Complex Analysis

16SCCMM13

UNIT-I

1) Definition limits:

A function $w=f(z)$ is said to have the limit l as z tends to z_0. If given $\varepsilon>0$ there exist $\delta>0$, such that $0<|z-z_0|<\delta$

$$\Rightarrow |f(z)-l|<\varepsilon$$

In case we write

$$\boxed{\lim_{z\to z_0} f(z)=l}$$

2) $\lim_{z\to 2} \frac{z^2-4}{z-2}=4$.

<u>Solution</u>:- $f(z)=\lim_{z\to 2}\frac{z^2-4}{z-2}$

$$=\lim_{z\to 2}\frac{z^2-2^2}{z-2}$$

$$=\lim_{z\to 2}\frac{(z+2)\cancel{(z-2)}}{\cancel{(z-2)}}$$

$$=4$$

Now given $\varepsilon>0$, we choose

$\delta=\varepsilon$

$$|f(z)-4|<\varepsilon$$

$$\therefore \lim_{z\to 2} f(z)=4.$$

3) The function $f(z)=\frac{\bar{z}}{z}$ does not have a limit as $z\to 0$.

Soln:- $f(z)=\frac{\bar{z}}{z}=\frac{x-iy}{x+iy}$

$z\to 0,\ y=mx$

$$=\frac{x-i(mx)}{x+i(mx)}$$

$$=\frac{\cancel{x}(1-im)}{\cancel{x}(1+im)}$$

$$=\frac{1-im}{1+im},\ x\neq 0$$

Hence, $z\to 0$ along the path $y=mx$, $f(z)$ tends to $\frac{1-im}{1+im}$ which is difference from different value m.

Hence, $f(z)$ does not have a limit as $z\to 0$.

4) Let $f(z)=\frac{x^2y^2}{(x+y^2)^3}$, $z\neq 0$ Thus does not have a limit as $z\to 0$.

Soln!- $f(z)=\frac{x^2y^2}{(x+y^2)^3}$

$z\to 0,\ y^2=mx$.

$$=\frac{x^2mx}{(x+mx)^3}=\frac{x^3m}{x^3(1+m)^3}$$

$$=\frac{m}{(1+m)^3}$$

Hence, $z \to 0$ along the parabola $y^2 = mx$, $f(z)$ tends to $\frac{m}{(1+m^3)}$ which depend on m.

Hence, $f(z)$ does not have a limit as $z \to 0$.

5. Continous function :-

Let f be a complex valued function defined on a region D of the complex plane. Let $z_0 \in D$, Then f is said to be continous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

Thus f is continous at z_0.

If given z_0, If given $\varepsilon > 0$ there exist $\delta > 0$ show that $|z - z_0| < \delta$ $\Rightarrow |f(z) - f(z_0)| < \varepsilon$.

6) Differentiablity :-

Let f be a complex valued function defined in a region D and let $z \in D$. Then f is said to be differentiable at z if $\boxed{\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}}$ exists and is finite. This limit is denoted by $f'(z)$ (or) $\frac{df}{dz}$ and is called the derivative of $f(z)$ at z.

7) Let $f(z) = e^x(\cos y + i\sin y)$

Soln:- $u(x,y) = e^x \cdot \cos y$

$v(x,y) = e^x \cdot \sin y$

$u_x(x,y) = e^x \cdot \cos y = v_y(x,y)$

$u_y(x,y) = -e^x \cdot \sin y = -v_x(x,y)$

The first order derivatives of u and v satisfy the cauchy Riemann equation.

8) Let $f(z) = |z|^2$

Soln:- $f(z) = x^2+y^2$

$u_x(x,y) = 2x$

$v_y(x,y) = 2y$

$u_x = v_y$

$\cancel{2}x = \cancel{2}y$

$x = y$

Clearly the C-R equations are satisfied at $z=0$.

9) Analytical function :-

If f is analytic at every point of a region D then f is said to be analytic in D. A functions which is analytic at every point of the complex plane is called an entie function (or) integral function.

16) Harmonic Function :-

Let $u(x,y)$ be a function of two real variables x, y defined in a region D. $u(x,y)$ is called to be a harmonic function.

If $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and this equation is called Laplace equation.

5 Mark

1) When the limit of a function $f(z)$ exists as z tends to z_0 then the limit has a unique value.

Proof :- Suppose that,

$\lim_{z \to z_0} f(z)$ has two value l_1 and l_2. Then,

$\epsilon > 0$ there δ_1 and $\delta_2 > 0$.

Such that,

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - l_1| < \epsilon/2$$

and $0 < |z - z_0| < \delta_2 \Rightarrow |f(z) - l_2| < \epsilon/2$

Now, let $\delta = \min\{\delta_1, \delta_2\}$

Then,

If $0 < |z - z_0| < \delta$

we have, $|l_1 - l_2| = |l_1 - f(z) + f(z) - l_2|$

$\leq |f(z) - l_1| + |f(z) - l_2|$

$< \epsilon/2 + \epsilon/2$

$= \epsilon$

Since $\epsilon > 0$ is arbitrary

$|l_1 - l_2| = 0$

So that $l_1 = l_2$

Hence proved.

2) Complex from of C-R equation.

Statement:- Let $f(z) = u(x,y) + iv(x,y)$ be differentiable. Then C-R equation can be put in the complex from as $f_x = -if_y$.

Proof:- Then

$f_x = u_x + iv_x$ and $f_y = u_y + iv_y$

Hence,

$f(x) = -if_y$

$\Leftrightarrow u_x + iv_x = -i(u_y + iv_y)$

$\Leftrightarrow u_x + iv_x = v_y - iu_y$

$\Leftrightarrow u_x = v_y$ and $v_x = -u_y$

These the two C-R equations are equivalent to the equation

$f_x = -if_y$ //.

