PAVENDAR BHARATHIDASAN COLLEGE OF ARTS & SCIENCE

VECTOR CALCULAS AND FOURIER SERIES

SUBJECT CODE: 16SCCMM7

2 MARKS

1. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $P(1,-2,-1)$ in the direction $\vec{i}+\vec{j}-\vec{k}$

Solution:

Given $\phi = x^2yz + 4xz^2$

Directional Derivative = $\text{grad}\,\phi \cdot \hat{n}$

unit Normal vector $\hat{n}$ in the direction $\vec{i}+\vec{j}-\vec{k}$ is $= \dfrac{\vec{i}+\vec{j}-\vec{k}}{\sqrt{3}}$

Now, $\text{grad}\,\phi = \nabla\phi$

$$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \vec{j}\frac{\partial}{\partial y}(x^2yz + 4xz^2) + \vec{k}\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$= \vec{i}(2xyz + 4z^2) + \vec{j}(x^2z) + \vec{k}(x^2y + 8xz)$$

at the point $(1,-2,-1)$

$$= \vec{i}\,[2(1)(-2)(-1) + 4(-1)^2] + \vec{j}\,[(1)^2(-1)] + \vec{k}\,[(1)^2(-2) + 8(1)(-1)]$$

$$= 8\vec{i} - \vec{j} - 10\vec{k}$$

directional derivative $= 8\vec{i} - \vec{j} - 10\vec{k} \cdot \dfrac{\vec{i} + \vec{j} - \vec{k}}{\sqrt{3}}$

$$= \frac{8-1+10}{\sqrt{3}} = \frac{3}{\sqrt{3}}$$

$$= \frac{\sqrt{3}\cdot\cancel{\sqrt{3}}}{\cancel{\sqrt{3}}} = \sqrt{3}$$

directional derivative $= \sqrt{3}$ //

2) find The value of "a" so that the vector $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$ is solenoidal given $\vec{F}$ is solenoidal.

Solution:

Given $\vec{F}$ is solenoidal

$$\text{div}.\vec{F} = 0$$

$$\nabla.\vec{F} = 0$$

$$\Rightarrow \quad \vec{i}\frac{\partial \vec{F}}{\partial x} + \vec{j}\frac{\partial \vec{F}}{\partial y} + \vec{k}\frac{\partial \vec{F}}{\partial z} = 0$$

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left((x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}\right) = 0$$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$$1 + 1 + a = 0$$

$$2 + a = 0 \qquad \boxed{a = -2}$$

3] If $\vec{a}$ is a constant vector, show that $\nabla(\vec{a}\times\vec{r})=0$

Solution:

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Now, $$\vec{a}\times\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(a_2 z - a_3 y) - \vec{j}(a_1 z - a_3 x) + \vec{k}(a_1 y - a_2 x)$$

$$\nabla\cdot(\vec{a}\times\vec{r}) = \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_1 z - a_3 x) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$$

$$\boxed{\nabla\cdot(\vec{a}\times\vec{r}) = 0}$$

4] prove that grad $\phi(u) = \phi'(u)$ grad u.

$$\text{grad}\,\phi(u) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\phi(u)$$

$$= \vec{i}\frac{\partial}{\partial x}\phi(u) + \vec{j}\frac{\partial}{\partial y}\phi(u) + \vec{k}\frac{\partial}{\partial z}\phi(u)$$

$$= \vec{i}\,\phi'(u)\frac{\partial u}{\partial x} + \vec{j}\,\phi'(u)\frac{\partial u}{\partial y} + \vec{k}\,\phi'(u)\frac{\partial u}{\partial z}$$

$$= \left(\vec{i}\frac{\partial u}{\partial x} + \vec{j}\frac{\partial u}{\partial y} + \vec{k}\frac{\partial u}{\partial z}\right)\cdot\phi'(u)$$

$$\text{grad}\,\phi(u) = \phi'(u)\cdot\text{grad}\,u$$

5] If $\vec{F} = 3xy\,\vec{i} - y^3\vec{j}$ evaluate $\int_C \vec{F}\cdot d\vec{r}$ where c is the curve $y = 2x^2$ in the xy plane from $(0,0)$ to $(1,2)$

Solution:

$$\vec{F} = 3xy\,\vec{i} - y^3\vec{j}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$$\vec{F}\cdot d\vec{r} = (3xy\,\vec{i} - y^3\vec{j})\cdot(dx\,\vec{i} + dy\,\vec{j})$$

$$= 3xy\,dx - y^3\,dy$$

given $y = 2x^2$

$$\vec{F}\cdot d\vec{r} = (3x)\cdot(2x^2)\,dx - y^3\,dy$$

$$= 6x^3\,dx - y^3\,dy$$

$$\int_C \vec{F}\cdot d\vec{r} = 6\int_{x=0}^{1} x^3\,dx - \int_{y=0}^{2} y^3\,dy$$

$$= 6\left[\frac{x^4}{4}\right]_0^1 - \left[\frac{y^4}{4}\right]_0^2 = \cancel{6}^{3}\left[\frac{1^4}{\cancel{4}_2} - 0\right] - \left[\frac{16}{4} - 0\right]$$

$$= \frac{3}{2} - 4 = \frac{3-8}{2} = -5/2$$

$$\int_C \vec{F}\cdot d\vec{r} = \frac{-5}{2}$$

6] Show that the $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ is a conservative vector field

Solution:

If $\vec{F}$ is conservative then

$$\nabla\times\vec{F} = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= 0+0+0 = 0$$

$\therefore$ $\vec{F}$ is a conservative vector field.

