Subject: Differential Equations and Laplace Transforms

Subject code: 16SCCMM3

I Section: A (2 mark question and answers)

1. What is Order and degree of the differential Equation.

The Order of the differential equation is the Order of the highest derivative occuring in the equation.

The degree of the differential equation is the degree of the highest order derivative occuring in the equation.

2. If $y''+4y=6x$, find order and degree of the differential equation.

The given equation is $y''+4y=6x$

Order $= 2$

degree $= 1$

3. Solve $p^2-5p+6=0$

$$p^2-5p+6=(p-3)(p-2)=0$$

$$\Rightarrow (p-3)=0,\ p-2=0$$

$$p=3,\ (\text{or})\ p=2$$

$$\text{ie. } \frac{dy}{dx}=3\ (\text{or})\ \frac{dy}{dx}=2$$

$+6$

-5

$-3 \quad -2$

$dy = 3dx$ (or) $dy = 2dx$

$\int dy = \int 3dx$ (or) $\int dy = \int 2dx$

$y = 3x + c$ (or) $y = 2x + c$

The general solution is

$$(y - 3x - c)(y - 2x - c) = 0.$$

4. What is clairaut's form.

The equation $y = px + f(p)$ is known as clairaut's form.

5. Solve $p = \log(px - y)$

Sol:

$$p = \log(px - y)$$

Take exponential on both side

$$e^p = e^{\log(px - y)}$$

$$e^p = px - y$$

$$\therefore y = px - e^p$$

This is of the form $y = px + f(p)$

The general solution is $y = cx - e^c$ (putting $p = c$).

6. Define Ordinary Differential Equations

A differential equation which contains the derivative or differentials of one or more dependent variables with respect to a single independent variable is called an

Ordinary differential Equation.

Ex:

(i) $\frac{dy}{dx} = x+y$

(ii) $(y-1)dx + xdy = 1$.

7. Solve $(D^2+D-6)y=0$.

Sol:

-6
1
-2 +3

$m^2+m-6=0$

$(m-2)(m+3)=0$

$m-2=0$ (or) $(m+3)=0$

$m=+2$ (or) $m=-3$.

Complimentary function is

$y = Ae^{2x} + Be^{-3x}$.

8. What is total differential Equation.

An eqnation of the form $Pdx + Qdy + Rdz = 0$ where P, Q, R are functions of x, y, z is called single (or) total differential equation.

9. Solve $(D^2+4D+3)y = e^{-3x}$.

Sol:

3
4
1 3

$m^2+4m+3=0$

$(m+1)(m+3)=0$

$m+1=0$, $m+3=0$

$m=-1$ (or) $m=-3$

$C.F = Ae^{-x} + Be^{-3x}$.

$$P.I = \frac{1}{D^2+4D+3} e^{-3x}$$

$$= \frac{1}{(-3)^2+4(-3)+3} e^{-3x}$$

$$= \frac{1}{9-12+3} e^{-3x}$$

$$= \frac{1}{(D+1)(D+3)} e^{-3x}$$

$$= \frac{1}{(-3+1)(-3+3)} e^{-3x}$$

$$= \frac{1}{-2} x e^{-3x}$$

$$= -x/2 \, e^{-3x}$$

$$\therefore y = Ae^{-x} + Be^{-3x} - x/2 \, e^{-3x}.$$

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10. Solve $(x^2D^2 - 3xD + 4)y = 0$

Sol:

Put $z = \log x$

$(x^2D^2 - 3xD + 4)y = 0$ becomes

$(\theta(\theta-1) - 3\theta + 4)y = 0$

$(\theta^2 - \theta - 3\theta + 4)y = 0$

$(\theta^2 - 4\theta + 4)y = 0$

The auxillary equation is

$m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$(m-2)^2 = 0$

+4
-4
-2 -2

$m = 2, 2$

$\therefore y = (A + Bz)e^{2z}$

$y = (A + B \log x)x^2$.

11. What is Partial differential equation.

Partial differential Equation is one which contains partial derivatives of the dependent variable with respect to two or more independent variables.

12. What is Order and degree of the Partial Differential Equation.

The Order of the PDE is the Order of the highest derivative occuring in it.

The degree of the PDE is the degree of the highest partial derivative occuring in it.

13. Form the PDE by eliminating the arbitrary constants a and b from $z = (x+a)(y+b)$.

Sol:

$z = (x+a)(y+b) \longrightarrow (1)$

Differentiate partially w.r.to 'x',

$\partial z/\partial x = (y+b)(1+0)$

$p = y + b \longrightarrow (2)$

Differentiate partially w.r.to 'y',

$\frac{\partial z}{\partial y} = (x+a)(1+0)$

$$q = x + a \longrightarrow (3)$$

Equations (2) and (3) substitute in (1),

$$\boxed{z = pq}$$

14. Find the PDE of all spheres of radius r units having their centres on XOY plane

Sol:

The equation of a sphere with centre at (a, b, c) and radius r is given by

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

On XOY plane, z-coordinate is zero. If the center lies on XOY plane then $c = 0$.

Hence, the above eqn becomes

$$(x-a)^2 + (y-b)^2 + (z-0)^2 = r^2 \longrightarrow (1)$$

Differentiate (1) partially w.r to x

$$2(x-a) + 0 + 2z \, \partial z/\partial x = 0$$

$$2(x-a) + 2zp = 0$$

$$\cancel{2}(x-a) = -\cancel{2}zp$$

$$x - a = -zp \longrightarrow (2)$$

Differentiate (1) partially w.r to y

$$2(y-b) + 2z \, \partial z/\partial y = 0$$

$$2(y-b) + 2zq = 0$$

$$\cancel{2}(y-b) = -\cancel{2}zq$$

$$y - b = -zq \longrightarrow (3)$$

Using the equations (2) and (3) in (1) we get

$$(-zp)^2 + (-zq)^2 + z^2 = r^2$$

$$z^2p^2 + z^2q^2 + z^2 = r^2$$

$$z^2(p^2+q^2+1) = r^2$$

15. Eliminate the arbitrary function ϕ from $z = x^2\phi(x-y)$.

Sol:

$$z = x^2\phi(x-y) \longrightarrow (1)$$

Differentiating (1) partially w.r to 'x'

$$\frac{\partial z}{\partial x} = x^2\phi'(x-y) + \phi(x-y)(2x) \longrightarrow (2)$$

(ie.,) $p = x^2\phi(x-y) + \phi(x-y)(2x) \longrightarrow (2)$

Differentiating (1) partially w.r to 'y'

$$\frac{\partial z}{\partial y} = x^2\phi'(x-y)(-1)$$

$$q = x^2\phi(x-y)(-1) \longrightarrow (3)$$

Using eqrations (1) and (2) in (3) we get

$$2z = (p+q)x.$$

16. What is clairaut's form.

An equation of the form $z = px + qy + f(p,q)$ is known as clairaut's form.

The complete integral will be

$$z = ax + by + f(a,b).$$

17. What is Lagrange's linear partial differential equation.

An eqration of the form $Pp + qQ = R$ where P, Q and R are functions of x, y and z is called Lagrange's linear partial differential equation.

18. Find Lagrange's auxillary equation?

Lagrange's auxillary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

19. Define Laplace Transform.

Let $f(t)$ be a function of t defined for $t \geq 0$. The Laplace transform of $f(t)$ associates a function of s defined by the equation

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

The symbol L is called the Laplace transform operator and s is a parameter.

20. Find $L[\sinh at]$.

Sol:

$$L(\sinh at) = L\left[\frac{e^{at} - e^{-at}}{2}\right]$$

$$\frac{1}{2}\left[L(e^{at}) - L(e^{-at})\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right].$$

$$= \frac{1}{2}\left[\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}.$$

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21. Evaluate $L[t^2-3t+2]$.