3) C-R equations in polar co-ordinates

Statement:- Let $f(z) = u(r,\theta) + iv(r,\theta)$ be differentiable at $re^{i\theta} = z \neq 0$. Then

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ Further

$f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

Proof:- We have $x = r\cos\theta$ and $y = r\sin\theta$

Hence,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}.$$

$$= \frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$$

$$= r\frac{\partial v}{\partial x}(-\sin\theta) + \frac{\partial v}{\partial y}\cos\theta)$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta.\ (\text{C-R eqn})$$

$$= \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta\ [\text{using (1)}]$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r}$$

IIIly we can prove $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

now,

$$r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = r\left[\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r}\right)\right]$$

$$= r\left[\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + i\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\right].$$

$$= r\cos\theta\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + r\sin\theta\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$= x\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x}\right] + iy\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$= x f'(z) + iy f'(z) = f'(z)(x+iy)$

$= f'(z)(z)$

$f'(z) = r/z \left(\frac{\partial u}{\partial r} + i \frac{\partial u}{\partial \theta} \right)$

3) If $f(z)$ is a differentiable function the C-R equation can be put the from $\frac{\partial f}{\partial \bar{z}} = 0$.

Proof:- $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(-\frac{1}{2i}\right)$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y}\right)$$

Then $\frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}$

which is the complex from of the C-R equation.

Thus C-R equation can be put in the form $\frac{\partial f}{\partial \bar{z}} = 0$.

4) If $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$ prove that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$

Soln:- Let $z = x + iy$

$\therefore x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$

Hence, $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$

$= \frac{1}{2}(\partial/\partial x) - \frac{1}{2i}(\partial/\partial y)$ [divide & mult i]

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{2}\left[\left(\frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial x \partial y}\right)\left(\frac{1}{2}\right) + \left(\frac{\partial^2}{\partial y \partial x} + i\frac{\partial^2}{\partial y^2}\right)\left(\frac{1}{2i}\right)\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + i\frac{\partial^2}{\partial x \partial y} + \frac{1}{i}\frac{\partial^2}{\partial y \partial x}\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x \partial y}\left(\frac{i^2+1}{i}\right)\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] + 0$$

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

5) Milne Thompson method:-

Let $u(x,y)$ be a given harmon[ic] function

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function.

Then, $f'(z) = u_x(x,y) + iv_x(x,y)$

$= u_x(x,y) - i u_y(x,y)$

let $\phi_1(x,y) = u_x(x,y)$ and

$\phi_2(x,y) = u_y(x,y)$

we have,

$$x = \frac{z+\bar{z}}{2} \text{ and } y = \frac{z-\bar{z}}{2i}$$

Hence $f'(z) = \phi_1\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) - i\phi_2\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$

Putting $z = \bar{z}$ we obtain

$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$,

Hence, $f(z) = \int (\phi(z,0) - i\phi_2(z,0))dz + C$

10 Mark

1) The Cauchy - Riemann Equations :-

Statement :-

Let $f(z) = u(x,y) + iv(x,y)$ be differentiable at a point $z_0 = x_0 + iy_0$. Then $u(x,y)$ and $v(x,y)$ have first order partial derivatives $u_x(x_0,y_0)$, $u_y(x_0,y_0)$, $v_x(x_0,y_0)$ and $v_y(x_0,y_0)$ at (x_0,y_0) and these partial derivative satisfy the cauchy Riemann equation (C-R equation) given by

$u_x(x_0,y_0) = v_y(x_0,y_0)$ and

$u_y(x_0,y_0) = -v_x(x_0,y_0)$

Also

$$\boxed{\begin{aligned} f'(z) &= u_x(x_0, y_0) + i v_x(x_0, y_0) \\ &= v_y(x_0, y_0) - i u_y(x_0, y_0) \end{aligned}}$$

Proof:- Since $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$

$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$ exists and hence the limit is independent of the path in which h approaches zero.

Now, $\frac{f(z_0+h) - f(z_0)}{h}$

$$= \frac{u(x_0+h_1, y_0+h_2) + iv(x_0+h_1, y_0+h_2) - u(x_0, y_0) - iv(x_0, y_0)}{h_1 + ih_2}$$

$$= \frac{u(x_0+h_1, y_0+h_2) - u(x_0, y_0)}{h_1 + ih_2} + \frac{i\,v(x_0+h_1, y_0+h_2) - v(x_0, y_0)}{h_1 + ih_2}$$

Suppose $h \to 0$ along the real axis so that $h = h_1$

Then

$$f'(z_0) = \lim_{h_1 \to 0} \left[\frac{f(z_0+h_1) - f(z_0)}{h_1} \right]$$

$$= \lim_{h_1 \to 0}\left[\frac{u(x_0+h_1, y_0) - u(x_0, y_0)}{h_1}\right]$$

$$+ i \lim_{h_1 \to 0}\left[\frac{v(x_0+h_1, y_0) - v(x_0, y_0)}{h_1}\right]$$

$$= u_x(x_0, y_0) + i v_x(x_0, y_0) \longrightarrow (1)$$

Now, $h \to 0$ and the imaginary axis $h = ih_2$.

$$f'(z_0) = \lim_{ih_2 \to 0}\left[\frac{f(z_0+h_2) - f(z_0)}{ih_2}\right]$$

$$= \lim_{ih_2 \to 0}\left[\frac{u(x_0, y_0+h_2) - u(x_0, y_0)}{ih_2}\right] + i \lim_{ih_2 \to 0} \frac{v(x_0, y_0+h_2) - v(x_0, y_0)}{ih_2}$$

$$= \left(\frac{u_y(x_0, y_0)}{i}\right) + i\left(\frac{v_y(x_0, y_0)}{i}\right)$$

$$= -i u_y(x_0, y_0) + v_y(x_0, y_0)$$

$$= -i u_y(x_0, y_0) + v_y(x_0, y_0) \longrightarrow (2)$$

Equating real and imaginary part we get,

$$u_x(x_0, y_0) = v_y(x_0, y_0)$$

$$u_y(x_0, y_0) = -v_x(x_0, y_0)$$

Hence proved //.