7] Define Surface Integrals

Defn:

The limit of the sum (1) extended over all sub-domain ΔS_i as the diameters of all such sub-domains approach zero is called the surface integral and is denoted by the symbol

$$\iint_S \vec{F}\cdot\hat{n}\, ds$$

8] Define volume Integrals

Defn:

Let us consider a scalar field F defined within and on the boundary of a domain V.

we form the sum $\sum_{i=1}^{n} F(x\vec{i}, y\vec{j}, z\vec{j})\,\delta v_i$

and define the volume integral of F over v if it exists to be

$$\iiint_V f(x,y,z)\, dv = \lim \sum_{i=1}^{n} F(x\vec{i}, y\vec{i}, z\vec{i})\,\delta v_i$$

9] Define Gauss divergence theorem

Defn

If $\vec{F}$ is a vector point function finite and differentiable in a region R bounded by a closed surface S, then the surface integral of the normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ taken over V.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

10] Using green's theorem to evaluate $\int_C [(2x-y)\,dx + (x+y)\,dy]$ where C is the boundary of the circle $x^2+y^2=a^2$ in the xoy plane.

Solution:

Here $P = 2x-y$; $Q = x+y$

$$\frac{\partial P}{\partial y} = -1 \quad ; \quad \frac{\partial Q}{\partial x} = 1$$

By green's theorem, we have

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\oint_C [(2x-y)\,dx + (x+y)\,dy] = \iint_R (1-(-1))\,dx\,dy$$

$$= 2\iint_R dx\,dy$$

$= 2$ (area of the region R)

$= 2\pi a^2$ (area of the circle is πa^2)

13] Define: FOURIER SERIES

Defn

If $f(x)$ is a periodic function, then it can be represented by an infinite series called fourier series as

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx;$$

14] SOLVE $\sin x$, $\cos x$, and $\tan x$ are periodic

Solution

$f(x) = \sin x = \sin(x+2\pi) = \sin(x+4\pi) = \ldots$ is period 2π

Similarly $f(x) = \cos x = \cos(x+2\pi) = \cos(x+4\pi) = \ldots$ is period 2π

Similarly $\tan x$ has period π

15] A function $f(x)$ is odd if $F(-x) = -F(x)$

(a) $F(x) = x^3$ (b) $F(x) = \sin x$.

(a) $F(x) = x^3$

$F(-x) = (-x)^3 = -x^3 = -f(x)$

(b) $F(x) = \sin x$

$F(-x) = (\sin(-x)) = -\sin x = -F(x)$

16] What are the odd and Even functions:

odd functions:

A function $F(x)$ is said to be odd if

$f(-x) = -f(x)$

11] Define Stokes Theorem:

Defn

The line Integral of the tangential component of a vector function $\vec{F}$ (finite and differentiable) around a simple closed curve c is equal to the surface Integral of the normal component of curl $\vec{F}$ over of the normal component any surface having as its boundary

$$\int \vec{F}\cdot d\vec{r} = \iint (\nabla \times \vec{F})\cdot \hat{n}\, ds.$$

12] using Stokes theorem prove that

$$\int_C \phi \cdot \nabla\psi \cdot d\vec{r} = -\int_C \psi \nabla\phi\, d\vec{r}.$$

We know that

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\therefore \int_C \nabla(\phi\psi)\, d\vec{r} = \iint_C \nabla\times\nabla(\phi\psi)\, \hat{n}\, ds$$

$$= 0 \qquad [\text{curl grad } \phi\psi = 0]$$

$$\therefore \int_C \nabla(\phi\psi)\cdot d\vec{r} = 0$$

$$\therefore \int_C (\phi\nabla\psi + \psi\nabla\phi)\cdot d\vec{r} = 0$$

$$\therefore \int_C \phi\nabla\psi\cdot d\vec{r} + \int_C \psi\nabla\phi\cdot d\vec{r} = 0$$

$$\therefore \int_C \phi\nabla\psi\, d\vec{r} = -\int \psi\nabla\phi\cdot d\vec{r}$$

Hence proved.

Even Functions:

A function $f(x)$ is said to be even if $f(-x) = f(x)$

17) Define : Half range cosine series

The half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$$

18) Define : Half range sine series

The Half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where $$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\,dx.$$

19. find a_0 in the cosine series of $f(x) = x$ in $0<x<2$

Solution:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx \qquad \boxed{\pi = 2}$$

$$a_0 = \frac{2}{\pi} \int_0^{2} x\,dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2}\right]_0^2 = \frac{2}{\pi} \left[\frac{4^2}{2} - 0\right]$$

$$\boxed{a_0 = \frac{4}{\pi}}$$

20) Define: Half Range series

Defn

If the given function is defined in the interval $(0,\pi)$ the series obtained is of period π and is called Fourier series. If it is expressed interms of cosine and sine series. If expressed in terms of sine of the multiple Angles In the interval $0<x<\pi$

5 Marks

1. Find the Angle of intersection at the point $(2,-1,2)$
$$x^2+y^2+z^2=9; \quad z=x^2+y^2-3$$

Solution:

Let, $\phi_1 = x^2+y^2+z^2-9$; $\phi_2 = x^2+y^2-z-3$

at the point $(2,-1,2)$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

$$\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2-9) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2-9)$$

$(2,-1,2)$ $$+\vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-9)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$= \vec{i}\,2(2) + \vec{j}\,2(-1) + \vec{k}\,2(2)$$

2) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

prove that

(i) $\nabla r = \frac{\vec{r}}{r}$

(ii) $\nabla r^n = n r^{n-2} \vec{r}$

where $r = |\vec{r}|$

Solution:

$$r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

differentiate with respect to x

$$\cancel{2}r \frac{\partial r}{\partial x} = \cancel{2}x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

diff. with y respectively

$$\cancel{2}r \frac{\partial r}{\partial y} = \cancel{2}y$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

diff. with z respectively

$$\cancel{2}r \frac{\partial r}{\partial z} = \cancel{2}z$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