Solution:

$$L[t^2-3t+2] = L[t^2] - 3L[t] + 2L[1]$$

$$= \frac{3!}{s^4} - 3\left(\frac{1}{s^2}\right) + 2\left(\frac{1}{s}\right)$$

$$= \frac{6-3s^2+2s^3}{s^4}$$

$$L(t^3-3t+2) = \frac{2s^3-3s^2+6}{s^4}.$$

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22. Find $L(t\cos 3t)$

Sol:

$$f(t) = \cos 3t$$

$$F(s) = L(f(t)) = L(\cos 3t) = \frac{s}{s^2+9};$$

$$F'(s) = \frac{d}{ds}\left(\frac{s}{s^2+9}\right).$$

$$= \frac{(s^2+9)(1) - s(2s)}{(s^2+9)^2}$$

$$F'(s) = \frac{-s^2+9}{(s^2+9)^2}$$

$$-F'(s) = \frac{s^2-9}{(s^2+9)^2} = L(t\cos 3t).$$

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23. Evaluate $L\left(\frac{1-e^{-t}}{t}\right)$.

Sol:

$$L(f(t)) = L(1-e^{-t})$$

$$= L(1) - L(e^{-t}) = \frac{1}{s} - \frac{1}{s+1}.$$

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty \frac{1}{s}\,ds - \int_s^\infty \frac{1}{s+1}\,ds$$

$$= \left[\log s - \log(s+1)\right]_s^\infty$$

$$= -\log s + \log(s+1)$$

$$= \log\left(\frac{s+1}{s}\right).$$

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24. Find $L^{-1}\left[\frac{1}{s^2+2s+2}\right]$.

Sol:

$$L^{-1}\left[\frac{1}{s^2+2s+2}\right] = L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$Dr = s^2+2s+1+1 = s^2+2s+2$

$$= e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right].$$ [$\because$ s is shifted to s+1 using shifting Theorem]

$$= e^{-t}\sin t.$$

25. Evaluate $L^{-1}\left[\frac{s}{(s+2)^2}\right]$.

Sol:

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}(sF(s)) = \frac{d}{dt}L^{-1}(F(s)).$$

where $F(s) = \frac{1}{(s+2)^2}$ $\left[\because L(t^n) = \frac{n!}{s^{n+1}}\right]$.

$$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{(s+2)^2}\right).$$

$$F(s) = e^{-2t}L^{-1}\left[\frac{1}{s^2}\right] = e^{-2t}t.$$

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = \frac{d}{dt}\left[e^{-2t}t\right]$$

$$= t(-2e^{-2t}) + e^{-2t}$$

$$= e^{-2t}(1-2t).$$

II Section - B (5 mark question and answers)

1. Solve $2\left(\frac{dy}{dx}\right)^2 + xy^2 = (x+2y^2)\frac{dy}{dx}$.

Sol:

Since $p = \frac{dy}{dx}$, the given equation is written as

$$2p^2 + xy^2 = (x+2y^2)p.$$

$$2p^2 + xy^2 = xp + 2y^2p$$

$$2p^2 + xy^2 - xp - 2y^2p = 0$$

$$2p^2 - 2y^2p - xp + xy^2 = 0$$

$$2p(p-y^2) - x(p-y^2) = 0.$$

$$(p-y^2)(2p-x) = 0.$$

$p = y^2$ (or) $p = x/2$

$\frac{dy}{dx} = y^2$ (or) $\frac{dy}{dx} = x/2$

$\frac{dy}{y^2} = dx$ (or) $2dy = x\,dx$

$\int \frac{dy}{y^2} = \int dx$ (or) $\int 2dy = \int x\,dx$

$-\frac{1}{y} = x + c$ (or) $2y = x^2/2 + c$

The general solution is

$$\left(-\frac{1}{y} - x - c\right)\left(2y - x^2/2 - c\right) = 0.$$

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2. Solve $p^2 - 2p\cosh x + 1 = 0$.

Sol:

$p^2 - 2p\cosh x + 1 = 0.$

$p^2 - 2p\left(\frac{e^x + e^{-x}}{2}\right) + 1 = 0.$

$p^2 - p(e^x + e^{-x}) + 1 = 0$

$(p - e^x)(p - e^{-x}) = 0.$

$p = e^x$ (or) $p = e^{-x}$

$\frac{dy}{dx} = e^x$ (or) $\frac{dy}{dx} = e^{-x}$

$dy = e^x dx$ (or) $dy = e^{-x} dx$

$\int dy = \int e^x dx$ (or) $\int dy = \int e^{-x} dx$

$y = e^x + c$ (or) $y = -e^{-x} + c$

The solution of the given equation is

$$(y - e^x - c)(y + e^{-x} - c) = 0.$$

3. Solve $y = x + p^2 - 2p$.

Solution:

$$y = x + p^2 - 2p \longrightarrow (1)$$

Differentiate (1) w.r.to 'x' we get

$$\frac{dy}{dx} = 1 + 2p\frac{dp}{dx} - \frac{2dp}{dx}.$$

$$p = 1 + 2p\frac{dp}{dx} - 2\frac{dp}{dx}.$$

$$(p-1) = 2(p-1)\frac{dp}{dx}.$$

$$\frac{\cancel{p-1}}{\cancel{p-1}} = 2\frac{dp}{dx}$$

$$1 = 2\frac{dp}{dx},$$

$$dx = 2dp.$$

Integrating, we get

$$\int dx = 2\int dp + c$$

$$x = 2p + c$$

$$x - c = 2p$$

$$p = \frac{x-c}{2} \longrightarrow (2)$$

put $p = \frac{x-c}{2}$ in (1) we get

$$y = x + \left(\frac{x-c}{2}\right)^2 - \cancel{2}\left(\frac{x-c}{\cancel{2}}\right).$$

$$y = x + \frac{(x-c)^2}{4} - (x-c)$$

$$y = \cancel{x} - \cancel{x} + c + \frac{(x-c)^2}{4}$$

$$y = c + \frac{(x-c)^2}{4}.$$

$$4y = (x-c)^2 + 4c.$$

4. Solve $y = \sin p - p\cos p$.

Solution:

$$y = \sin p - p\cos p \longrightarrow (1)$$

Differentiate (1) w.r.to 'x' we get

$$\frac{dy}{dx} = \cos p\frac{dp}{dx} - \left[p(-\sin p)\frac{dp}{dx} + \cos p\frac{dp}{dx}\right].$$

$$= \cancel{\cos p\frac{dp}{dx}} + p\sin p\frac{dp}{dx} - \cancel{\cos p\frac{dp}{dx}}.$$

$$\frac{dy}{dx} = p\sin p\frac{dp}{dx}$$

$$p = p\sin p\frac{dp}{dx}.$$

$$1 = \sin p\frac{dp}{dx}$$

$$dx = \sin p\, dp.$$

Integrating we get

$$\int dx = \int \sin p\, dp$$

$$x = -\cos p + c$$

$$\cos p = -(x+c)$$

$$p = \cos^{-1}(-(x+c)).$$

$$\sin p = \sqrt{1-\cos^2 p} = \sqrt{1-(x+c)^2}$$

$y = \sin p - p\cos p$ becomes

$$y = \sqrt{1-(x+c)^2} + \cos^{-1}[-(x+c)](x+c).$$

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5. Solve $y = 2px + y^2p^3$.