2) Find the analytic function $f(z) = u+iv$ given that $u - v = e^x(\cos y - \sin y)$.

Soln:- $u - v = e^x \cos y - e^x . \sin y$

$u_x - v_x = e^x \cos y - e^x . \sin y \longrightarrow (1)$

$u_y - v_y = -e^x . \sin y - e^x \cos y \longrightarrow (2)$.

C-R equation $\Rightarrow$ $u_x = v_y$; $u_y = -v_x$

$u_x + u_y = e^x \cos y - e^x \sin y \longrightarrow (3)$

$u_y - u_x = -e^x . \sin y - e^x \cos y \longrightarrow (4)$

$(3) + (4) \Rightarrow \cancel{2} u_y = -\cancel{2} e^x . \sin y$

$u_y = -e^x . \sin y$

$(3) - (4) \Rightarrow \cancel{2} u_x = \cancel{2} e^x . \cos y$.

$u_x = e^x . \cos y \Rightarrow \boxed{u = e^x . \cos y + c_1}$

$v_y - v_x = e^x . \cos y - e^x . \sin y$

$-v_x - v_y = -e^x . \sin y - e^x . \cos y$.

$-\cancel{2} v_x = -\cancel{2} e^x . \sin y$

$v_x = e^x . \sin y$

$\cancel{2} v_y = \cancel{2} e^x \cos y$

$\int v_y = \int e^x . \cos y . dy$

$\boxed{v = -e^x . \sin y + c_2}$

$u + iv = e^x(\cos y - i \sin y) + c$

$f(z, 0) = e^z + c$ //

UNIT-II

2 Mark

1) Elementey Transformation translation :-

$w = z + b$

If $z = x + iy$, $w = u + iv$ and $b = b_1 + ib_2$,

The point (x, y) in the z-plane is the point $(x + b_1, y + b_2)$. when $b \neq 0$.

2) Inversion of $w = 1/z$.

Soln:- Consider the transformation $w = 1/z$.

put $z = re^{i\theta}$

$w = (1/r)e^{-i\theta}$

$T_1(z) = (1/r)e^{i\theta}$, $T_2(z) = re^{-i\theta} = \bar{z}$

$(T_1 \circ T_2)(z) = T_1(T_2(z))$

$= 1/r(e^{-i\theta}) = 1/z$.

The transformation $T_1(z) = (1/r)e^{i\theta}$ replents the inversion.

3) Under the transformation $w = iz + i$ and the half plane $x > 0$ maps onto half plane $v > 1$.

Soln:- $z = x + iy$, $w = u + iv$, $w = iz + i$

$u + iv = i(x + iy) + i = ix - y + i$

$= -y + i(x + 1)$

$u = -y$

$v = x + 1$

$\therefore x>0 \Leftrightarrow v>1$

$\therefore$ The half plane $x>0$ is maped into the half plane $v>1$.

4) Bilinear transformation :-

$$W = T(z) = \frac{az+b}{cz+d}$$

where a, b, c, d are complex constant and $ad-bc \neq 0$ is called bilinear (or) mobius transformation

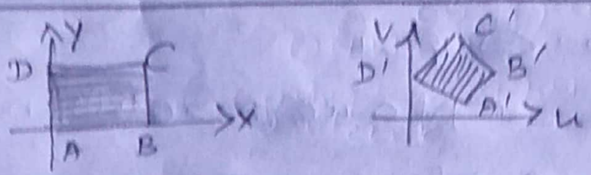
5) Cross ratio :-

The cross ratio of these four points denoted by (z_1, z_2, z_3, z_4) is defined by

$$(z_1, z_2, z_3, z_4) = \begin{cases} \dfrac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} & \text{if none of } z_1, z_2, z_3, z_4 \text{ is } \alpha \\ \dfrac{z_1-z_3}{z_1-z_4} & \text{if } z_2 \text{ is } \alpha \\ \dfrac{z_2-z_4}{z_1-z_4} & \text{if } z_3 \text{ is } \alpha \\ \dfrac{z_1-z_3}{z_2-z_3} & \text{if } z_4 \text{ is } \alpha \\ \dfrac{z_2-z_4}{z_2-z_3} & \text{if } z_1 \text{ is } \alpha. \end{cases}$$

5 Mark

1) Find the image of the square region vertices $(0,0), (2,0), (2,2), (0,2)$ under the transformation $W = (1+i)z + (2+i)$



z - plane

Soln:- $w = (1+i)z + (2+i)$

A(0, 0) is mapped into A'

$w = (1+i)(0+0i) + (2+i)$

$w = 2+i$

B(2, 0) is mapped into B'

$w = (1+i)(2+0i) + 2+i$

$= 4+3i$.