(i) $\nabla r = \sum \vec{i} \frac{\partial}{\partial x} \cdot r$

$$= \vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}$$

$$= \frac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla\phi_1 = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$|\nabla\phi_1| = \sqrt{(4)^2 + (-2)^2 + (4)^2} = \sqrt{16+4+16} = \sqrt{36} = 6$$

$$|\nabla\phi_1| = 6$$

$$\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2-z-3) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2-z-3)$$

$$+ \vec{k}\frac{\partial}{\partial z}(x^2+y^2-z-3)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(-1)$$

at $(2,-1,2)$

$$= \vec{i}\,2(2) + \vec{j}\,2(-1) + \vec{k}(-1)$$

$$\nabla\phi_2 = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$|\nabla\phi_2| = \sqrt{(4)^2 + (-2)^2 + (-1)^2} = \sqrt{16+4+1} = \sqrt{21}$$

$$|\nabla\phi_2| = \sqrt{21}$$

$$\cos\theta = \frac{(4\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (4\vec{i} - 2\vec{j} - \vec{k})}{6 \cdot \sqrt{21}}$$

$$= \frac{(16+4-4)}{6\sqrt{21}} = \frac{\cancel{16}\,8}{\cancel{6}_3\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

The Angle of Intersection is $\frac{8}{3\sqrt{21}}$

$$= \frac{1}{r}\vec{r}$$

$$\nabla r = \frac{\vec{r}}{r}$$

(ii) $\nabla r^n = \sum \vec{i}\, \frac{\partial r^n}{\partial x}$

$$\nabla r^n = \sum \vec{i}\, \frac{\partial}{\partial x}(r^n)$$

$$= \sum \vec{i}\, (n r^{n-1}) \frac{\partial r}{\partial x}$$

$$= n r^{n-1} \left[\sum \vec{i}\, \frac{x}{r} \right]$$

$$= \frac{n r^{n-1}}{r} [x\vec{i} + y\vec{j} + z\vec{k}]$$

$$= \frac{n r^{n-1}}{r} \vec{r}$$

$$\nabla r^n = n r^{n-2} \vec{r}$$

3) $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$ (or) curl. curl $\vec{F}$ = grad.div$\vec{F}$ - Laplacian $\vec{F}$

PROOF:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \vec{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\text{curl}(\text{curl}\,\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) & \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) & \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \end{vmatrix}$$

$$= \sum \left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\right]\vec{i}$$

$$= \sum \left[\frac{\partial^2 F_2}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial z \partial x} - \left(\frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]\vec{i}$$

add $\frac{\partial^2 F_1}{\partial x^2}$

$$\Rightarrow = \sum \left[\left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z}\right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right] \cdot \vec{i}$$

$$= \sum \left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial^2 F_3}{\partial z}\right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]\vec{i}$$

$$= \sum \left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial^2 F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)F_1\right]\vec{i}$$

$$= \sum \left[\frac{\partial}{\partial x}(\nabla \cdot \vec{F}) - \nabla^2 F_1\right]\vec{i}$$

$$= \left[\vec{i}\frac{\partial}{\partial x}(\nabla \cdot \vec{F}) + \vec{j}\frac{\partial}{\partial y}(\nabla \cdot \vec{F}) + \vec{k}\frac{\partial}{\partial z}(\nabla \cdot \vec{F})\right]$$
$$- \nabla^2\left[F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right]$$

$$= \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

Hence proved

4) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = 3x^2\vec{i} + (2xz-y)\vec{j} + z\vec{k}$ and C is the straight line from $A(0,0,0)$ to $B(2,1,3)$

Solution:

The Eqn of AB is $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$ (say)

(i.e)

$x = 2t$; $y = t$; $z = 3t$

$dx = 2dt$; $dy = dt$; $dz = 3dt$

The points $(0,0,0)$ and $(2,1,3)$ correspond to $t=0$ and $t=1$ respectively

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (3x^2\vec{i} + (2xz-y)\vec{j} + 2\vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C (3x^2 dx + (2xz-y)dy + z\,dz$$

$$= \int_0^1 3(2t)^2 \cdot 2dt + [2(2t)(3t) - t]\,dt + 3t\cdot 3dt$$

$$= \int_0^1 [24t^2 dt + (12t^2 - t)\,dt + 9t]\,dt$$

$$= \int_0^1 (36t^2 + 8t)\,dt = 36\left[\frac{t^3}{3}\right]_0^1 + 8\left[\frac{t^2}{2}\right]_0^1$$

$$= 12(1-0) + 4(1-0) = 12+4 = 16$$

$$\int \vec{F}\cdot d\vec{r} = 16$$

5) Find the total work done in moving a particle in a force field given by $\vec{F} = (2x-y+z)\vec{i} + (x+y-z)\vec{j} + (3x-2y-5z)\vec{k}$ along a circle c in the xy plane $x^2+y^2=9$, $z=0$

Solution:

$$\text{Work done} = \int_c \vec{F} \cdot d\vec{r}$$

$$= \int_c [(2x-y+z)\vec{i} + (x+y-z)\vec{j} + (3x-2y-5z)\vec{k}]\,[dx\vec{i} + dy\vec{j} + dz\vec{k}]$$

$$= \int_c [(2x-y+z)dx + (x+y-z)dy + (3x-2y-5z)dz]$$

$$= \int_c (2x-y)dx + (x+y)dy$$

For the circle $x^2+y^2=9$ the parametric eqn are $x = 3\cos\theta$; $y = 3\sin\theta$; θ varies from 0 to 2π

$$= \int_0^{2\pi} [(6\cos\theta - 3\sin\theta)(-3\sin\theta\,d\theta) + (3\cos\theta + 3\sin\theta)(3\cos\theta\,d\theta)]$$

$$= \int_0^{2\pi} (9 - 9\sin\theta\cos\theta)\,d\theta$$

$$= \left[9\theta - \tfrac{9}{2}\sin^2\theta\right]_0^{2\pi}$$

$$= 18\pi$$

6) If $\vec{F} = 2xz\vec{i} - x\vec{j} + y^2\vec{k}$ then evaluate $\iiint_V \vec{F}\, d\vec{v}$ where v is the region bounded by the surface $x=0$; $y=0$; $z=x^2$; $z=4$