Sol:

$$y = 2px + y^2p^3 \longrightarrow (1)$$

$$y - y^2p^3 = 2px$$

$$x = \frac{y - y^2p^3}{2p}$$

Differentiate w.r.to y

$$\frac{dx}{dy} = \frac{1}{2}\left[\frac{p\left(1 - 3y^2p^2\frac{dp}{dy} - 2yp^3\right) - (y - y^2p^3)\frac{dp}{dy}}{p^2}\right].$$

$$\frac{1}{p} = \frac{p - 3y^2p^3\frac{dp}{dy} - 2yp^4 - y\frac{dp}{dy} + y^2p^3\frac{dp}{dy}}{2p^2}$$

$$2p = p - 3y^2p^3\frac{dp}{dy} - 2yp^4 - y\frac{dp}{dy} + y^2p^3\frac{dp}{dy}.$$

$$2p - p + 3y^2p^3\frac{dp}{dy} + 2yp^4 + y\frac{dp}{dy} - y^2p^3\frac{dp}{dy} = 0.$$

$$p + 3y^2p^3\frac{dp}{dy} + 2yp^4 + y\frac{dp}{dy} - y^2p^3\frac{dp}{dy} = 0.$$

$$p + 2y^2p^3\frac{dp}{dy} + 2yp^4 + y\frac{dp}{dy} = 0.$$

$$p.(1 + 2yp^3) + y\,{}^{dp}/_{dy}(1 + 2yp^3) = 0.$$

$$(1+2yp^3)\left(p+y\frac{dp}{dy}\right)=0.$$

$p+y\frac{dp}{dy}=0$ gives

$$\frac{d}{dy}(py)=0$$

$$py=0 \longrightarrow (2)$$

put $p=c/y$ in (1) we get

$$y=\frac{2c}{y}x+y^2\frac{c^3}{y^3}.$$

$$y=\frac{2cx}{y}+\frac{y^2c^3}{y^3}.$$

$$\boxed{y^2=2cx+c^3}$$

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6. Solve $x^2(y-px)=yp^2$

Solution:

$$x^2(y-px)=yp^2$$

put $x^2=u$ and $y^2=v$

$$x=\sqrt{u} \text{ and } y=\sqrt{v}$$

$$dx=\frac{1}{2\sqrt{u}}du \text{ and } dy=\frac{1}{2\sqrt{v}}dv$$

$$p=\frac{dy}{dx}=\frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}.$$

$x^2(y-px)=yp^2$ becomes

$$u\left(\sqrt{v} - \frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}\sqrt{u}\right) = \sqrt{v}\,\frac{u}{v}\left(\frac{dv}{du}\right)^2$$

$$u\sqrt{v} - \frac{u^2}{\sqrt{v}}\frac{dv}{du} = \frac{u}{\sqrt{v}}\left(\frac{dv}{du}\right)^2$$

$$uv - u^2\frac{dv}{du} = u\left(\frac{dv}{du}\right)^2$$

$$v - u\frac{dv}{du} = \left(\frac{dv}{du}\right)^2$$

$$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$$

$$v = pu + p^2 \text{ where } p = \frac{dv}{du}.$$

This is of the form $v = pu + f(p)$.

The general solution is

$$v = cu + c^2 \quad (p = c)$$

$$y^2 = cx^2 + c^2$$

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7. Solve the boundary value problem

$y'' - 10y' + 25y = 0$, $y(0) = 1$, $y(1) = 0$.

Solution:

$$(D^2 - 10D + 25)y = 0.$$

The auxiliary equation is $m^2 - 10m + 25 = 0$

$+25$

-10

$-5 \quad -5$

$$(m-5)(m-5) = 0$$

$$m - 5 = 0 \text{ (or) } m - 5 = 0$$

$$m = 5, 5$$

The solution is $y = (A + Bx)e^{5x} \longrightarrow (1)$

When $x = 0$, $y = 1$ and when $x = 1$, $y = 0$.

$$1 = (A + 0)e^{0}$$

$$\boxed{A = 1}$$

$$0 = (A + B(1))e^{5(1)}$$

$$0 = (A + B)e^{5}$$

$$\boxed{B = -1}$$

$\therefore$ (1) becomes $y = (1 - x)e^{5x}$

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8. Solve $(D^2 - 6D + 9)y = 3^x + \log 2$.

Solution:

To find C.F

$$(D^2 - 6D + 9)y = 0$$

The auxiliary equation is

$+9$, -6, -3, -3

$$m^2 - 6m + 9 = 0$$

$$(m - 3)(m - 3) = 0$$

$$(m - 3)^2 = 0$$

$$m = 3, 3.$$

$$C.F = (A + Bx)e^{3x}.$$

To find P.I

$$P.I_1 = \frac{1}{D^2 - 6D + 9} 3^x$$

$$= \frac{1}{(\log 3)^2 - 6\log 3 + 9} 3^x.$$

$$P.I_2 = \frac{1}{D^2-6D+9}\log 2$$

$$= \frac{1}{9}\log 2 \quad (\because D = 0)$$

The solution is

$$y = (A+Bx)e^{3x} + \frac{1}{(\log 3)^2 - 6\log 3 + 9}3x + \frac{1}{9}\log 2.$$

9. Solve $(D^2-3D+2)y = \cos 3x \cos 2x$

Solution:

To find C.F

$$(D^2-3D+2)y = 0.$$

The auxiliary equation is

$$m^2-3m+2=0$$

$$(m-1)(m-2)=0$$

$$m = 1, 2.$$

$$C.F = Ae^{x} + Be^{2x}.$$

To find P.I

$$P.I = \frac{1}{D^2-3D+2}\cos 3x \cos 2x \quad \left[\because \cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]\right].$$

$$= \frac{1}{2}\left[\frac{1}{D^2-3D+2}(\cos 5x + \cos x)\right]$$

$$= \frac{1}{2}\frac{1}{D^2-3D+2}\cos 5x + \frac{1}{2}\frac{1}{D^2-3D+2}\cos x$$

$$= \frac{1}{2}\frac{1}{-25-3D+2}\cos 5x + \frac{1}{2}\frac{1}{-1-3D+2}\cos x$$

$$= \frac{1}{2}\frac{1}{-3D-23}\cos 5x + \frac{1}{2}\frac{1}{1-3D}\cos x$$

$$= -\frac{1}{2}\frac{1}{3D+23}\cos 5x + \frac{1}{2}\left(\frac{1}{1-3D}\right)\cos x.$$

$$= -\frac{1}{2}\frac{(3D-23)}{9D^2-529}\cos 5x + \frac{1}{2}\left[\frac{1}{1-3D}\times\frac{1+3D}{1+3D}\right)\cos x$$

$$= \frac{-1}{2}\frac{(3D-23)}{9(-25)-529}\cos 5x + \frac{1}{2}\left[\frac{1+3D}{1-9D^2}\right]\cos x$$

$$= \frac{-1}{1508}\left(-15\sin 5x - 23\cos 5x\right) + \frac{1}{20}\left(\cos x - 3\sin x\right)$$

$$= \frac{1}{1508}\left(15\sin 5x + 23\cos 5x\right) + \frac{1}{20}\left(\cos x - 3\sin x\right).$$

The solution is

$$y = Ae^{x} + Be^{2x} + \frac{1}{1508}\left(15\sin 5x + 23\cos 5x\right) + \frac{1}{20}\left(\cos x - 3\sin x\right)$$

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10. Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2\log x.$

Sol:

$$(x^2D^2 - xD - 3)y = x^2\log x.$$

Put $z = \log x$

Equation (1) becomes

$$[\theta(\theta-1) - \theta - 3]y = e^{2z}z$$

$$(\theta^2 - 2\theta - 3)y = e^{2z}z$$

To find C.F

$$(\theta^2 - 2\theta - 3)y = 0$$

-3

-2

$-3 \quad +1$

The auxiliary equation is

$$m^2 - 2m - 3 = 0.$$

$$(m-3)(m+1) = 0$$

$$m - 3 = 0, \quad m + 1 = 0$$

$$m = 3, \quad m = -1$$

$$C.F = Ae^{3z} + Be^{-z}.$$

To find P.I $= \dfrac{1}{\theta^2 - 2\theta - 3} e^{2z} z$

$$= e^{2z} \frac{1}{(\theta+2)^2 - 2(\theta+2) - 3} z$$

$$= e^{2z} \frac{1}{\theta^2 + 2\theta - 3} z$$

$$= e^{2z} \frac{1}{-3\left(1 - \frac{2\theta}{3} - \frac{\theta^2}{3}\right)} z$$

$$= \frac{-e^{2z}}{3}\left(1 - \frac{2\theta}{3}\right)^{-1} z$$

$$= \frac{-e^{2z}}{3}\left(1 + \frac{2\theta}{3} + \frac{4\theta^2}{9} + \cdots\right) z$$

$$= -\frac{e^{2z}}{3}\left(z + \frac{2}{3}\right).$$

The solution is $y = Ae^{3z} + Be^{-z} - \dfrac{e^{2z}}{3}\left(z + 2/3\right).$

$$y = Ax^3 + \frac{B}{x} - \frac{x^2}{3}\left(\log x + 2/3\right).$$

11. Solve $(x^3D^2 + 3x^2D + 5x)y = 2$.

Solution:

$$(x^3D^2 + 3x^2D + 5x)y = 2$$

Dividing by x

$$(x^2D^2 + 3xD + 5)y = \frac{2}{x}.$$

Put $z = \log x$

$$(\theta(\theta-1) + 3\theta + 5)y = 2e^{-z}.$$

$$(\theta^2 + 2\theta + 5)y = 2e^{-z}.$$

To find C.F.