C(2, 2) is mapped into C'

$= (1+i)(2+2i) + 2+i$

$= 2+2i+2i-2+2+i$

$= 2+i5$

D(0, 2) is mapped into D'

$= (1+i)(0+2i) + 2+i$

$= 2i - 2 + 2 + i$

$= 0+3i$ //

2) Find the image of the strip $2 < x < 3$

Centre $w = 1/z$

Soln:- $w = 1/z$ and $x = \frac{u}{u^2+v^2}$ and $y = \frac{v}{u^2+v^2}$

Now $x > 2 \Rightarrow \frac{u}{u^2+v^2} > 2$

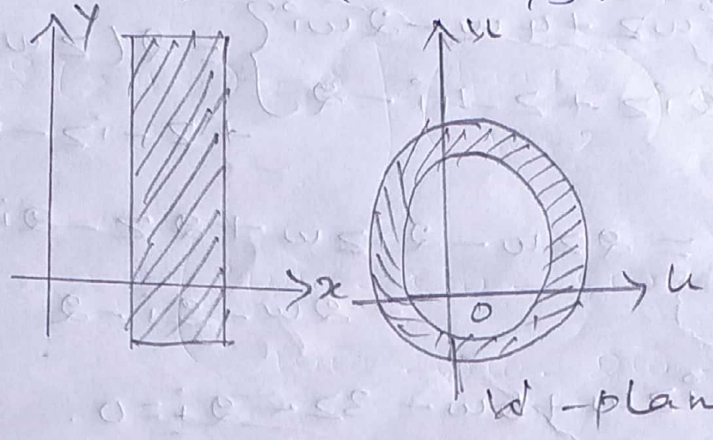
$\Rightarrow 2(u^2+v^2) - u < 0$

$\Rightarrow u^2+v^2 - u/2 < 0$

The region $x > 2$ is mapped region represented by $u^2+v^2-u/2 < 0$ the interior of the circle centre $C(1/4, 0)$ and radius $1/2$.

$$x < 3 \Rightarrow \frac{u}{u^2+v^2} < 3$$

$$\Rightarrow u^2+v^2-u/3 \geq 0.$$



The region $x < 3$ is mapped onto the exterior of the circle centre $(1/6, 0)$ and radius $1/\sqrt{6}$.

The strip $2 < x < x$ is mapped onto the region bounded by the circle $u^2+v^2 = u/2$ and $u^2 = v^2 = u/3$ in the w-plane.

3) Find the bilinear transformation points $z_1 = 2, z_2 = i, z_3 = -2$ onto $w_1 =$ $w_2 = i, w_3 = -1$.

Soln:- $\frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)} = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$

$$\frac{(w-i)(1+1)}{(w+1)(1-i)} = \frac{(z-i)(2+2)}{(z+2)(2-i)}$$

$$(w-i)(z+2)(2-i) = 2(z-i)(w+1)(1-i)$$

$$\left.\begin{array}{l}2wz - iwz + 4w - 2wi \\ \quad - 2iz + z - 4i - 2\end{array}\right\} = 2(zw - izw + z - iz - iw - w - i - 1)$$

$$= 2zw - 2izw + 2z - 2iz - 2iw - 2w - 2i - 2$$

$$izw + 6w - 3z - 2i = 0.$$

$$izw - 6w = 3z + 2i.$$

$$w(6+iz) = 3z + 2i$$

$$w = \frac{3z+2i}{6+iz}.$$

10 Mark

1) Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $z\bar{z} - 2(z+\bar{z}) = 0$ into a straight line given by $2(w+\bar{w}) + 3 = 0$.

Soln:- $w = \frac{2z+3}{z-4}$

$w(z-4) = 2z+3$

$wz - 4w - 2z - 3 = 0.$

$z(w-2) - (4w+3) = 0$

$z(w-2) = 4w+3$

$$z = \frac{4w+3}{w-2}$$

$$\bar{z} = \frac{4\bar{w}+3}{\bar{w}-2}$$

$$z\bar{z} = \left(\frac{4w+3}{w-2}\right)\left(\frac{4\bar{w}+3}{\bar{w}-2}\right)$$

$$= \frac{16\bar{w}w + 12w + 12\bar{w} + 9}{w\bar{w} - 2w - 2\bar{w} + 4}$$

$$2(z+\bar{z}) = 2\left[\left(\frac{4w+3}{w-2}\right) + \left(\frac{4\bar{w}+3}{\bar{w}-2}\right)\right]$$

$$= 2\left(\frac{8w\bar{w} - 5\bar{w} - 5w - 12}{w\bar{w} - 2w - 2\bar{w} + 4}\right)$$

$$= \frac{16w\bar{w} - 10\bar{w} - 10w - 24}{w\bar{w} - 2w - 2\bar{w} + 4}$$

$$z\bar{z} - 2(z+\bar{z}) = \frac{-22\bar{w} - 24w - 33}{w\bar{w} - 2w - 2\bar{w} + 4}$$

$$-22\bar{w} - 24w - 33 = 0$$

$$22\bar{w} + 24w + 33 = 0$$

$$2\bar{w} + 2w + 3 = 0$$

$$2(\bar{w} + w) + 3 = 0 \,//$$

UNIT-III

1) Definition of Integral :-

Let $f(t) = u(t)$ be a continous complex valued function defined on $[a, b]$

$$\int_a^b f(t) \cdot dt = \int_a^b u(t)\,dt + i\int_a^b v(t) \cdot dt.$$

2) piecewise differentiable curve :-

Let C_1 be a Differentiable curve with origion and terminus z_2, be another curve z_2 and z_3. Then the curve with consits of C_1 followed by C_2 is a piecewise differentiable curve with origion z_1 and terminus z_3. This curve is denoted by $C_1 + C_2$.