Solution

To cover the region v we first keep x and y fixed and integrated from $z=x^2$ to $z=4$ then keep x fixed and integrate from $x=0$ to $x=2$

$$\therefore \iiint_V \vec{F}.dv = \iiint_V (2xz\vec{i} - x\vec{j} + y^2\vec{k})\, dx\, dy\, dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\left(\int_{z=x^2}^{4} (2xz\,\vec{i} - x\vec{j} + y^2\vec{k})\, dz\right) dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{6} [(16x - x^5)\vec{i} - (x^3 - 4x)\vec{j} + (4-x^2)y^2\vec{k}]\, dy\, dx$$

$$= \int_{x=0}^{2} (96x - 6x^5)\vec{i} + (6x^3 - 24x)\vec{j} + 72(4-x^2)\vec{k}\, .dx$$

$$= 128\vec{i} - 24\vec{j} + 384\vec{k}$$

7) Use divergence theorem to evaluate $\iint_S \vec{F}.\hat{n}\, ds$ where $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the surface of the sphere $x^2+y^2+z^2 = a^2$

Solution

By divergence theorem we have

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint \nabla\cdot \vec{F}.dv$$

given $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^3\vec{i} + y^3\vec{j} + z^3\vec{k})$$

$$= 3x^2 + 3y^2 + 3z^2$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

$$= \iiint_V (3x^2 + 3y^2 + 3z^2)\,dv$$

$$= 3\iiint_V (x^2 + y^2 + z^2)\,dv$$

$$= 3\iiint_V (x^2 + y^2 + z^2)\,dx\,dy\,dz$$

To evaluate this volume integral we have to change cartesian to spherical polar

$$x^2 + y^2 + z^2 = r^2; \quad dx\,dy\,dz = r^2 \sin\theta\,dr\,d\theta\cdot d\phi$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = 3\int_0^a \int_0^\pi \int_0^{2\pi} r^2 (r^2 \sin\theta)\,d\phi\,d\theta\cdot dr$$

$$= 3\int_0^a \int_0^\pi r^4 \sin\theta\,[\phi]_0^{2\pi}\,d\theta\cdot dr$$

$$= 6\pi \int_0^a r^4 [-\cos\theta]_0^\pi\,dr$$

$$= 12\pi \int_0^a r^4\,dr$$

$$= \frac{12}{5}\pi a^5,$$

8) By applying green's theorem that the area bounded by a simple closed curve c is

$= \frac{1}{2}\int_C (x\,dy - y\,dx)$ and hence find the area of the ellipse

$$\frac{1}{2}\int_C (x\,dy - y\,dx) = \int_C \left(-\frac{y}{2}\,dx + \frac{x}{2}\,dy\right)$$

Here $P = -y/2$; $Q = x/2$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) dx\,dy = \iint_R dx\,dy$$

= Area of the closed region R

Hence $\frac{1}{2}\int_C (x\,dy - y\,dx)$ = Area of the closed region R

The Eqn of the ellipse is

$x = a\cos\theta$; $y = b\sin\theta$

$dx = -a\sin\theta\,d\theta$; $dy = b\cos\theta\,d\theta$

W.K.T

$$A = \frac{1}{2}\int_C (x\,dy - y\,dx)$$

$$= \frac{1}{2}\int_C [a\cos\theta\,(b\cos\theta\,d\theta - b\sin\theta - a\sin\theta\,\boxed{d\theta}]$$

($\therefore \theta$ varies to 0 to 2π)

$$= \frac{1}{2}\int_0^{2\pi} ab\,(\cos^2\theta + \sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\, d\theta = \frac{ab}{2}[\theta]_0^{2\pi} = \frac{ab \cdot \cancel{2}\pi}{\cancel{2}}$$

$$= \pi ab$$

The area of the ellipse is πab

2) using stoke's theorem evaluate $\int_C (2x-y)dx - yz^2 dy - y^2 z\, dz$ where C is the circle $x^2+y^2=1$; corresponding to the surface of the sphere of unit radius.

Solution

By stoke's theorem.

we have $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot \hat{n}\, ds.$

given $\vec{F}\cdot d\vec{r} = [(2x-y)\,dx - yz^2 dy - y^2 z\, d\vec{k}]$

$$(\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= (-2yz + 2yz)\vec{i} - (0-0)\vec{j} + (0+1)\vec{k}$$

$$= \vec{k}$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot \hat{n}\, ds$$

$$= \iint_R \vec{k}\cdot\vec{k}\ \frac{dx \cdot dy}{|\hat{n}\cdot\vec{k}|}$$

$$= \iint_R dx \cdot dy$$

$[\therefore\ |\hat{n}\cdot\vec{k}| = 1$ also $\vec{k}\cdot\vec{k} = 1]$

xy plane

$$= \iint_R dx\, dy$$

Where R denotes the projection of the surface S on xy plane which is clearly a circle of unit radius.

$$= \text{area of unit circle}$$

$$= \pi.$$

10] Find The Fourier series for f(x) is $(-\pi, \pi)$ is

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi & 0 < x < \pi \end{cases}$$

<u>Solution</u>

The Fourier series for the function f(x) in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} \pi\, dx \right]$$

$$= \frac{1}{\pi} \left[\pi (x)_0^{\pi} \right] = \pi$$

$$a_0 = \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \cos nx\, dx + \int_{0}^{\pi} \pi \cos nx\, dx \right]$$

$$= \frac{1}{\pi} \left[\pi \left[\frac{\sin nx}{n} \right]_0^{\pi} \right]$$

$$= \frac{\sin \pi x}{n} - \frac{\sin 0x}{n}$$

$$= 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \sin nx \, dx + \int_{0}^{\pi} \pi \sin nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\pi \left[-\frac{\cos nx}{n} \right]_0^{\pi} \right] = -\left[\frac{\cos \pi x}{n} - \frac{\cos 0}{n} \right]$$

$$= -\left[\frac{(-1)^n}{n} - \frac{1}{n} \right]$$

$$= \frac{1}{n} (1 - (-1)^n) \text{ when } n \text{ is odd}$$

$b_n = 0$ when n is even

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \, dx + \sum_{n=1}^{\infty} b_n \sin nx \, dx$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} (0) \cos nx \, dx + \sum_{n=1}^{\infty} \frac{1}{n} (1 - (-1)^n) \sin nx \, dx$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{1}{n} (1 - (-1)^n) \sin nx \, dx$$