$$(\theta^2 + 2\theta + 5)y = 0.$$

The auxiliary equation is

$$m^2 + 2m + 5 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $a = 1$, $b = 2$, $c = 5$

$$m = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2 \times 1}$$

$$= \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$m = -1 \pm 2i$

$C.F = e^{-z}(A\cos 2z + B\sin 2z)$

To find P.I.

$$P.I = \frac{1}{\theta^2 + 2\theta + 5} 2e^{-z}$$

$$= \frac{1}{2} e^{-z} \quad (\theta = -1)$$

The solution is

$$y = e^{-z}(A\cos 2z + B\sin 2z) + \frac{1}{2}e^{-z}.$$

$$y = \frac{1}{x}(A\cos(2\log x) + B\sin(2\log x)) + \frac{1}{2x}.$$

---

12. Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + a^2y = \sec ax$$

Solution:

$$(D^2 + a^2)y = \sec ax.$$

To find C.F

$$(D^2 + a^2)y = 0.$$

The auxiliary equation is

$$m^2 + a^2 = 0$$

$$m^2 = -a^2, \quad m = \pm ia$$

$C.F = \cos ax + C_2 \sin ax.$

Let $f_1 = \cos ax$, $f_2 = \sin ax.$

$f_1' = -a\sin ax$, $f_2 = a\cos ax.$

$$W = f_1 f_2' - f_2 f_1' = a$$

$$P = -\int \frac{\sin ax \sec ax}{a}\,dx$$

$$= -\frac{1}{a}\int \tan ax\,dx$$

$$= -\frac{1}{a^2}\log \sec ax + C_1$$

$$Q = \int \frac{\cos ax \sec ax}{a}\,dx = \frac{1}{a}\int dx$$

$$= \frac{x}{a} + C_2.$$

The solution is

$$y = \left(-\frac{1}{a^2}\log \sec ax + C_1\right)\cos ax + \left(\frac{x}{a} + C_2\right)\sin ax$$

$$y = C_1 \cos ax + C_2 \sin ax + \frac{x}{a}\sin ax - \frac{\cos ax}{a^2}\log \sec ax.$$

---

13. Solve $z = px + qy + \sqrt{1+p^2+q^2}$.

<u>Solution:-</u>

Complete integral is $z = ax + by + \sqrt{1+a^2+b^2}$

<u>Singular integral</u>

$$z = ax + by + \sqrt{1+a^2+b^2} \longrightarrow (1)$$

Differentiating (1) partially w.r.to a

$$0 = x + \frac{a}{\sqrt{1+a^2+b^2}}$$

$$x = -\frac{a}{\sqrt{1+a^2+b^2}}$$

Differentiating ① partially w.r.to b

$$0 = y + \frac{b}{\sqrt{1+a^2+b^2}}$$

$$y = \frac{-b}{\sqrt{1+a^2+b^2}}$$

$$x^2 + y^2 = \frac{a^2}{1+a^2+b^2} + \frac{b^2}{1+a^2+b^2}$$

$$x^2+y^2 = \frac{a^2+b^2}{1+a^2+b^2} = 1 - \frac{1}{1+a^2+b^2}$$

$$\frac{1}{1+a^2+b^2} = 1 - x^2 - y^2$$

$$1+a^2+b^2 = \frac{1}{1-x^2-y^2}$$

$$a = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

Substituting the values of a and b in ①

We get

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

General integral

$$z = ax + by + \sqrt{1+a^2+b^2}$$

Put $b = g(a)$

$$z = ax + g(a)y + \sqrt{1+a^2+(g(a))^2}$$

Differentiating partially w.r to a

$$0 = x + yg'(a) + \frac{a + g(a)g'(a)}{\sqrt{1+a^2+(g(a))^2}} \longrightarrow (3)$$

Eliminating a between (2) and (3) we get the general integral.

---

14. Solve $p - q = \log(x+y)$

Sol:

given: $p - q = \log(x+y)$

The auxiliary equation is

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)}$$

Consider $dx = -dy$

Integrating on both sides

$$\int dx = -\int dy$$

$$x = -y + a$$

$$x + y = a \longrightarrow (1)$$

Taking $-dy = \dfrac{dz}{\log(x+y)}$

Integrating on both sides

$$-\int dy = \int \frac{dz}{\log(x+y)}$$

$$-y = \frac{z}{\log(x+y)} + b$$

Since $\log(x+y)$ is a constant.

$$-y - \frac{z}{\log(x+y)} = b \longrightarrow (2)$$

The solution of the given equation is

$$f\left(x+y,\ -y-\frac{z}{\log(x+y)}\right) = 0.$$

---

15. Solve $x^2(y-z)p + y^2(z-x) = z^2(x-y)$.

Solution:

The auxiliary equation is

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Let the multipliers be $\frac{1}{x}, \frac{1}{y}, 1/z$.

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{x(y-z) + y(z-x) + z(x-y)}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \quad \left[\because \frac{1}{x}P + \frac{1}{y}Q + \frac{1}{z}R = 0\right].$$

Integrating on both sides

$$\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log a$$

$$\log(xyz) = \log a$$

$$xyz = a \longrightarrow (1)$$

Let the multipliers be $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}}{\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y}}$$

$$\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2} = 0 \cdot \left(\because \frac{1}{x^2}P+\frac{1}{y^2}Q+\frac{1}{z^2}R=0\right)$$

Integrating on both sides

$$\int\frac{dx}{x^2}+\int\frac{dy}{y^2}+\int\frac{dz}{z^2} = 0$$

$$-\frac{1}{x}-\frac{1}{y}-\frac{1}{z} = -b$$

$$+\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = +b$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = b \longrightarrow (2)$$

The solution of the given equation is

$$f\left(xyz, \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 0.$$

---

16. ~~Solve~~ Find the Laplace transform of $\left(\frac{e^{-at}-e^{-bt}}{t}\right)$.

Solution:

$$L\left(\frac{e^{-at}-e^{-bt}}{t}\right) = L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)\,ds$$

$$f(t) = e^{-at}-e^{-bt}$$

$$L(f(t)) = L\left(e^{-at}-e^{-bt}\right).$$

$$L(f(t)) = L(e^{-at}) - L(e^{-bt})$$

$$= \frac{1}{s+a} - \frac{1}{s+b}$$

$$\therefore L\left(\frac{f(t)}{t}\right) = \int_s^\infty L(f(t))\,ds$$

$$= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \int_s^\infty \frac{ds}{s+a} - \int_s^\infty \frac{ds}{s+b}.$$

$$= \left(\log(s+a) - \log(s+b)\right)_s^\infty$$

$$= \log\left(\frac{s+a}{s+b}\right)_s^\infty$$

$$= 0 - \log\left(\frac{s+a}{s+b}\right).$$

$$= \log\left(\frac{s+b}{s+a}\right).$$

---

17. Solve $L^{-1}\left(\frac{s^2-s+2}{s(s+2)(s-3)}\right)$.

Solution:

Let us resolve into partial fractions

$$\frac{s^2-s+2}{s(s+2)(s-3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-3}.$$

$$\frac{s^2-s+2}{s(s+2)(s-3)} = \frac{A(s+2)(s-3)+Bs(s-3)+Cs(s+2)}{s(s+2)(s-3)}$$

$$s^2-s+2 = A(s+2)(s-3)+Bs(s-3)+Cs(s+2).$$

To find the value of A, B, C

put $s=0$,

$$2 = A(-6)$$

$$A = \frac{-2}{6} = \frac{-1}{3}.$$

Put $s=3$,

$$9-3+2 = C3(3+2)$$

$$8 = 15C$$

$$C = 8/15$$

Put $s=-2$

$$(-2)^2+2+2 = B(-2)(-2-3).$$

$$4+4 = 10B$$

$$8 = 10B$$

$$B = 8/10 = 4/5$$

Substituting the values of A, B and C we get

$$\frac{s^2-s+2}{s(s+2)(s+3)} = \frac{-1}{3}\frac{1}{s} + \frac{4}{5}\frac{1}{s+2} + \frac{8}{15}\frac{1}{s-3}.$$

$$L^{-1}\left(\frac{s^2-s+2}{s(s+2)(s+3)}\right) = \frac{-1}{3}L^{-1}\left(\frac{1}{s}\right) + \frac{4}{5}L^{-1}\left(\frac{1}{s+2}\right) + 8/15\, L^{-1}\left(\frac{1}{s-3}\right).$$

$$= \frac{-1}{3}(+1) + \frac{4}{5}(e^{-2t}) + \frac{8}{15}(e^{3t})$$

$$f(t) = \frac{8}{15}e^{3t} + \frac{4}{5}e^{-2t} - \frac{1}{3}.$$

---

18 Find $L^{-1}\left(\frac{s^2}{(s-2)^3}\right)$.

Solution:

$$L^{-1}\left(\frac{s^2}{(s-2)^3}\right) = L^{-1}(s^2F(s)) = \frac{d^2}{dt^2}L^{-1}(F(s)).$$

$$F(s) = \frac{1}{(s-2)^3}.$$

where $L^{-1}(F(s)) = L^{-1}\left(\frac{1}{(s-2)^3}\right)$.

$$= e^{2t}L^{-1}\left(\frac{1}{s^3}\right) \text{ (using Type II)}$$

$$= \frac{e^{2t}}{2!}L^{-1}\left(\frac{2!}{s^3}\right). \quad \left(\because L(t^2) = \frac{2!}{s^{2+1}}\right).$$

$$L^{-1}(F(s)) = \frac{e^{2t}}{2!}t^2$$

$$\therefore L^{-1}\left(\frac{s^2}{(s-2)^3}\right) = \frac{d^2}{dt^2}(L^{-1}(F(s))) = \frac{d^2}{dt^2}\left(\frac{e^{-2t}t^2}{2!}\right)$$

$$= \frac{1}{2!}\frac{d}{dt}(2te^{-2t} - 2e^{-2t}t^2).$$

$$= \frac{1}{2!}(2e^{-2t} - 4te^{-2t} - 4te^{-2t} + 4t^2e^{-2t}).$$

$$= \frac{e^{-2t}}{2!}\left(2-8t+4t^2\right).$$

$$= \frac{e^{-2t}}{\cancel{2}} \times \cancel{2}\left(1-4t+2t^2\right).$$

$$= e^{-2t}\left(1-4t+2t^2\right).$$

III Section: C (10 mark question and answers)

1. Solve $p(p+y) = x(x+y)$

Solution:

$$P(P+y) = x(x+y)$$

$$p^2+py = x^2+xy.$$

$$p^2+py-x^2-xy = 0.$$

$$p = \frac{-y \pm \sqrt{y^2+4(x^2+xy)}}{2}$$

$$= \frac{-y \pm \sqrt{4x^2+4xy+y^2}}{2}$$

$$= \frac{-y \pm \sqrt{(2x+y)^2}}{2}$$

$$\frac{dy}{dx} = \frac{-y \pm (2x+y)}{2}$$

$$\frac{dy}{dx} = \frac{-y+(2x+y)}{2} \quad (\text{or}) \quad \frac{dy}{dx} = \frac{-y-(2x+y)}{2}.$$

$$\frac{dy}{dx} = \frac{\cancel{-y} + 2x + \cancel{y}}{2} \quad (\text{or}) \quad \frac{dy}{dx} = \frac{-y - 2x - y}{2}$$

$$\frac{dy}{dx} = \frac{\cancel{2}x}{\cancel{2}} \quad (\text{or}) \quad \frac{dy}{dx} = \frac{-2y - 2x}{2}$$

$$\frac{dy}{dx} = x \quad (\text{or}) \quad \frac{dy}{dx} = \frac{+\cancel{2}(-y - x)}{\cancel{2}}$$

$$dy = x\,dx \quad (\text{or}) \quad \frac{dy}{dx} + y = -x$$

Take $dy = x\,dx$

$$\int dy = \int x\,dx$$

$$y = \frac{x^2}{2} + c \longrightarrow (1)$$

Take $\frac{dy}{dx} + y = -x$

Assume the solution of the equation

$\frac{dy}{dx} + Py = Q$ as

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

Here $P = 1$, $Q = -x$

$$e^{\int P dx} = e^{\int dx} = e^{x}$$

$\therefore$ The solution of $\frac{dy}{dx} + y = -x$ is

$$y e^{x} = \int (-x) e^{x} dx$$

$$= -[x e^{x} - e^{x}] + c$$

$$= -xe^x + e^x + c$$

$$\therefore ye^x + xe^x - e^x - c = 0. \longrightarrow (2)$$

From (1) and (2) the solution of the given equation is

$$\left(y - x^2/2 - c\right)\left(ye^x + xe^x - e^x - c\right) = 0.$$

2) Solve $xp^2 - 2yp + x = 0$.

Solution:

$$xp^2 - 2yp + x = 0 \longrightarrow (1)$$

$$xp^2 + x = 2yp$$

$$y = \frac{xp^2 + x}{2p}$$

$$y = \frac{x(1+p^2)}{2p}.$$

Differentiating w.r to x we get

$$p = \frac{2p\left[1 + 2xp\frac{dp}{dx} + p^2\right] - x(1+p^2)}{4p^2} \times 2\frac{dp}{dx}.$$

$$4p^3 = 2p + 4xp^2\frac{dp}{dx} + 2p^3 - 2x\frac{dp}{dx} - 2xp^2\frac{dp}{dx}.$$

$$2p^3 - 2p - 2xp^2\frac{dp}{dx} + 2x\frac{dp}{dx} = 0.$$

$$2p(p^2-1) - 2x\frac{dp}{dx}(p^2-1) = 0.$$

Dividing by (p^2-1) we get

$$2p - 2x\frac{dp}{dx} = 0.$$

$$2p = 2x\frac{dp}{dx}.$$

$$p = x\frac{dp}{dx}.$$

$$\frac{dx}{x} = \frac{dp}{p}$$

Integrating,

$$\log x = \log p + \log c$$

$$\log x = \log(pc)$$

$$x = pc \longrightarrow (2)$$

put $p = x/c$ in (1),

$$x\left(\frac{x^2}{c^2}\right) - 2y\left(\frac{x}{c}\right) + x = 0.$$

$$x^3 - 2xyc + xc^2 = 0.$$

$$x(x^2 - 2yc + c^2) = 0.$$

$$\Rightarrow \boxed{x^2 - 2yc + c^2 = 0}$$

---

3) Solve $p^3 - 4xyp + 8y^2 = 0.$

Solution:

$$p^3 - 4xyp + 8y^2 = 0 \longrightarrow (1)$$

Solving for x

$$4x = \frac{p^2}{y} + \frac{8y}{p}.$$

Differentiate w.r.to y

$$4\frac{dx}{dy} = \frac{-p^2}{y^2} + \frac{2p}{y}\frac{dp}{dy} + \frac{8}{p} - \frac{8y}{p^2}\frac{dp}{dy}$$

$$\frac{4}{p} = -\frac{p^2}{y^2} + \frac{2p}{y}\frac{dp}{dy} + \frac{8}{p} - \frac{8y}{p^2}\frac{dp}{dy} \quad \frac{4}{p} - \frac{p^2}{y^2}$$