3) Length :-

$z = z(t)$ where $a \leq t \leq b$. Then the length l of C is defined by

$$l = \int_a^b |z'(t)| \cdot dt.$$

4) Higher derivates :-

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(\zeta)}{\zeta - z}\, d(\zeta)$$

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(e_1)}{(e_1-z)^2}\cdot de_1$$

In general

$$f^n(z) = \frac{n!}{2\pi i}\frac{f(e_1)}{(e_1-z)^{n+1}}\cdot de_1.$$

5) $\int_C \frac{e^z}{z^n} dz = \frac{2\pi i}{(n-1)!}$ where C is the circle $|z|=1$

Soln:- $f^n(z) = \frac{n!}{2\pi i}\int_C \frac{f(e_1)}{(e_1-z)^{n+1}} de_1$

$$\int_C \frac{e^z}{(z-0)^n} dz, \quad f(z) = e^z$$

$$= \frac{2\pi i}{(n-1)!} e^z, \text{ poles } z=0$$

$$= \frac{2\pi i}{(n-1)!} e^0$$

$$= \frac{2\pi i}{(n-1)!}$$

6) Morera's theorem:-

If $f(z)$ is continous in a simply connected in domain in a simply D and if $\int_C f(z)dz = 0$ for ever simply closed curve in C lying in D. Then $f(z)$ is analytic in D.

4) Cauchy's inequality:-

Let $f(z)$ be analytic inside and on the circle c with centre z_0 and radius r. Let M denote the maximum of $|f(z)|$ on c. Then

$$|f^n(z_0)| \le \frac{n!M}{r^n}.$$

5 Mark

1) $\left|\int_C f(z)\,dz\right| \le Ml$ where $M = \max\{|f(z)|\ \text{ec}\}$ and l is the length of C.

Proof:- Suppose C is given by the equation $z(t) = z$, where $a \le t \le b$.

$$|f(z(t))| \le M \text{ for all } t;\ a \le t \le b \quad \text{--- (1)}$$

now $\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right|$

$$\le \int_a^b f(z(t))\,z'(t)\,dt$$

$$= \int_a^b |f(z(t))|\,|z'(t)|\,dt.$$

$$\le \int_a^b M|z'(t)|\,dt \text{ using (1)}$$

$$= M\int_a^b |z'(t)|\,dt = Ml$$

$$\left|\int_C f(z)\,dz\right| \le Ml.$$

2) Fundamental theorem of algebra :-

St :- Every polynomial of degree ≥ 1 has at least one zero (root) in C.

proof :- Let $f(z)$ be a polynomial of degree ≥ 1. Then $f(z) \neq 0 \forall z$.

Further $f(z) \neq 0 \forall z$.

$\therefore \frac{1}{f(z)}$ is also an entire function

Also as $z \to \infty$, $f(z) \to \infty$

$\therefore \frac{1}{f(z)} \to 0$ as $z \to \infty$

$\therefore \frac{1}{f(z)}$ is bounded function

and constant. polynomial degree is zero.

$f(z)$ is atleast one root in C.

3) Let C denote the unit circle $|z|=1$ then $\int_C \frac{e^z}{z} dz$.

Soln :- $f(z) = e^z$, $z_0 = 0$.

By using cauchy's integral theorem

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz.$$

$$2\pi i f(z_0) = \int_C \frac{f(z)}{z-z_0} dz$$

$$2\pi i f(z_0) = \int_C \frac{e^z}{z} \, dz$$

$$= 2\pi i f(0)$$

$$= 2\pi i e^0$$

$$= 2\pi i \;//.$$

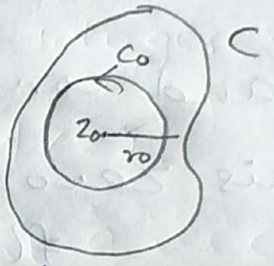
10 Mark

1) Cauchy's Integral formula:-

St:- Let $f(z)$ be function which is analytic inside and on a simple closed curve C, let z_0 be any point interior of C then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, dz$$

Proof:-



$$\int_C \frac{f(z)\,dz}{z - z_0} = \int_{C_0} \frac{f(z)\,dz}{z - z_0}.$$

$$= \int_{C_0} \left(\frac{f(z) - f(z_0) + f(z_0)}{z - z_0} \right) dz.$$

$$= \int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) . dz + \int_{C_0} \frac{f(z_0)}{z - z_0} \, dz$$

$$= \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0} dz + f(z_0) 2\pi i$$

Thus $\displaystyle\int_C \frac{f(z)\,dz}{z - z_0} = \int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz + 2\pi i f(z_0)$ → (1).

We claim that $\int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) dz = 0$.

Given $\epsilon > 0$ there exist $\delta > 0$ such that:

If we choose $r_0 < \delta$ then,

$|z - z_0| < r_0$

$\Rightarrow |f(z) - f(z_0)| < \epsilon$

Hence $\left| \int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) \cdot dz \right| < \epsilon/r_0 (2\pi r_0)$

$= 2\pi\epsilon$

Since ϵ is arbitrary we have

$$\int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) \cdot dz = 0.$$

From (1) we get $\int_C \frac{f(z)}{z-z_0} \, dz = 2\pi i f(z_0)$

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} \cdot dz.$$

—— x ——

UNIT - IV

2 Mark

1) Find the Laurents series for $\frac{z}{(z+1)(z+2)}$ about $z = -2$.

Soln:- Let $f(z) = \frac{z}{(z+1)(z+2)}$

$$= \frac{-1}{z+1} + \frac{2}{z+2} = \frac{-1}{(z+2)-1} + \frac{2}{z+2}$$

$$= \frac{1}{1-(z+2)} + \frac{2}{z+2}$$

$$= (1-(2+z))^{-1} + \frac{2}{z+2}$$

$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + (z+2)^3 + \cdots$$

Zero's of an analytic function :-

Let $f(z)$ be a function which is analytic in a region D. $a \in D$ then a is said to be a zero of order r. r is positive integer for $f(z)$ is. $f(z) = (z-a)^r \phi(z)$.