11] Obtain the Fourier series for the functions $F(x) = x^2$; $-\pi \leq x \leq \pi$ specify the sum of the series at the end points $x = \pi, -\pi$, Deduce the sum of the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ & $1 - \frac{1}{2^2} + \frac{1}{3^2} + \dots$

Given $F(x) = x^2$

Here $F(x) = F(-x) = x^2$

Here $f(x)$ is an even functions

$\therefore$ The Fourier co-efficients $b_n = 0$; Now the Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad [\because b_n = 0]$$

Now, we have to find a_0 and a_n

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx .$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3}\pi^2 \rightarrow (2)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi \cos n\pi}{n^2}\right] = \frac{4}{n^2}(-1)^2$$

$$a_n = \frac{4}{n^2}(-1)^n \longrightarrow (3)$$

Substituting (2) & (3) in (1), we get

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right] \rightarrow (4)$$

putting $x=\pi$ and $-\pi$ in (4) we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \cdots\right]$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

putting $x=0$ in (4) we get

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

Here 0 is a point of continuity in $(-\pi,\pi)$

(Q) Show that the Fourier series for $f(x)=x$, $-\pi < x < \pi$ is given by $f(x) = 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$

Solution

given $f(x)=x$

$f(-x) = -x$

$f(x) = f(-x) = -f(x)$

$\therefore f(x)=x$ is an odd function

Hence, $a_0 = 0$; $a_n = 0$

Therefore the Fourier series for the function $f(x)$ is given by

$$F(x) = \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left[\frac{1}{2} (\sin (n+1) x + \sin (n-1) x) \right] dx$$

$$= \frac{1}{\pi} \left[-\frac{\cos (n+1) x}{n+1} - \frac{\cos (n-1) x}{n-1} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1} \frac{-(-1)^{n+1}}{n-1} + \frac{1}{n-1} \right]$$

$$= \frac{1}{\pi} \left[-\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n-1} + \frac{1}{n+1} + \frac{1}{n-1} \right]$$

$$b_n = \frac{2n}{\pi(n^2-1)} [1-(-1)^{n+1}] \longrightarrow (2).$$

Substituting (2) in (1) we get

$$f(x) = \sum_{n=1}^{\infty} \frac{2n}{\pi(n^2-1)} (1-(-1)^{n+1}) \sin nx$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[1-(-1)^{n+1}]}{n^2-1} \sin nx$$

14] Prove that the series cosine series for $f(x) = \sin x$ in $0 < x < \pi$ is $f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{2,4,6\ldots}^{\infty} \frac{\cos nx}{n^2-1}$

Solution

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin x \, dx$$

$$= \frac{2}{\pi} (-\cos x)_0^{\pi} = \frac{4}{\pi} \quad \boxed{a_0 = 4/\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx = \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi \cos n\pi}{n} - (-\pi) - \left(\frac{-\cos n\pi}{n}\right)\right]$$

$$= \frac{1}{n}\left[-2 \cos n\pi\right] = \frac{2}{n}(-1)^{n+1}$$

$$b_n = \frac{2}{n}(-1)^{n+1} \quad \ldots\ldots (2).$$

Substituting (2) in (1), we get

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$$

13] EXPAND the $\cos x$ in a half-range sine series $(0, \pi)$

Solution

The sine series for the function $f(x)$ in $(0, \pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \rightarrow (1)$$

Now, $b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx\, dx$

[$\because$ Here $f(x) = \cos x$]

$$= \frac{2}{\pi}\int_0^{\pi} \sin nx \cos x\, dx$$

[$\because \sin(A+B) + \sin(A-B) = 2\sin A \cos B$]

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{(n+1)}\right]_0^{\pi} - \frac{1}{\pi}\left[-\frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1}\right] - \frac{1}{\pi}\left[-\frac{(-1)^{n-1}}{n-1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

Since $\cos n\pi = (-1)^n$ and $\cos 0 = 1$

$$a_n = \frac{-2}{\pi}\left(\frac{(-1)^n+1}{n^2-1}\right).$$

$$a_n = \frac{-4}{\pi}\left(\frac{1}{n^2-1}\right)$$ when n is even and zero when n is odd

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{2,4,6\ldots}^{\infty} \frac{\cos nx}{n^2-1}$$

5) Find the Half-range sine series for the function

$$f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases}$$

Solution:

The Sine Series for the function $f(x)$ in $(0,\pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

10 MARKS

1) If $\vec{F} = xyz\,\vec{i} + xz^2\,\vec{j} - y^3\,\vec{k}$ and $\vec{G} = x^3\vec{i} - xyz\,\vec{j} + x^2z\,\vec{k}$ find the value of (i) $\frac{\partial^2 F}{\partial x \partial y}$ at the origin

(ii) $\frac{\partial^2 \vec{F}}{\partial y^2} \times \frac{\partial^2 \vec{G}}{\partial x^2}$ at $(1,1,0)$

Solution:

given $\vec{F} = xyz\,\vec{i} + xz^2\,\vec{j} + y^3\,\vec{k}$

$\vec{G} = x^3\,\vec{i} - xyz\,\vec{j} + x^2z\,\vec{k}$

(i) $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{F}}{\partial y}\right)$

$\frac{\partial \vec{F}}{\partial y} = \frac{\partial}{\partial y}(\vec{F})$

$= \frac{\partial}{\partial y}(xyz\,\vec{i} + xz^2\,\vec{j} - y^3\,\vec{k})$

$= xz\,\vec{i} - 3y^2\,\vec{k}$

$\frac{\partial^2 \vec{F}}{\partial x \partial y} = \frac{\partial}{\partial x}(xz\,\vec{i} - 3y^2\,\vec{k})$