$$+ \frac{2p}{y}\frac{dp}{dy} - \frac{8y}{p^2}\frac{dp}{dy} = 0.$$

$$\frac{(4y^2 - p^3)}{py^2} + \frac{dp}{dy}\left(\frac{2p}{y} - \frac{8y}{p^2}\right) = 0.$$

$$\frac{4y^2 - p^3}{py^2} + \frac{dp}{dy}\left(\frac{2p^3 - 8y^2}{yp^2}\right) = 0.$$

$$\frac{4y^2 - p^3}{py^2} + \frac{2}{p^2 y}\frac{dp}{dy}\left(p^3 - 4y^2\right) = 0.$$

$$\frac{4y^2 - p^3}{py^2} - \frac{2}{p^2 y}\frac{dp}{dy}\left(4y^2 - p^3\right) = 0.$$

$$(4y^2 - p^3)\left(\frac{1}{py^2} - \frac{2}{p^2y}\frac{dp}{dy}\right) = 0.$$

$\frac{1}{py^2} - \frac{2}{p^2y}\frac{dp}{dy} = 0$ gives

$$\frac{dp}{dy} = \frac{p^2y}{2}\frac{1}{py^2} = \frac{p}{2y}$$

$$\frac{dp}{p} = \frac{dy}{2y}.$$

Integrating on both sides

$$\int\frac{dp}{p} = \frac{1}{2}\int\frac{dy}{y}$$

$$\log p = \frac{1}{2}\log y + \log c$$

$$\log p = \log y^{1/2} + \log c$$

$$\log p = \log\sqrt{y} + \log c$$

$$\log p = \log(c\sqrt{y}).$$

$$\cancel{\log} p = c\sqrt{y} \longrightarrow (2)$$

Put $p = c\sqrt{y}$ in (1),

$$(c\sqrt{y})^3 - 4xy(c\sqrt{y}) + 8y^2 = 0$$

$$c^3 y\sqrt{y} - 4cxy\sqrt{y} + 8y^2 = 0$$

$$cy\sqrt{y}(c^2 - 4x) = -8y^2$$

Squaring,

$$c^2(c^2-4x)^2 = 64y.$$

---

4. Solve $(D^3+3D^2+2D)y = x^2$.

<u>Solution:</u>

<u>To find C.F.</u>

$$(D^3+3D^2+2D)y = 0.$$

The auxiliary equation is

$$m^3+3m^2+2m = 0.$$

$$m(m^2+3m+2) = 0.$$

$$m(m+1)(m+2) = 0$$

$$m=0,\ m=-1,\ m=-2.$$

$$C.F = A + Be^{-x} + Ce^{-2x}.$$

<u>To find P.I:</u>

$$P.I = \frac{1}{D^3+3D^2+2D}x^2$$

$$= \frac{1}{2D\left(1+\frac{3D}{2}+\frac{D^2}{2}\right)}x^2.$$

$$= \frac{1}{2D}\left(1+\frac{3D}{2}+\frac{D^2}{2}\right)^{-1}x^2.$$

$$= \frac{1}{2D}\left[1-\left(\frac{3D}{2}+\frac{D^2}{2}\right)+\left(\frac{3D}{2}+\frac{D^2}{2}\right)^2-\cdots\right]x^2.$$

$$= \frac{1}{2D}\left[1-\frac{3D}{2}-\frac{D^2}{2}+\frac{9D^2}{4}\right]x^2$$

$$=\frac{1}{2D}\left[1-\frac{3D}{2}+\frac{7D^2}{4}\right]x^2$$

$$=\frac{1}{2D}\left[x^2-\frac{3}{2}(2x)+\frac{7}{4}(2)\right]$$

$$=\frac{1}{2D}\left[x^2-3x+\frac{7}{2}\right]$$

$$=\frac{1}{2}\left[\frac{x^3}{3}-3\frac{x^2}{2}+\frac{7x}{2}\right]$$

The solution is $y=A+Be^{-2x}+Ce^{-x}+\frac{x^3}{6}-\frac{3x^2}{4}+\frac{7x}{4}$.

---

5. Solve $(D^2-4D+4)y=8x^2e^{2x}\sin 2x$.

Solution:

To find C.F

$$(D^2-4D+4)y=0$$

The auxiliary equation is

$+4$

-4

$-2 \quad -2$

$$(m^2-4m+4)=0$$

$$(m-2)(m-2)=0$$

$$m=2,2.$$

$$C.F=(A+Bx)e^{2x}.$$

To find P.I:

$$P.I=\frac{1}{(D-2)^2}8x^2e^{2x}\sin 2x$$

$$=8e^{2x}\frac{1}{(D+2-2)^2}x^2\sin 2x$$

$$P.I=8e^{2x}\frac{1}{D^2}x^2\sin 2x \longrightarrow (1)$$

$$\frac{1}{D} x^2 \sin 2x = \int x^2 \sin 2x \, dx$$

$$= x^2\left(\frac{-\cos 2x}{2}\right) - \cancel{2}x\left(\frac{-\sin 2x}{\cancel{4}\,2}\right) + 2\left(\frac{\cos 2x}{8}\right).$$

$$\frac{1}{D^2} x^2 \sin 2x = \int \frac{-x^2 \cos 2x}{2}\, dx + \int \frac{x \sin 2x}{2}\, dx + \int \frac{\cos 2x}{4}\, dx$$

$$= -\frac{1}{2}\left[x^2\left(\frac{\sin 2x}{2}\right) - \cancel{2}x\left(\frac{-\cos 2x}{\cancel{4}\,2}\right) + \cancel{2}\left(\frac{-\sin 2x}{\cancel{8}\,4}\right)\right].$$

$$+\frac{1}{2}\left[x\left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{4}\right)\right] + \frac{\sin 2x}{8}$$

$$= -\frac{x^2 \sin 2x}{4} - \frac{x \cos 2x}{4} + \frac{\sin 2x}{8} - \frac{x \cos 2x}{4}$$

$$+ \frac{\sin 2x}{8} + \frac{\sin 2x}{8}.$$

$$= -\frac{x^2 \sin 2x}{4} - \frac{\cancel{2}x \cos 2x}{\cancel{4}\,2} + \frac{3 \sin 2x}{8}.$$

$$= \frac{-x^2 \sin 2x}{4} - \frac{x \cos 2x}{2} + \frac{3 \sin 2x}{8}.$$

① $\Rightarrow$ $P.I = 8e^{2x}\left[\frac{-x^2 \sin 2x}{4} - \frac{x \cos 2x}{2} + \frac{3 \sin 2x}{8}\right]$

$$= \frac{\cancel{8}^{2}}{\cancel{4}_{1}} e^{2x} \times -x^2 \sin 2x - \frac{\cancel{8}^{4} x e^{2x} \cos 2x}{\cancel{2}_{1}}$$

$$+ \cancel{8} e^{2x} \times \frac{3 \sin 2x}{\cancel{8}}$$

$$= -2x^2 e^{2x} \sin 2x - 4x e^{2x} \cos 2x$$

$$+ 3e^{2x} \sin 2x.$$

$= e^{2x}(-2x^2 \sin 2x - 4x\cos 2x + 3\sin 2x)$.

The solution is

$y = (A + Bx)e^{2x} + e^{2x}(-2x^2 \sin 2x - 4x\cos 2x + 3\sin 2x]$.

---

6. Solve the differential eqration

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)].$$

<u>Solution:</u>

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)]. \longrightarrow (1)$$

Put $z = \log(1+x)$; $1+x = e^z$

Equation (1) becomes

$$(\theta(\theta-1)+\theta+1)y = 4\cos z$$

$$(\theta^2 - \cancel{\theta} + \cancel{\theta} + 1)y = 4\cos z$$

$$(\theta^2+1)y = 4\cos z$$

<u>To find C.F</u>

$$(\theta^2+1)y = 0.$$

The auxiliary eqration is

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = i, -i$$

$C.F = A\cos z + B\sin z$.

