = \mathbf{0.81873089}$

Tabular values are:

x	*Second order*	*Third order*	*Fourth order*	*Exact value* $y = e^{-x}$
0·1	0·905	0·91	0·9048375	0·904837418
0·2	0·819025	0·823366	0·81873089	0·818730753

Fourth order values are more closer to exact values.

Example 3. *Compute* $y(0.3)$ *given* $\frac{dy}{dx} + y + xy^2 = 0$, $y(0) = 1$ *by taking* $h = 0.1$ *using R.K method of fourth order (correct to 4 decimals).*

Solution. $y' = -(xy^2 + y) = f(x, y)$; $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $y_3 = ?$

For 1st interval:

$$k_1 = hf(x_0, y_0) = (0.1)[-(x_0y_0^2 + y_0)] = -0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right) = (0.1)f(0.05, 0.95)$$

$$= -0.1[(0.05)(0.95)^2 + 0.95] = -0.0995$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right) = (0.1)f(0.05, 0.95025)$$

$$= (0.1)[-(0.05 \times 0.95025 + 1)(0.95025)]$$

$$= -0.09953987 \approx -0.0995$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= (0.1)f(0.1, 0.9005) = -0.0982$$

$$y_1 = 1 + \frac{1}{6}[-0.1 + 2(-0.0995) + 2(-0.0995) - 0.0982]$$

$y(0.1) = \mathbf{0.9006}$

Again taking (x_1, y_1) in place of (x_0, y_0) repeat the process.

$$k_1 = hf(x_1, y_1) = (0.1) f(0.1, 0.9006)$$
$$= -0.0982$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2} k_1\right) = (0.1) f(0.15, 0.8515)$$
$$= -0.0960$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2} k_2\right) = (0.1) f(0.15, 0.8526)$$
$$= -0.0962$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) f(0.2, 0.8044)$$
$$= -0.0934$$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.9006 + \frac{1}{6}[-0.0982 + 2 \times (-0.0960) + 2(-0.0962) + (-0.0934)]$$

$y(0.2) = \mathbf{0.8046}$

Again, starting from (x_2, y_2) in place of (x_0, y_0)

$$k_1 = -0.0934, \quad k_2 = -0.0902, \quad k_3 = -0.0904, \quad k_4 = -0.0867$$

$$\therefore \quad y_3 = y_2 + \frac{1}{6}\Delta y = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$y(0.3) = \mathbf{0.7144}$.

Example 4. *Using R.K. method of fourth order, find $y(0.8)$ correct to 4 decimal places if $y' = y - x^2$, $y(0.6) = 1.7379$.* *(April 1991)*

Solution. Here, $x_0 = 0.6$, $y_0 = 1.7379$, $h = 0.1$, $x_1 = 0.7$, $x_2 = 0.8$

$$f(x, y) = y - x^2$$

By R.K. method of 4th order

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \ldots(1)$$

where $k_1 = hf(x_0, y_0) = (0.1) f(0.6, 1.7379)$

$$= (0.1)[1.7379 - (0.6)^2] = 0.1378$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2} k_1\right)$$

$$= (0.1) f(0.65, 1.8068) = (0.1)[1.8068 - (0.65)^2] = 0.1384$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2} k_2\right)$$

$$= (0.1) f(0.65, 1.8071)$$

$$= (0.1)\,[1.8071 - (0.65)^2] = 0.1385$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)\, f(0.7,\ 1.8764)$$

$$= (0.1)\,[(1.8764) - (0.7)^2] = 0.1386$$

Hence, using (1),

$$y(0.7) = y_1 = 1.7379 + \frac{1}{6}[0.1378 + 2\,(0.1384) + 2\,(0.1385) + 0.1386]$$

$$\mathbf{y\,(0.7) = 1.8763.}$$

To find $y_2 = y(0.8)$, we again start from $(x_1, y_1) = (0.7,\ 1.8763)$

Now, $$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \quad \text{...(2)}$$

where $$k_1 = hf(x_1, y_1) = (0.1)\,[1.8763 - (0.7)^2] = 0.1386$$

$$k_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{1}{2}k_1\right) = (0.1)\, f(0.75,\ 1.9456)$$