$= z\,\vec{i}$

$\left(\frac{\partial^2 F}{\partial x \partial y}\right)_{(1,1,0)} = 0\,\vec{i} = 0.$

(ii) $\frac{\partial^2 \vec{F}}{\partial y^2} \times \frac{\partial^2 \vec{G}}{\partial x^2}$

$\frac{\partial \vec{F}}{\partial y} = xz\,\vec{i} - 3y^2\,\vec{k}$

Now $b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx\,dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx\,dx\right]$

$$= \frac{2}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(-\frac{\cos nx}{n}\right) + \left(-\frac{\sin nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left\{\left[\frac{-\pi/2 \cos n\pi/2}{n} + \frac{\sin n\pi/2}{n^2}\right\} - (0-0)\right\} + \left[\left\{(\pi-\pi)\left(-\frac{\cos nx}{n}\right) + \left(-\frac{\sin n\pi}{n^2}\right)\right\}\right] - \left\{(\pi - \pi/2) - \left(-\frac{\cos n\pi/2}{n}\right) + \left(\frac{-\sin n\pi/2}{n^2}\right)\right\}\right]$$

$$= \frac{2}{\pi}\left[\cancel{\frac{-\pi/2 \cos n\pi/2}{n}} + \frac{\sin n\pi/2}{n^2} + \cancel{\frac{\pi/2 \cos n\pi/2}{n}} + \frac{\sin n\pi/2}{n^2}\right]$$

$$= \frac{2}{\pi} \cdot \frac{2 \sin n\pi/2}{n^2}$$

$$= \frac{4}{\pi} \frac{\sin n\pi/2}{n^2}$$

When n is even $b_n = 0$.

When n is odd, then

$$f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$$

$$\frac{\partial^2 \vec{F}}{\partial y^2} = -6y\vec{k}$$

$$\frac{\partial \vec{G}}{\partial x} = \frac{\partial}{\partial x}(x^3\vec{i} - xyz\vec{j} + x^2z\vec{k})$$

$$= 3x^2\vec{i} - yz\vec{j} + 2xz\vec{k}$$

$$\frac{\partial^2 G}{\partial x^2} = 6x\vec{i} + 2z\vec{k}$$

$$\therefore \frac{\partial^2 \vec{F}}{\partial y^2} \times \frac{\partial^2 \vec{G}}{\partial x^2} = (-6y\vec{k}) \times (6x\vec{i} + 2z\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & -6y \\ 6x & 0 & 2z \end{vmatrix}$$

$$= 0\vec{i} - \vec{j}[0 + 36xy] + \vec{k}(0)$$

$$= -36xy\vec{j}$$

$$\left[\frac{\partial^2 \vec{F}}{\partial y^2} \times \frac{\partial^2 \vec{G}}{\partial x^2}\right]_{at(1,1,0)} = -36(1)(1)\vec{j}$$

$$= -36\vec{j}$$

2) prove that (i) $\nabla \cdot (\vec{F} \pm \vec{G}) = \nabla \cdot \vec{F} \pm \nabla \cdot \vec{G}$

(ii) $\nabla \times (\vec{F} \pm \vec{G}) = \nabla \times \vec{F} \pm \nabla \times \vec{G}$

Proof:

(i) $\nabla \cdot (\vec{F} \pm \vec{G})$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\vec{F} \pm \vec{G})$$

$$= \vec{i}\,\frac{\partial}{\partial x}(\vec{F} \pm \vec{G}) + \vec{j}\,\frac{\partial}{\partial y}(\vec{F} \pm \vec{G}) + \vec{k}\,\frac{\partial}{\partial z}(\vec{F} \pm \vec{G})$$

$$= \vec{i}\left(\frac{\partial \vec{F}}{\partial x} \pm \frac{\partial \vec{G}}{\partial x}\right) + \vec{j}\left(\frac{\partial \vec{F}}{\partial y} \pm \frac{\partial \vec{G}}{\partial y}\right) + \vec{k}\left(\frac{\partial \vec{F}}{\partial z} \pm \frac{\partial \vec{G}}{\partial z}\right)$$

$$= \left(\vec{i}\,\frac{\partial \vec{F}}{\partial x} + \vec{j}\,\frac{\partial \vec{F}}{\partial y} + \vec{k}\,\frac{\partial \vec{F}}{\partial z}\right) \pm \left(\vec{i}\,\frac{\partial \vec{G}}{\partial x} + \vec{j}\,\frac{\partial \vec{G}}{\partial y} + \vec{k}\,\frac{\partial \vec{G}}{\partial z}\right)$$

$$\nabla\cdot(\vec{F} \pm \vec{G}) = \nabla\cdot\vec{F} \pm \nabla\cdot\vec{G}$$

(ii) $\nabla \times (\vec{F} \pm \vec{G})$

$$= \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \times (\vec{F} \pm \vec{G})$$

$$= \vec{i} \times \frac{\partial}{\partial x}(\vec{F} \pm \vec{G}) + \vec{j} \times \frac{\partial}{\partial y}(\vec{F} \pm \vec{G}) + \vec{k} \times \frac{\partial}{\partial z}(\vec{F} \pm \vec{G})$$

$$= \left(\vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}\right) \pm \left(\vec{i} \times \frac{\partial \vec{G}}{\partial x} + \vec{j} \times \frac{\partial \vec{G}}{\partial y} + \vec{k} \times \frac{\partial \vec{G}}{\partial z}\right)$$

$$\nabla \times (\vec{F} \pm \vec{G}) = \nabla \times \vec{F} \pm \nabla \cdot \vec{G}$$

3) Evaluate $\iint_S \phi \cdot \hat{n}\, ds$ where $\phi = \frac{3}{8}xyz$ and S is the surface of the cylinder $x^2+y^2=16$ included in the surface first octant b/w $z=0$ and $z=5$