To find P.I :

$$P.I = \frac{1}{\theta^2+1} 4\cos z \quad (\text{when } \theta^2 = -1, Dr = 0)$$

$$= 4\frac{z}{2\theta}\cos z$$

$$= 2z\int \cos z\, dz$$

$$= 2z\sin z$$

The solution is

$$y = A\cos z + B\sin z + 2z\sin z$$

$$y = A\cos(\log(x+1)) + B\sin(\log(x+1))$$
$$+ 2\log(x+1)\sin(\log(x+1)).$$

7. Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + y = x\sin x$.

Solution :

$$(D^2+1)y = x\sin x$$

To find C.F

$$(D^2+1)y = 0.$$

The auxiliary equation is

$$m^2+1 = 0$$

$$m^2 = -1$$

$$m = i, -i.$$

$$C.F = C_1\cos x + C_2\sin x$$

$f_1 = \cos x$; $f_2 = \sin x$

$f_1' = -\sin x$; $f_2' = \cos x$

$W = f_1 f_2' - f_2 f_1'$

$= \cos x \times \cos x - \sin x \times -\sin x$

$= \cos^2 x + \sin^2 x$

$= 1$

$$P = -\int \frac{f_2 \phi}{W} dx$$

$$= -\int \sin x (x \sin x) dx$$

$$= -\int x \sin^2 x \, dx$$

$$= -\int x \left(\frac{1 - \cos 2x}{2}\right) dx$$

$$= -\frac{1}{2}\int x \, dx + \frac{1}{2}\int x \cos 2x \, dx$$

$$= -\frac{1}{2}\frac{x^2}{2} + \frac{1}{2}\left(x \frac{\sin 2x}{2} - \left(\frac{-\cos 2x}{4}\right)\right) + c_1$$

$$= \frac{-x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + c_1$$

$$Q = \int \frac{f_1 \phi}{W} dx$$

$$= \int \cos x (x \sin x) dx$$

$$= \frac{1}{2}\int x \sin 2x \, dx$$

$$= \frac{1}{2}\left(x\left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{4}\right)\right) + c_2$$

$$= \frac{-x\cos 2x}{4} + \frac{\sin 2x}{8} + c_2$$

The solution is

$$y = Pf_1 + Qf_2$$

$$= \left(\frac{-x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8} + c_1\right)\cos x$$

$$+ \left(\frac{-x\cos 2x}{4} + \frac{\sin 2x}{8} + c_2\right)\sin x$$

$$y = c_1\cos x + c_2\sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin x$$

---

8. Solve $(x^2-y^2-z^2)p + 2xyq - 2zx = 0$.

Solution:

The auxiliary equation is

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2zx}.$$

Consider $\dfrac{dy}{\cancel{2}\cancel{x}y} = \dfrac{dz}{\cancel{2}z\cancel{x}}$

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating on both sides

$$\log y = \log z + \log b$$

$$\cancel{\log} y = \cancel{\log} bz$$

$$y = bz$$

$$b = y/z \longrightarrow (1)$$

Choose the multipliers x, y, z.

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2zx} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2-y^2-z^2)+2xy^2+2xz^2}$$

Consider

$$\frac{dy}{2xy} = \frac{x\,dx+y\,dy+z\,dz}{x^3-xy^2-xz^2+2xy^2+2xz^2}$$

$$\frac{dy}{2xy} = \frac{x\,dx+y\,dy+z\,dz}{x^3+xy^2+xz^2}$$

$$\frac{dy}{2xy} = \frac{x\,dx+y\,dy+z\,dz}{x(x^2+y^2+z^2)}$$

$$\frac{dy}{2y} = \frac{x\,dx+y\,dy+z\,dz}{x^2+y^2+z^2}.$$

$$\frac{dy}{y} = \frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2}$$

Integrating on both sides

$$\log y = \log(x^2+y^2+z^2) + \log b.$$

$$\log y = \log(b(x^2+y^2+z^2)).$$

$$y = b(x^2+y^2+z^2)$$

$$b = \frac{y}{x^2+y^2+z^2} \longrightarrow (2)$$

The solution of the given equation is

$$f\left(\frac{y}{z}, \frac{y}{x^2+y^2+z^2}\right) = 0.$$

Q. Solve $(p^2+q^2)y = qz$.

Solution:

Let $f(x,y,z,p,q) = (p^2+q^2)y - qz$,

$$f_x = 0;\ f_y = p^2+q^2;\ f_z = -q.$$

$$f_p = 2py;\ f_q = 2qy - z.$$

The auxiliary equation is

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q}$$

$$= \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{df}{0}.$$

$$\frac{dp}{-pq} = \frac{dq}{p^2+q^2-q^2} = \frac{dz}{-2p^2z - 2q^2y + qz}$$

$$= \frac{dx}{-2py} = \frac{dy}{z-2qy} = \frac{dF}{0} \longrightarrow (1)$$

$$\frac{dp}{-pq} = \frac{dq}{p^2}$$

$$\frac{dp}{-q} = \frac{dq}{p}.$$

$$p\,dp = -q\,dq.$$

$$p\,dp + q\,dq = 0$$

Integrating, $\frac{p^2}{2} + \frac{q^2}{2} = \frac{c^2}{2}$.

$$p^2+q^2=c^2 \longrightarrow (2)$$

From the given equation $(p^2+q^2)y = qz$.

we get $c^2y = qz$.

$$q = \frac{c^2y}{z}.$$

$$p^2 = c^2 - q^2.$$

$$p^2 = c^2 - \frac{c^4y^2}{z^2} = \frac{c^2(z^2-c^2y^2)}{z^2}.$$

Since $dz = p\,dx + q\,dy$

$$dz = \frac{c}{z}\sqrt{z^2-c^2y^2}\,dx + \frac{c^2y}{z}\,dy.$$

$$z\,dz = c\sqrt{z^2-c^2y^2}\,dx + c^2y\,dy.$$

$$z\,dz - c^2y\,dy = c\sqrt{z^2-c^2y^2}\,dx$$

$$\frac{z\,dz - c^2y\,dy}{\sqrt{z^2-c^2y^2}} = c\,dx$$

Integrating on both sides

$$\sqrt{z^2-c^2y^2} = cx+d.$$

$$z^2 = (cx+d)^2 + c^2y^2.$$

---

10. Solve $2zx - px^2 - 2qxy + pq = 0$.

Solution:

Let $f(x,y,z,p,q) = 2zx - px^2 - 2qxy + pq$.

$$f_x = 2z - 2px - 2qy \,;\; f_y = -2qx \,;$$

$$f_z = 2x.$$

$$f_p = -x^2 + q \,;\; f_q = -2xy + p.$$

The auxiliary equation is

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0}.$$

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dF}{0}$$

$$dq = 0.$$

$$\therefore q = a$$

Put $q = a$ in $2zx - px^2 - 2qxy + pq = 0$

$$2zx - px^2 - 2axy + ap = 0$$

$$2x(z - ay) = p(x^2 - a).$$

$$p = \frac{2x(z - ay)}{x^2 - a}$$

$dz = pdx + qdy$ becomes

$$dz = \frac{2x(z - ay)}{x^2 - a}\,dx + a\,dy$$

$$dz - a\,dy = \frac{2x(z-ay)\,dx}{x^2-a}.$$

$$\frac{d(z-ay)}{z-ay} = \frac{2x\,dx}{x^2-a}.$$

Integrating on both sides

$$\log(z-ay) = \log(x^2-a) + \log b$$

$$\cancel{\log}(z-ay) = \cancel{\log}(b(x^2-a)).$$

$$z - ay = b(x^2-a)$$

$$z = ay + b(x^2-a).$$

11: Find $L(te^{-t}\cosh 2t)$.