$$= (0.1)\,[1.9456 - (0.75)^2] = 0.1383$$

$$k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{1}{2}k_2\right)$$

$$= (0.1)\, f(0.75,\ 1.9455)$$

$$= (0.1)\,[1.9455 - (0.75)^2] = 0.1383$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= (0.1)\, f(0.8,\ 2.0146)$$

$$= (0.1)\,[2.0146 - (0.8)^2] = 0.1375$$

Using (2),

$$y_2 = y(0.8) = 1.8763 + \frac{1}{6}[0.1386 + 2\,(1.1383) + 2\,(1.1383) + 0.1375]$$

$$= 2.0145$$

$$\mathbf{y_2 = y\,(0.8) = 2.0145.}$$

Example 5. *Using Runge-Kutta method of fourth order, solve* $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ *given* $y(0) = 1$ *at* $x = 0.2,\ 0.4.$ *(MS. April '92)*

(Anna Ap. 2005)

(Anna Nov. 2004)

Solution. $y' = f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$;

Here $x_0 = 0,\ h = 0.2,\ x_1 = 0.2,\ x_2 = 0.4,\ y_0 = 1$

$$f(x_0, y_0) = f(0, 1) = \frac{1 - 0}{1 + 0} = 1$$

$$k_1 = hf(x_0, y_0) = (0.2) \times 1 = 0.2$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right) = (0.2)\, f(0.1,\ 1.1)$$

$$= (0\cdot2)\left[\frac{(1\cdot1)^2-(0\cdot1)^2}{(1\cdot1)^2+(0\cdot1)^2}\right] = (0\cdot2)\left[\frac{1\cdot21-0\cdot01}{1\cdot21+0\cdot01}\right]$$

$$= 0\cdot1967213$$

$$k_3 = hf\left(x_0+\frac{1}{2}h,\ y_0+\frac{1}{2}k_2\right)$$

$$= (0\cdot2)f\left(0\cdot1,\ 1+\frac{1}{2}(0\cdot1967213)\right)$$

$$= (0\cdot2)f(0\cdot1,\ 1\cdot0983606)$$

$$= (0\cdot2)\left[\frac{(1\cdot0983606)^2-(0\cdot01)}{(1\cdot0983606)^2+(0\cdot01)}\right] = 0\cdot1967$$

$$k_4 = hf(x_0+h,\ y_0+k_3)$$

$$= (0\cdot2)f(0\cdot2,\ 1\cdot1967)$$

$$= (0\cdot2)\left[\frac{(1\cdot1967)^2-(0\cdot2)^2}{(1\cdot1967)^2+(0\cdot2)^2}\right] = 0\cdot1891$$

$$\therefore \quad \Delta y = \frac{1}{6}[k_1+2k_2+2k_3+k_4]$$

$$= \frac{1}{6}[0\cdot2+2\,(0\cdot19672)+2\,(1\cdot1967)+0\cdot1891]$$

$$= 0\cdot19598$$

$$\mathbf{y\,(0\cdot2) = y_1 = y_0 + \Delta y = 1\cdot19598.}$$

Again to find $y\,(0\cdot4)$, start from $(x_1, y_1) = (0\cdot2,\ 1\cdot19598)$.

Now,

$$\therefore \quad k_1 = hf(x_1, y_1) = (0\cdot2)\left[\frac{(1\cdot19598)^2-(0\cdot2)^2}{(1\cdot19598)^2+(0\cdot2)^2}\right] = 0\cdot1891$$

$$k_2 = hf\left(x_1+\frac{1}{2}h,\ y_1+\frac{1}{2}k_1\right) = (0\cdot2)f(0\cdot3,\ 1\cdot29055)$$

$$= (0\cdot2)\left[\frac{(1\cdot29055)^2-(0\cdot3)^2}{(1\cdot29055)^2+(0\cdot3)^2}\right] = 0\cdot17949$$

$$k_3 = (0\cdot2)f(0\cdot3,\ 1\cdot28572) = 0\cdot1793$$

$$k_4 = (0\cdot2)f(0\cdot4,\ y_1+k_3) = (0\cdot2)f(0\cdot4,\ 1\cdot37528)$$

$$= 0\cdot1687$$

$$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= \frac{1}{6}[0\cdot1891+2\,(0\cdot1795)+2\,(0\cdot1793)+0\cdot1687]$$