Solution:

$$\iint_S \phi \cdot \hat{n}\, ds = \iint_R \phi \cdot \hat{n}\, \frac{dx\,dz}{|\hat{n}\cdot\vec{j}|}$$

where R is the projection of the surfaces on xz plane

$$\nabla(x^2+y^2) = 2x\vec{i} + 2y\vec{j}$$

$$|\nabla(x^2+y^2)| = \sqrt{(2x)^2+(2y)^2} = \sqrt{4x^2+4y^2}$$

$$= \sqrt{4(x^2+y^2)} = \sqrt{64} = 8$$

$$\therefore \hat{n} = \frac{2x\vec{i}+2y\vec{j}}{8} = \frac{x\vec{i}+y\vec{j}}{4}$$

$$\hat{n}\cdot\vec{j} = \left(\frac{x\vec{i}+y\vec{j}}{4}\right)\cdot\vec{j} = y/4$$

$$\therefore \iint_S \phi\cdot\hat{n}\,ds = \iint_R \phi\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot j|} \rightarrow (1)$$

$$= \iint_R 3/8\, xyz\left(\frac{x\vec{i}+y\vec{j}}{4}\right)\frac{dx\,dy}{y/4}$$

$$= \iint_R 3/8\, xz\,(x\vec{i}+y\vec{j})\,dx\,dz$$

$$= 3/8\iint_R (x^2z\vec{i} + xzy\vec{j})\,dx\,dz$$

$$= 3/8\iint_R [x^2z\vec{i} + xzy\vec{j}]\,dx\,dz$$

$$= 3/8\iint_R [x^2z\vec{i} + xz\sqrt{16-x^2}\,\vec{j}]\,dx\,dz$$

$$= 3/8\iint_R [x^2z\vec{i} + xz\sqrt{16-x^2}\,\vec{j}]\,dx\,dz$$

$$= 3/8\int_{z=0}^{5}\int_{x=0}^{4} [x^2z\vec{i} + xz\sqrt{16-x^2}\,\vec{j}]\,dx\,dz$$

4) If $\vec{F} = (2x^2-3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$ then evaluate

(i) $\iiint_V \nabla\times\vec{F}\,dv$ (ii) $\iiint_V \nabla\cdot\vec{F}\,dv$ where

V is the region bounded by $x=0$; $y=0$; $z=0$ and $2x+2y+z=4$

Solution:

(i) $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(-4+3) + \vec{k}(-2y-0)$$

$$= \vec{j} - 2y\vec{k}$$

$$\therefore \iiint_V \nabla\times\vec{F}\,dv = \iiint_V (\vec{j}-2y\vec{k})\,dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} (\vec{j}-2y\vec{k})\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} (\vec{j}-2y\vec{k})\,[z]_0^{4-2x-2y}\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} (\vec{j}-2y\vec{k})(4-2x-2y)\,dy\,dx$$

$$= \int_{x=0}^{2} \left[(4y-2xy-y^2)\vec{j} - 2(2y^2-xy^2-\tfrac{2}{3}y^3)\vec{k}\right]_0^{2-x} dx$$

$$= \frac{3}{8}\int_{z=0}^{5}\left\{\left[\frac{x^3 z}{3}\vec{i}\right]_0^4 + z\int_0^4 x\sqrt{16-x^2}\,\vec{j}\,dx\right\}dz$$

$$= \frac{3}{8}\int_{z=0}^{5}\left[\frac{64z}{3}\vec{i} + z\vec{j}\,I_1\right]dz \longrightarrow (2)$$

where

$$I_1 = \int_0^4 x\sqrt{16-x^2}\,dx$$

put $x = 4\sin\theta$; $dx = 4\cos\theta\,d\theta$

when $x=0$; $\theta = 0$

$x = 4$; $\theta = \pi/2$

$$\therefore I = \int_0^{\pi/2} 4\sin\theta\sqrt{16-16\sin^2\theta}\,.4\cos\theta\,.d\theta$$

$$= 64\int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta$$

$$= 64\int_0^{\pi/2}\cos^2\theta\,d(-\cos\theta)$$

$$I_1 = 64\left[\frac{-\cos^3\theta}{3}\right]_0^{\pi/2}$$

$$I_1 = 64/3$$

$$\iint_S \phi\cdot\hat{n}\,ds = \frac{3}{8}\int_{z=0}^{5}\left[\frac{64z}{3}\vec{i} + \frac{64z}{3}\vec{j}\right]dz$$

substituting (2) in (1)

$$= \frac{3}{8}\left(100\vec{i} + 100\vec{j}\right)$$

$$= 2\int_0^2 (x^3 - 4x^2 + 4x)\,dx$$

$$= 2\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{4x^2}{2}\right]_0^1$$

$$= 2\left[4 - \frac{32}{3} + 8\right]$$

$$= 8/3 \,'$$

5) verify the gauss divergence theorem $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x=0;\ x=1;\ y=0;\ y=1;\ z=0;\ z=1$

Solution:

gauss divergence theorem is

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

given $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\vec{F}$$

$$= \frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y$$

$$= 4z - y$$

Now,

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 (4z-y)\,[x]_0^1\,dy\,dz$$

$$= \int_{x=0}^{2} \left[(2-x^2)\vec{j} - \tfrac{2}{3}(2-x)^3\vec{k}\right] dx$$

$$= \left[-\frac{(2-x)^3}{3}\vec{j} + \frac{2}{3}\frac{(2-x)^4}{4}\right]_0^2$$

$$= \tfrac{8}{3}(\vec{j}-\vec{k}).$$

(ii) $\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left[(2x^2-3z) - 2xy - 4x\right]$

$$= \frac{\partial}{\partial x}(2x^2-3z) - \frac{\partial}{\partial y}(2xy) - \frac{\partial}{\partial z}(4x$$