Singularities :- A point a is called as singular point of $f(z)$. If $f(z)$ is not analytic at a and f is analytic at some point of every disc $|z-a|<r$.

poles :- Let a be an isolate singularity of $f(z)$ the point a is called a pole.

$$\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \dots + \frac{b_r}{(z-a)^r}$$

5) Meromorphic function :-

A function $f(z)$ is said to be a meromorphic function. If it is analytic except at a finite number of points and these finite set of points are poles.

5 Mark :-

1) Riemann's theorem :-

Let f be a function a domain $0 < |z-z_0| < \delta$, then either f is analytic at z_0 or else z_0 is a removable singular point of f

Proof :- $b_n = \frac{1}{2\pi i} \int_c \frac{f(z) \, dz}{(z-z_0)^{n+1}}$

where C is the circle $|z-z_0| = r$ where $r < \delta$.

Now, $|f(z)| \leq M$

$\therefore b_n \leq \frac{1}{2\pi} \frac{M(2\pi r)}{r^{-n+1}} = Mr^n$

$0 < r < \delta$ is true for every r. Such that

Hence the laurents series for $f(z)$ has no principle part hence the theorem Riemann's theorem.

2) Expand $\frac{1}{z(z-1)}$ as laurents series

i) about $z=0$ in powers of z and

ii) about $z=1$.

Soln:- Let $f(z)$ is not analytical $0 \& 1$

$0<|z|<1$

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z}\left(\frac{1}{1-z}\right)$$

$$= -\frac{1}{z}(1-z)^{-1}$$

$$= -\frac{1}{z}(1+z+z^2+\dots)$$

$$= -\left(+\frac{1}{z}+1+z+z^2+\dots\right)$$

This is laurents expands of $f(z)$ in $0<|z|<1$.

(ii) $$\frac{1}{z(z-1)} = \frac{1}{z-1}\left[\frac{1}{1+(z-1)}\right]$$

$$= \frac{1}{z-1}\left[1-(z-1)+(z-1)^2-(z-1)^3+\dots\right]$$

$$= \frac{1}{z-1}-1+(z-1)-(z-1)^2+\dots$$

$0<|z-1|<1$

3) Expand $\frac{1}{z^2-3z+2}$ in laurents series the region $1<|z|<2$.

Soln:- Let $f(z) = \frac{1}{z^2-3z+2}$

$$= \frac{1}{(z-1)(z-2)}$$

$$= \frac{1}{z-1} - \frac{1}{z-2}.$$

$1<|z|<2$

$$f(z) = \frac{1}{-2(1-z/2)} - \frac{1}{z(1-1/z)}$$

$$= -1/2(1-z/2)^{-1} - 1/z(1-1/z)^{-1}$$

$$= -1/2(1+z/2+(z/2)^2+(z/2)^3+\dots)$$
$$- 1/z(1+1/z+(1/z)^2+(1/z)^3+\dots)$$

$$= -1/2\sum_{n=1}^{\alpha}(z/2)^n - 1/z\sum_{n=1}^{\alpha}(1/z)^n$$

$$= -\sum_{n=1}^{\alpha}\frac{z^n}{2^{n+1}} - \sum_{n=1}^{\alpha}\frac{1}{z^{n+1}}$$ //

10 Mark

Expand $\frac{-1}{(z-1)(z-2)}$ as a power series in z in the regions

(i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|<2$.

Soln:- Let $f(z) = \frac{-1}{(z-1)(z-2)}$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}.$$

i) F(z) is not analytic 1 & c.

$|z|<1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}.$$

$$= -\frac{1}{1-z} + \frac{1}{2-z}$$

$$= -(1-z)^{-1} + (2-z)^{-1}$$

$$= -(1-z^{-1}) + 2^{-1}(1-z/2)^{-1}$$

$$= \sum_{n=0}^{\infty}\left[-z^n + 1/2\left(z/2\right)^n\right]$$

$$= \sum_{n=0}^{\infty}\left(-z^n + \frac{z^n}{2^{n+1}}\right) = \sum_{n=0}^{\infty} z^n\left(\frac{1}{2^{n+1}}-1\right)$$

(ii) $1<|z|<2$.

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{1}{z(1-1/z)} + \frac{1}{2(1-z/2)}$$

$$= 1/z(1-1/z)^{-1} + 1/2(1-z/2)^{-1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}.$$

This gives the Laurents series $1<|z|<2$.

(iii) $|z|>2$ and this domain $|2/z|<1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}.$$

$$= \frac{1}{z(1-1/z)} - \frac{1}{z(1-2/z)}$$

$$= \frac{1}{z}\left[(1-1/z)^{-1} - (1-2/z)^{-1}\right]$$

$$= \frac{1}{z}\left[\sum_{n=0}^{\infty} (1/z)^n - (2/z)^n\right]$$

$$= \frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} - \frac{2^n}{z^2}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty} \frac{1-2^n}{z^n}$$

$$= \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

Let

2) $f(z)$ be a function having a as isolated singular point. Then the following are i) a is pole of order for $f(z)$.

(ii) $f(z) = \frac{1}{(z-a)^r}$

(iii) a is a zero of order r for

<u>proof</u>:- (i) ⇒ (ii)

$$f(z) = \sum_{n=1}^{r} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} a_n (z-a)^n$$

where $b_r \neq 0$.

$$\therefore f(z) = \frac{1}{(z-a)^r} [b_r + b_{r-1}(z-a) + \dots$$
$$+ b_0 (z-a)^{r-} + a_0 (z-a$$

$$= \frac{1}{(z-a)^r} \theta(z) \text{ where,}$$

$$\theta(z) = b_r + b_{r-1}(z-a) + \dots$$

$$\lim_{z \to a} \theta(z) = b_r \neq 0.$$

(ii) ⇒ (iii)

Let $f(z) = \frac{1}{(z-a)^r} . \theta(z)$.

$$\therefore \frac{1}{f(z)} = (z-a)^r \frac{1}{\theta(z)} \text{ and } \frac{1}{\theta(z)}$$

analytic at a and $\frac{1}{\theta(a)} \neq 0$.