$$= 0\cdot1792$$

$\therefore\ y_2 = y\,(0{\cdot}4) = y_1 + \Delta y = 1{\cdot}3751.$

11·14. Runge-Kutta method for simultaneous first order differential equations

AIM. To solve numerically the simultaneous equations $\frac{dy}{dx} = f_1\,(x, y, z)$ and $\frac{dz}{dx} = f_2\,(x, y, z)$ given the initial conditions $y\,(x_0) = y_0$, $z\,(x_0) = z_0$.

[Here, x is independent variable while y and z are dependent.]

Now, starting from (x_0, y_0, z_0) the increments Δy and Δz in y and z respectively are given by formulae,

$$k_1 = hf_1\,(x_0, y_0, z_0)$$

$$k_2 = hf_1\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1,\ z_0 + \frac{1}{2}l_1\right)$$

$$k_3 = hf_1\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2,\ z_0 + \frac{1}{2}l_2\right)$$

$$k_4 = hf_1\,(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$\Delta y = \frac{1}{6}\,(k_1 + 2\,k_2 + 2\,k_3 + k_4) \text{ where } h = \Delta x$$

$$l_1 = hf_2\,(x_0, y_0, z_0)$$

$$l_2 = hf_2\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1,\ z_0 + \frac{1}{2}l_1\right)$$

$$l_3 = hf_2\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2,\ z_0 + \frac{1}{2}l_2\right)$$

$$l_4 = hf_2\,(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$\Delta z = \frac{1}{6}\,(l_1 + 2l_2 + 2l_3 + l_4)$$

$$y_1 = y_0 + \Delta y \text{ and } z_1 = z_0 + \Delta z.$$

Having got (x_1, y_1, z_1) we get (x_2, y_2, z_2) by repeating the above algorithm once again starting from (x_1, y_1, z_1).

If we consider the *second order Runge-Kutta method*, then

$$k_1 = hf_1\,(x_0, y_0, z_0)$$

$$k_2 = hf_1\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1,\ z_0 + \frac{1}{2}l_1\right)$$

$$\Delta y = k_2$$

and $$l_1 = hf_2(x_0, y_0, z_0)$$

$$l_2 = hf_2\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1,\ z_0 + \frac{1}{2}l_1\right)$$

$$\Delta z = l_2$$

Then $x_1 = x_0 + h,\ y_1 = y_0 + \Delta y,\ z_1 = z_0 + \Delta z.$

Example 6. *Find* $y(0\cdot 1), z(0\cdot 1)$ *from the system of equations,* $\frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y^2$ *given* $y(0) = 2, z(0) = 1$ *using Runge-Kutta method of fourth order.*

Solution. Now $\frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y^2$

$\therefore\ f_1(x, y, z) = x + z, f_2(x, y, z) = x - y^2$

$x_0 = 0, y_0 = 2, z_0 = 1, h = 0\cdot 1$

We use

$k_1 = hf_1(x_0, y_0, z_0)$

$k_2 = hf_1\left(x_0 + \frac{1}{2}h, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$l_1 = hf_2(x_0, y_0, z_0)$

$l_2 = hf_2\left(x_0 + \frac{1}{2}h, y_0 + \frac{k_1}{2}, z_0 + \frac{1}{2}l_1\right)$

$l_3 = hf_3\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$

$l_4 = hf_3(x_0 + h, y_0 + k_3, z_0 + l_3)$

$\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

We will calculate k_i and l_i and then to k_{i+1}.