Solution:

$$L(te^{-t}\cosh 2t) = L(t f(t)) = -F'(s)$$

(by result 1)

where $f(t) = e^{-t}\cosh 2t$.

$$F(s) = L(f(t)) = L(e^{-t}\cosh 2t).$$

$$= L(\cosh 2t)_{s\to s+1}$$ (using shifting Theorem)

$$= \left(\frac{s}{s^2-2^2}\right)_{s\to s+1} = \frac{s+1}{(s+1)^2-4}.$$

$$F(s) = \frac{s+1}{(s+1)^2-4} = \frac{s+1}{s^2+2s+1-4}$$

$$F(s) = \frac{s+1}{s^2+2s-3}.$$

$$F'(s) = \frac{d}{ds}\left(\frac{s+1}{s^2+2s-3}\right)$$

$$= \frac{(s^2+2s-3)(1) - (s+1)(2s+2)}{(s^2+2s-3)^2}$$

$$= \frac{(s^2+2s-3) - (2s^2+4s+2)}{(s^2+2s-3)^2}$$

$$= \frac{-s^2-2s-5}{(s^2+2s-3)^2}.$$

$$-F'(s) = \frac{s^2+2s+5}{(s^2+2s-3)^2}.$$

$$\therefore L(te^{-t}\cosh 2t) = \frac{s^2+2s+5}{(s^2+2s-3)^2}.$$

12. Find $L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right)$.

<u>Solution:-</u>

$$u = s^2+4s+5$$

$$du = (2s+4)\,ds = 2(s+2)\,ds.$$

$\frac{du}{2} = (s+2)ds.$

Since the Numerator is the direct derivative of Dr, we have

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty \frac{s+2}{(s^2+4s+5)^2}\,ds$$

$$= \int_s^\infty \frac{du/2}{u^2}$$

$$= \frac{1}{2}\int_s^\infty u^{-2}\,du.$$

$$= -\frac{1}{2}\left(\frac{1}{u}\right)_s^\infty = -\frac{1}{2}\left(\frac{1}{s^2+4s+5}\right)_s^\infty$$

$$= -\frac{1}{2}\left(0 - \frac{1}{s^2+4s+5}\right) \text{ As } s\to\infty, \frac{1}{s}\to 0.$$

$$= \frac{1}{2}\left(\frac{1}{s^2+4s+5}\right)$$

Taking inverse on both sides,

$$\frac{f(t)}{t} = \frac{1}{2}L^{-1}\left(\frac{1}{s^2+4s+5}\right)$$

$$= \frac{1}{2}L^{-1}\left(\frac{1}{(s+2)^2+1}\right)$$

$$= \frac{1}{2}e^{-2t}L^{-1}\left(\frac{1}{s^2+1}\right) \quad (\because s \to s+2)$$

(using shifting Theorem)

$$= \frac{1}{2}e^{-2t}\sin t.$$

$$f(t) = \frac{t}{2} e^{-2t} \sin t.$$

$$\therefore L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right) = \frac{t}{2} e^{-2t} \sin t.$$

---

13. Solve $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 20 \sin 2t$ given

$y(0) = 0,\ y'(0) = 2.$

Solution:-

$$y'' - y' - 2y = 20 \sin 2t.$$

Taking Laplace transform on both sides and applying the formula for transform of derivatives.

$$L(y'') - L(y') + 2L(y) = 20 L(\sin 2t).$$

$$s^2 L(y) - s y(0) - y'(0) - [s L(y) - y(0)] + 2L(y) = 20 \frac{2}{s^2+2^2}.$$

$$s^2 L(y) - 0 - 2 - (s L(y) - 0) + 2L(y) = \frac{40}{s^2+4}.$$

$$s^2 L(y) - 2 - s L(y) + 2L(y) = \frac{40}{s^2+4}.$$

$$L(y)(s^2 - s + 2) = \frac{40}{s^2+4} + 2$$

$$= \frac{40 + 2(s^2+4)}{s^2+4}$$

$$= \frac{2s^2 + 48}{s^2+4}$$

$$L(y) = \frac{2s^2 + 48}{(s^2+4)(s^2-s+2)}$$

$$y = L^{-1}\left(\frac{2s^2+48}{(s^2-s+2)(s^2+4)}\right).$$

First we split the function by method of partial fractions

$$\frac{2s^2+48}{(s^2-s+2)(s^2+4)} = \frac{As+B}{s^2-s+2} + \frac{Cs+D}{s^2+4}.$$

$$2s^2+48 = (As+B)(s^2+4) + (Cs+D)(s^2-s+2)$$

$$= A(s^3+4s) + B(s^2+4) + C(s^3-s^2+2s) + D(s^2-s+2).$$

To find the values of A, B, C and D we eqrate the coefficient of like Terms

Eqrating the coefficient of s^3:

$$0 = A + C$$

$$A = -C \longrightarrow (1)$$

Equating the coefficient of s^2

$$2 = B - C + D \longrightarrow (2)$$

Equating coefficient of s

$$0 = 4A + 2C - D \longrightarrow (3)$$

Equating constant term

$$48 = 4B + 2D$$

$$2D = 48 - 4B$$

$$D = 24 - 2B.$$

$$-2B = -24 + D$$

$$B = 12 - D/2 \longrightarrow (4)$$

using (4) in (2),

$$2 = (12 - D/2) - C + D$$

from (3) $0 = 4A + 2C - D$

$$4 = 24 - D + 2A + 2D \quad \text{since } C = -A$$

$$4 = 24 + D + A \quad \Rightarrow 4 - 24 = D + A$$

$$-20 = D + A$$

$$0 = 4A - D + 2(-A)$$

$$4A - D - 2A = 0$$

$$2A - D = 0$$

$$D + A = -20$$

$$-D + 2A = 0$$

$$3A = -20$$

$$A = \frac{-20}{3}. \qquad D - 20/3 = -20$$

$$D = -20 + 20/3$$

$$D = \frac{-40}{3}. \qquad D = \frac{-60 + 20}{3} = \frac{-40}{3}$$

$C = 20/3$

$B = 12 - \left(-\frac{40}{6}\right) = 56/3.$

$= 72 + 40/6$

$= 32/6 = 16/3.$

$$y = L^{-1}\left(\frac{2s^2+48}{(s^2+4)(s^2-s+2)}\right).$$

$$= L^{-1}\left(\frac{As+B}{s^2-s+2}\right) + L^{-1}\left(\frac{Cs+D}{s^2+4}\right).$$

$$= L^{-1}\left(\frac{-20/3\, s + 56/3}{s^2-s+2}\right) + L^{-1}\left(\frac{\frac{20}{3}s - \frac{40}{3}}{s^2+4}\right).$$

$$= \frac{1}{3}\left[L^{-1}\left(-4\left[\frac{5s-14}{s^2-s+2}\right)\right] + L^{-1}\left[20\left(\frac{s-2}{s^2+4}\right)\right]\right].$$

$$= \frac{1}{3}\left[L^{-1}4\left[\frac{5\left(\frac{s-1}{2}\right) + \frac{5}{2} - 4}{\left(s\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}\right] + L^{-1}\left(20\left[\frac{s}{s^2+4}\right]\right) - 20L^{-1}\left[\frac{2}{s^2+4}\right]\right],$$

$$= \frac{-4}{3}\left[L^{-1}\left(5\frac{s}{s^2+(\sqrt{7}/2)^2}\right) + L^{-1}\left(\frac{-3/2}{s^2+(\sqrt{7}/2)^2}\right)\right.$$

$$+\frac{20}{3}(\cos 2t - \sin 2t)\Bigg]$$

$$= \frac{-4}{3}\left[\left(5\cos\frac{\sqrt{7}}{2t}\right) - \frac{3}{\cancel{2}} \times \frac{\cancel{2}}{\sqrt{7}} L^{-1}\left[\frac{\sqrt{7}/2}{s^2 + \left(\frac{\sqrt{7}}{2t}\right)^2}\right]\right]$$

$$+\frac{20}{3}(\cos 2t - \sin 2t).$$

$$= \frac{-4}{3}\left[\left(5\cos\frac{\sqrt{7}}{2t}\right) - \frac{23}{\sqrt{7}}\sin\frac{\sqrt{7}}{2}t\right] + \frac{20}{3}(\cos 2t - \sin 2t)$$

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