Hence a is a zero of order r for $\frac{1}{f(z)}$.

(iii) $\Rightarrow$ (ii)

$$\frac{1}{f(z)} = (z-a)^r \cdot g(z)$$

where $g(z)$ is analytic at a and $g(a) \neq 0$.

$\therefore f(z) = \frac{g_1(z)}{(z-a)^r}$. where $g_1(z)$ is analytic at a and $g_1(a) \neq 0$

$g_1(z) = a_0 + a_1(z-a) + \dots + a_n(z-a)^n +$

$$\therefore f(z) = \frac{a_0}{(z-a)^r} + \frac{a_1}{(z-a)^{r-1}} + \dots + a_r$$

$$+ a_{r-1}(z-a) + \dots$$

in $0 < |z-a| < r$.

$\therefore f(z)$ at $z=a$

$$\frac{a_0}{(z-a)^r} + \frac{a_1}{(z-a)^{r-1}} + \dots + \frac{a_{r-1}}{z-a}$$

and $a_0 \neq 0$.

$\therefore$ a is a pole of order r for $f(z)$.

UNIT-V

2 Mark

1) Residues:-

Then residues of $f(z)$ at a is define to be the co-efficient of $\frac{1}{z-a}$ in the laurents series expansion of $f(z)$ and is denoted by $Res\{f(z);a\}$

Thus $Res\{f(z);a\} = \frac{1}{2\pi i}\int_C f(z)\,dz = b_1$.

C is a circle, $|z-a| = r$.

$0 < |z-a| < r$.

2) calculate the residue of $\frac{z+1}{z^2-2z}$ at poles:-

Soln:- $f(z) = \frac{h(z)}{k(z)}$

where $h(z) = z+1$ and $k(z) = z^2-2z$

$k'(z) = 2z-2$; The poles of point 0 & 2.

$Res\{f(z);0\} = \frac{h(z)}{k'(z)}$

$= \frac{z+1}{2z-2} = \frac{0+1}{0-2} = -1/2$.

$Res\{f(z);2\} = \frac{z+1}{2z-2} = \frac{2+1}{4-2} = 3/2$

3) Calculate Residue $\frac{1}{(z^2+a^2)^2}$ at $z = ai$, [illegible]

Soln:-

$h(z)=1$, $k(z) = (z^2+a^2)^2$

$= (z^2 + (ia)^2)^2$

$= (z^2 - ia^2)^2$

$= (z-ia)^2 (z+ia)^2$.

poles $\Rightarrow ia, -ia$.

$k(z) = (z+ia)^2$

$k'(z) = 2(z+ia)(1)$.

$k'(z) = 2(z+ia)$.

$$\text{Res}\{f(z), ia\} = \frac{1}{2(z+ia)}$$

$$= \frac{1}{2(2ia)}$$

$$= \frac{1}{4ia} \text{ //.}$$

4) Find the residue $\cot z$ at $z=0$.

Soln:- $z=0$ and $\cot z$.

$$\text{Let } f(z) = \frac{\cos z}{\sin z} = \frac{h(z)}{k(z)}$$

$h(z) = \cos z$

$k(z) = \sin z$, and $k'(z) = \cos z$

$$\text{Res}\{f(z), 0\} = \lim_{z\to 0} \frac{h(z)}{k'(z)} = \lim_{z\to 0} \frac{\cos}{\cos}$$

$= 1$ //

5) Evaluate $\int_C \frac{dz}{2z+3}$ where c is $|z|=2$.

Soln:- $Res\{f(z); z\} = \frac{h(z)}{k'(z)}$

$h(z) = 1$ | $2z+3=0$

$k(z) = 2z+3$ | $2z=-3$

$k'(z) = 2$ | $\boxed{z=-3/2}$

$Res\{f(z); -3/2\} = \lim_{z\to -3/2} 1/2$.

$= 1/2$.

By Residues theorem

$\int_C f(z)\cdot dz = 2\pi i \int_C f(z)\cdot dz$.

$= 2\pi i \times 1/2$

$= \pi i$.

6) Evaluate $\int_C \frac{dz}{z^2 e^z}$ where $C=\{z\ |z|=1\}$

Soln:- $f(z) = \frac{1}{z^2 e^z} = \frac{e^{-z}}{z^2}$

$g(z) = e^{-z}$

$g'(z) = -e^{-z}$

$\frac{g'(z)}{g(z)} = -\frac{e^{-z}}{e^{-z}}$

$z^2=0$. and $z=0$.

$\lim_{z\to 0} \frac{g'(z)}{g(z)} = \lim_{z\to 0} \frac{-e^{-z}}{e^{-z}} = -1$

6 Mark

) calculate the Residue of $\frac{2z+3}{z(z^2+1)}$ it its pole.

Soln:

$$f(z) = \frac{h(z)}{k(z)}$$

$h(z) = 2z+3$ and $k(z) = z(z^2+1)$
$= z^3+z$.

$k'(z) = 3z^2+1$.