$k_1 = (0\cdot 1) f_1(0, 2, 1)$
$= (0\cdot 1)(0 + 1)$
$= 0\cdot 1$

$k_2 = (0\cdot 1) f(0\cdot 05, 2\cdot 05, 0\cdot 8)$
$= (0\cdot 1)(0\cdot 05 + 0\cdot 8)$
$= 0\cdot 085$

$k_3 = (0\cdot 1) f(0\cdot 05, 2\cdot 0425, 0\cdot 79238)$
$= (0\cdot 1)(0\cdot 05 + 0\cdot 79238)$
$= 0\cdot 084238$

$l_1 = (0\cdot 1) f_2(0, 2, 1)$
$= (0\cdot 1)(0 - 2^2)$
$= -0\cdot 4$

$l_2 = (0\cdot 1) f_2(0\cdot 05, 2\cdot 05, 0\cdot 8)$
$= (0\cdot 1)[0\cdot 05 - (2\cdot 05)^2]$
$= -0\cdot 41525$

$l_3 = (0\cdot 1) f(0\cdot 05, 2\cdot 0425, 0\cdot 79238)$
$= (0\cdot 1)[0\cdot 05 - (2\cdot 0425)^2]$
$= -0\cdot 4122$

$k_4 = (0\cdot1) f(0\cdot1, 2\cdot084238, 0\cdot5878);$
$= (0\cdot1)(0\cdot1 + 0\cdot5878)$
$= 0\cdot06878$

$l_4 = (0\cdot1)(0\cdot1 - (2\cdot084238)^2)$
$= -0\cdot4244$

$$y_1 = 2 + \frac{1}{6}[0\cdot1 + 2(0\cdot085 + 0\cdot084238) + 0\cdot06878] = 2\cdot0845$$

$$z_1 = 1 + \frac{1}{6}[-0\cdot4 - (0\cdot41525 + 0\cdot4122) \times 2 - 0\cdot4244]$$
$$= 0\cdot5868$$

$y(0\cdot1) = 2\cdot0845$ and $z(0\cdot1) = 0\cdot5868$.

11·15. Runge-Kutta method for second order differential equation

AIM. *To solve $y'' = f(x, y, y')$, given $y(x_0) = y_0$, $y'(x_0) = y_0'$.*

Now, set $y' = z$ and $y'' = z'$

Hence, differential equation reduces to

$$\frac{dy}{dx} = y' = z$$

and $$\frac{dz}{dx} = z' = y'' = f(x, y, y') = f(x, y, z)$$

$\therefore \quad \dfrac{dy}{dx} = z$

and $\quad \dfrac{dz}{dx} = f(x, y, z)$

are simultaneous equations where $f_1(x, y, z) = z$, $f_2(x, y, z) = f(x, y, z)$ given. Also $y(0)$ and $z(0)$ are given.

Starting from these equations, we can use the previous article and solve the problem.

Example 7. *Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$, find the value of $y(0\cdot1)$ by using Runge-Kutta method of fourth order.*

Solution. $y'' = -xy' - y$, $y(0) = 1$, $y'(0) = 0$, $h = 0\cdot1$, $y_0 = 1$,

$x_0 = 0$, $y_1 = y(0\cdot1)$

Setting $\quad y' = z$

The equation becomes,

$$y'' = z' = -xz - y$$

$$\therefore \quad \frac{dy}{dx} = z = f_1(x, y, z) \quad \text{...(1)}$$

$$\frac{dz}{dx} = -xz - y = f_2(x, y, z) \quad \text{...(2)}$$

given $\quad y_0 = 1$, $z_0 = y_0' = 0$.

By algorithm,

$$k_1 = hf_1(x_0, y_0, z_0) = (0\cdot1) f_1(0, 1, 0) = (0\cdot1)(0) = 0$$

$$l_1 = hf_2(x_0, y_0, z_0) = (0\cdot1) f_2(0, 1, 0) = (-1)(0\cdot1) = -0\cdot1$$

$$k_2 = hf_1\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= (0\cdot1)\, f_1\,(0\cdot05, 1, -0\cdot05) = (0\cdot1)\,(-0\cdot05) = -0\cdot005$$

$$l_2 = (0\cdot1)\, f_2\,(0\cdot05, 1, -0\cdot05) = (0\cdot1)\,[+(0\cdot05)\,(0\cdot05) - 1]$$

$$= -0\cdot09975$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$= (0\cdot1)\, f_1\,(0\cdot05, 0\cdot9975, -0\cdot0499)$$

$$= (0\cdot1)\,(-0\cdot0499) = -0\cdot00499$$

$$l_3 = hf_2\,(0\cdot05, 0\cdot9975, -0\cdot0499)$$

$$= -(0\cdot1)\,[(0\cdot05)\,(-0\cdot04991) + 0\cdot9975]$$

$$= -0\cdot09950$$

$$k_4 = hf_1\,(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= (0\cdot1)\, f_1\,(0\cdot1, 0\cdot99511, -0\cdot0995)$$

$$= (0\cdot1)\,(-0\cdot0995) = -0\cdot00995$$

$$l_4 = hf_2\,(0\cdot1, 0\cdot99511, -0\cdot0995)$$

$$= (0\cdot1)\,[-\{(0\cdot1)\,(-0\cdot0995) + 0\cdot99511\}]$$

$$= -0\cdot0985$$

$$\therefore \quad y_1 = y_0 + \Delta y = 1 + \frac{1}{6}\,[0 + 2\,(-0\cdot005) + 2\,(-0\cdot00499) - 0\cdot00995]$$

$$= 0\cdot9950$$

y (0·1) = 0·9950.

EXERCISE 11.4

Evaluate using Runge-Kutta methods. Unless otherwise mentioned, use fourth order R.K. method.

1. Find $y\,(0\cdot2)$ given $\frac{dy}{dx} = y - x$, $y\,(0) = 2$ taking $h = 0\cdot1$.

2. Evaluate $y\,(1\cdot4)$ given $\frac{dy}{dx} = x + y$, $y\,(1\cdot2) = 2$.

3. Obtain the value of y at $x = 0\cdot2$ if y satisfies
$\frac{dy}{dx} - x^2 y = x$; $y\,(0) = 1$ taking $h = 0\cdot1$.

4. Solve $\frac{dy}{dx} = xy$ for $x = 1\cdot4$, taking $y\,(1) = 2$, $h = 0\cdot2$.

5. Solve: $y' = \frac{y - x}{y + x}$ given $y\,(0) = 1$, to obtain $y\,(0\cdot2)$.

6. Solve the initial value problem

$\frac{du}{dt} = -2tu^2$, $u(0) = 1$ with $h = 0{\cdot}2$ on the interval $(0, 0{\cdot}6)$ by using fourth order R.K. method. *(Nov. 1991)*

7. Evaluate for $y(0{\cdot}1), y(0{\cdot}2), y(0{\cdot}3)$ given
$$y' = \frac{1}{2}(1 + x)y^2, y(0) = 1.$$

8. Solve: $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$, $y(1) = 1$ for $y(1{\cdot}1)$ taking $h = 0{\cdot}05$.

9. Find $y(0{\cdot}5), y(1), y(1{\cdot}5), y(2)$ taking $h = 0{\cdot}5$ given $y' = \frac{1}{x+y}$, $y(0) = 1$.

10. Evaluate $y(1{\cdot}2)$ and $y(1{\cdot}4)$ given $y' = \frac{2xy + e^x}{x^2 + xe^x}$, $y(1) = 0$. *(MS. Ap. 1989)*

11. Find y for $x = 0{\cdot}2\,(0{\cdot}2)\,0{\cdot}6$ given $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$.

12. Find $y(0{\cdot}2)$ given $\frac{dy}{dx} = -xy$, $y(0) = 1$, taking $h = 0{\cdot}2$ by R.K. method of 4th order.

13. Find $y(0{\cdot}1), y(0{\cdot}2)$ given $y' = x - 2y$, $y(0) = 1$ taking $h = 0{\cdot}1$ by (1) second order, third order and fourth order R.K. method.

14. Determine y at $x = 0{\cdot}2\,(0{\cdot}2)\,(0{\cdot}6)$ by R.K method given $\frac{dy}{dx} = \frac{1}{1+x}$, given $y(0) = 0$.