$z(z^2+1) = 0$.

$$\text{Res}\{f(z); 0\} = \lim_{z\to 0} \frac{h(z)}{k'(z)}.$$

$$= \lim_{z\to 0} \frac{2z+3}{3z^2+1}.$$

$$= \frac{2(0)+3}{3(0)+1}$$

$$= 3$$

$$\text{Res}\{f(z); i\} = \lim_{z\to i} \frac{h(z)}{k'(z)}$$

$$= \lim_{z\to i} \frac{2z+3}{3z^2+1}.$$

$$= \frac{2i+3}{-2} \text{ //}$$

$$\text{Res}\{f(z), -i\} = \lim_{z \to -i} \frac{h(z)}{k'(z)}$$

$$= \frac{-2i+3}{+3(-i)^2+1} = \frac{-2i+3}{-2}$$

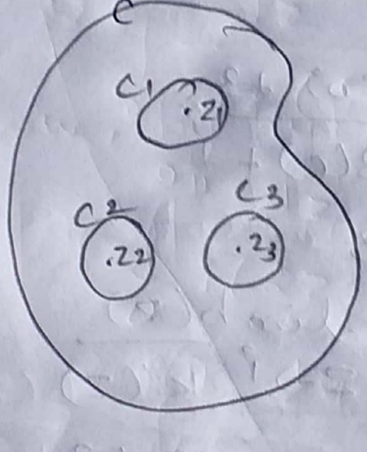
$$= +\left(\frac{-3+2i}{+2}\right)$$

$$= \frac{2i-3}{2}$$

2) Cauchy's Residue theorem:-

Let $f(z)$ be a function which is analytic inside and on a simple closed curve points $z_1, z_2 \ldots \ldots z_n$ inside C.

Then $\int_C f(z)\, dz = 2\pi i \text{ Res}\{f(z); z_j\}$.



Proof:- Let $C_1, C_2, \ldots C_n$ be circles with centres $z_1, z_2 \ldots z_n$.

By Cauchy's theorem for

$$\int_C f(z)\, dz = \int_{C_1} f(z) \cdot dz + \int_{C_2} f(z)\, dz + \ldots$$

$= 2\pi i \, Res\{f(z); z_1\} + 2\pi i \, Res\{f(z); z_2\}$
$+ \cdots + 2\pi i \, Res\{f(z), z_n\}.$

$= 2\pi i \sum_{j=1}^{n} Res\{f(z); z_j\}$

Hence proved//.

3) Fundamental theorem.

St:- A polynomial of degree n with complex co-efficients has n zero in c.

Proof :-

let $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z_n.$

where $a_n \neq 0$. polynomial of degree n.

let $f(z) = a_n z^n$.

$g(z) = a_0 + a_1 + \cdots + a_{n-1} z^{n-1}$

$\lim_{z \to \infty} \frac{g(z)}{f(z)} = 0$

such that $\left| \frac{g(z)}{f(z)} \right| < 1 \ \forall z$.

with $|z| > r$.

Hence by the Rouche's theorem,

$f(z)$, $g(z)$ and $f(z) + g(z)$.

Hence the given polynomial $f(z)+g(z)$ also has n zeros.

4) Evaluate $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$.

Soln:- $I = \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$.

$z = e^{i\theta}$

$dz = e^{i\theta}(i)\, d\theta$

$= iz\, d\theta$

$d\theta = dz/iz$

$I = \int_0^{2\pi} \frac{dz}{iz\left[5+4\left(\frac{z+z^{-1}}{2}\right)\right]}$

$= \int_0^{2\pi} \frac{dz}{iz(5+2z+2z^{-1})}$

$= \int_0^{2\pi} \frac{dz}{i(2z^2+5z+2)}$

$= \int_0^{2\pi} \frac{dz}{i[2z(z+2)+(2+z)]}$

$= \int_0^{2\pi} \frac{dz}{i(2z+1)(z+2)}$.

$2z+1=0 \Rightarrow z=-1/2$

$z=-2$.

$$\text{Res}\{f(z);-1/2\} = \lim_{z\to -1/2} \frac{1}{i(z+2)}.$$

$$= \frac{1}{i(3/2)}$$

$$= \frac{2}{3i}$$

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}(2/3i)$$

$$= -1/3\pi.$$

10 Marks

1) Rouche's theorem:

If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve c and if $|g(z)|<|f(z)|$ on c have the same number of zeros inside c.

proof:- $f(z)+g(z) = f(z)\left(1+\frac{g(z)}{f(z)}\right)$

$$= f(z)\phi(z) \longrightarrow (1)$$

$$\phi(z) = 1+\frac{g(z)}{f(z)}$$

Hence $[f(z)+g(z)]' = f'(z)+g'(z)$

$= f'(z)\phi(z) + g'(z)\phi'(z) \rightarrow (2)$

$\frac{(2)}{(1)} \Rightarrow \frac{f'(z)+g'(z)}{f(z)+g(z)} = \frac{f'(z)\phi(z)+g'(z)\phi'(z)}{f(z)\phi(z)}$

$= \frac{f'(z)\phi(z) + f(z)\phi(z)}{f(z)\phi(z)}$

$= \frac{f'(z)\cancel{\phi(z)}}{f(z)\cancel{\phi(z)}} + \frac{\cancel{f(z)}}{\cancel{f(z)}}\frac{\phi'(z)}{\phi(z)}$

$= \frac{f'(z)}{f(z)} + \frac{\phi'(z)}{\phi(z)}$

$\frac{1}{2\pi i}\int_C \left[\frac{f'(z)+g'(z)}{f(z)+g(z)}\right] dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}$

$+ \frac{1}{2\pi i}\int_C \frac{\phi'(z)}{\phi(z)} d$

$\rightarrow (3)$

now, by hypothesis,

$|g(z)| < |f(z)|$

$\frac{g(z)}{f(z)} < 1$ on c

$\therefore |\phi(z)-1| < 1$ on c.

$\therefore \phi(z) \neq 0$ for every point z

inside C:

Hence,

$$\int_C \frac{\phi'(z)}{\phi(z)}\,dz = \text{number of zeros of } \phi(z) \text{ with in } C$$

$$= 0.$$

The number of zeros of $f(z)+g(z)$ and $f(z)$ inside C.

Hence proved.