15. Find $y(0{\cdot}2)$ given
$y' = 3x + \frac{1}{2}y$, $y(0) = 1$ by using Runge-Kutta method of 4th order.

16. Solve $y' = xy + 1$ as $x = 0{\cdot}2, 0{\cdot}4, 0{\cdot}6$ given $y(0) = 2$, taking $h = 0{\cdot}2$.

17. Given $y' = x^3 + \frac{1}{2}y$, $y(1) = 2$, find $y(1{\cdot}1), y(1{\cdot}2)$.

18. Solve $10y' = x^2 + y^2$, given $y(0) = 1$ for $x = 0{\cdot}1\,(0{\cdot}1)\,(0{\cdot}3)$.

19. Solve $8y' = x + y^2$ given $y(0) = 0{\cdot}5$ for $x = 0{\cdot}1\,(0{\cdot}1)\,(0{\cdot}4)$.

20. Solve the system: $\frac{dy}{dx} = xz + 1$, $\frac{dz}{dx} = -xy$ for $x = 0{\cdot}3\,(0{\cdot}3)\,(0{\cdot}9)$ taking $x = 0$, $y = 0$, $z = 1$. *(MKU 1979)*

21. Solve: $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y$, given $y(0) = 0$, $z(0) = 1$ for $x = 0{\cdot}0$ to $0{\cdot}2$ taking $h = 0{\cdot}1$.

22. Solve $\frac{dy}{dx} = -xz$, $\frac{dz}{dx} = y^2$, given $y(0) = 1$, $z(0) = 1$ for $x = 0\,(0{\cdot}2)\,(0{\cdot}4)$.

23. Evaluate $y(1{\cdot}1), z(1{\cdot}1)$ given $\frac{dy}{dx} = xyz$, $\frac{dz}{dx} = \frac{xy}{z}$, $y(1) = 1/3$, $z(1) = 1$.

24. Using R.K. method determine $x(0{\cdot}1), y(0{\cdot}1)$ given $\frac{dx}{dt} = xy + t$, $x(0) = 1$

$\frac{dy}{dt} = ty + x, \quad y(0) = -1.$

25. Find $x(0{\cdot}1), y(0{\cdot}1)$ given $\frac{dx}{dt} = 2x + y, \frac{dy}{dt} = x - 3y$, given $x(0) = 0, y(0) = 0{\cdot}5$.

26. Solve $y'' - x(y')^2 + y^2 = 0$ using R.K. method for $x = 0{\cdot}2$ given $y(0) = 1, y'(0) = 0$, taking $h = 0{\cdot}2$.

27. Find $y(0{\cdot}1)$ given $y'' = y^3, y(0) = 10, y'(0) = 5$ by R.K. method.

28. Find $y(0{\cdot}1), y(0{\cdot}2)$ given $y'' - x^2y' - 2xy = 1, y(0) = 1, y'(0) = 0$.

29. Find $y(0{\cdot}1)$ given $y'' + 2xy' - 4y = 0, y(0) = 0{\cdot}2, y'(0) = 0{\cdot}5$.

30. Obtain the value of $x(0{\cdot}1)$ given

$\frac{d^2x}{dt^2} = \frac{tdx}{dt} - 4x, x(0) = 3, x'(0) = 0.$

31. Compute the value of $y(0{\cdot}2)$ given

$$y'' = -y, y(0) = 1, y'(0) = 0.$$

11·16. Predictor-Corrector methods

The methods which we have discussed so far are called single-step methods because they use only the information from the last step computed. The methods of Milne's predictor-corrector, Adams-Bashforth predictor corrector formulae are multi-step methods.

In solving the equation $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ we used Euler's formula

$$y_{i+1} = y_i + hf'(x_i, y_i), i = 0, 1, 2, \ldots \qquad \ldots(1)$$

We improved this value by Improved Euler method

$$y_{i+1} = y_i + \frac{1}{2} h [f(x_i, y_i) + f(x_{i+1}, y_{i+1})] \qquad \ldots(2)$$

In the equation (2), to get the value of y_{i+1} we require y_{i+1} on the R.H.S. To overcome this difficulty, we calculate y_{i+1} using Euler's formula (1) and then we use it on the R.H.S. of (2), to get the L.H.S. of (2). This y_{i+1} can be used further to get refined y_{i+1} on the L.H.S. Here, we *predict* a value of y_{i+1} from the rough formula (1) and use in (2) to correct the value. Every time, we improve using (2). Hence equation (1) Euler's formula is a *predictor* and (2) is a corrector. A predictor formula is used to *predict* the value of y at x_{i+1} and a *corrector* formula is used to correct the error and to improve that value of y_{i+1}.

11·17. Milne's Predictor Corrector Formulae

Suppose our aim is to solve $\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \qquad \ldots(1)$

numerically.