$\nabla\cdot\vec{F} = 4x - 2x = 2x$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \iiint_V 2x\,dx\,dy\,dz$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{4-2x-2y} x\,dz\,dy\,dx$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2} x\,[z]_0^{4-2x-2y}\,dy\,dx$$

$$= 2\int_{x=0}^{2}\int_{0}^{2}(4x-2x^2-2xy)\,dy\,dx$$

$$= 2\int_{x=0}^{2}\int_{0}^{2}[4x-2x^2-2xy]\,\cancel{[\quad]}\,dy\,dx$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(\vec{i})\,dy\,dz$$

$$= \int_0^1\int_0^1 4xz\,dy\,dz \qquad \boxed{x=1}$$

$$= \int_0^1\int_0^1 4z\,dy\,dz$$

$$= 4\int_0^1 \left[\frac{z^2}{2}\right]_0^1 dy = 2\int_0^1 dy$$

$$= 2[y]_0^1$$

Evaluation of

$$= 2 \longrightarrow (1).$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{i})\,dy\,dz$$

$$= \int_0^1\int_0^1 -4xz\,dy\,dz \qquad \boxed{x=0}$$

$$= -4\int_0^1\int_0^1 0\,dy\,dz$$

Evaluation of

$$= 0 \longrightarrow (2)$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\,(\vec{j})\,dx\,dz$$

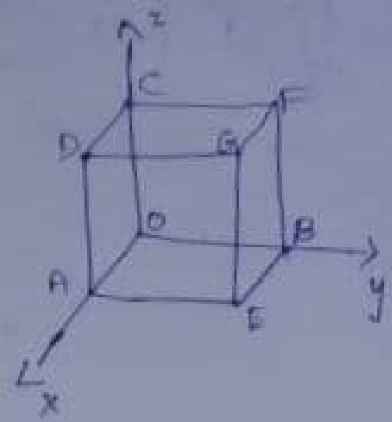
$$= \int_0^1\int_0^1 -y^2\,dx\,dz \qquad \boxed{y=1}$$

$$= -\int_0^1\int_0^1 dx\,dz$$

$$= -\int_0^1 [x]_0^1\,dz = -1[z]_0^1 = -1$$

$$= -1 \longrightarrow (3)$$

$$= \int_0^1 \int_0^1 (4z - y)\, dy\, dz$$



$$= \int_0^1 \left[4yz - \tfrac{1}{2} y^2\right]_0^1 dz$$

$$= \int_0^1 \left[4z - \tfrac{1}{2}\right] dz$$

$$= \left[\frac{4z^2}{2} - \tfrac{1}{2} z\right]_0^1$$

$$= \left[2z^2 - \tfrac{1}{2} z\right]_0^1 = 3/2$$

$$\iiint_V \nabla \cdot \vec{F}\, dv = 3/2$$

Now $\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

where

S_1 = Face AEGD $\Rightarrow$ Surface is $\vec{i}$

S_2 = Face OBFC $\Rightarrow$ surface is $-\vec{i}$

S_3 = Face EBFG $\Rightarrow$ Surface is $\vec{j}$

S_4 = Face OADC $\Rightarrow$ Surface is $-\vec{j}$

S_5 = Face DGFC $\Rightarrow$ Surface is $\vec{k}$

S_6 = Face OAEB $\Rightarrow$ Surface is $-\vec{k}$

Evaluation of $\iint_{S_1} \vec{F} \cdot \hat{n}\, ds$

Evaluation of $\iint_{S_4} \vec{F}\cdot\hat{n}\, ds$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\, ds = \iint (4xz\vec{i} - y^2\vec{j} + yz\vec{k})(-\vec{j})\, dx\, dz \quad \boxed{y=0}$$

$$= \int_0^1\int_0^1 y^2\, dx\, dz$$

$$= 0 \longrightarrow (4)$$

Evaluation of $\iint_{S_5} \vec{F}\cdot\hat{n}\, ds$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\, ds = \iint (4xz\vec{i} - y^2\vec{j} + yz\vec{k})(\vec{k})\, dy\, dx \quad \boxed{z=1}$$

$$= \int_0^1\int_0^1 yz\, dx\, dy$$

$$= \int_0^1\int_0^1 y\, dx\, dy$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx = \frac{1}{2}\int_0^1 dx = \frac{1}{2}[1]$$

$$= \frac{1}{2} \rightarrow (5)$$

Evaluation of $\iint_{S_6} \vec{F}\cdot\hat{n}\, ds$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\, ds = \iint (4xz\vec{i} - y^2\vec{j} + yz\vec{k})(-\vec{k})\, dx\, dy \quad \boxed{z=0}$$

$$= \int_0^1\int_0^1 -yz\, dx\, dy$$

$$= 0 \rightarrow (6)$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

$$= 2 + 0 - 1 + 0 + \frac{1}{2} + 0$$

$= 3/2$

from (A) & (B) we get

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

Hence gauss divergence theorem is verified.

6) verify stoke's Theorem for a vector field defined by $\vec{F} = (x^2-y^2)\vec{i} + 2xy\,\vec{j}$ in a rectangular region in the xoy plane bounded by the lines

$x=0;\ x=a;\ y=0;\ y=b$

Solution

Stoke's theorem is

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds$$

Evaluation of $\int_C \vec{F}\cdot d\vec{r}$

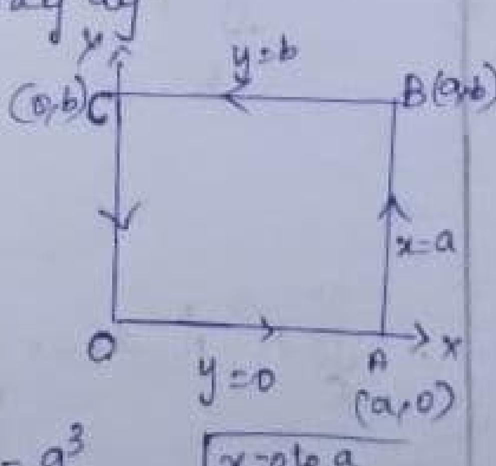
given $\vec{F} = (x^2-y^2)\vec{i} + 2xy\,\vec{j}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$$\vec{F}\cdot d\vec{r} = (x^2-y^2)\,dx + 2xy\,dy$$

now, $\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2-y^2)\,dx + 2xy\,dy$

$$= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$



Along OA $(y=0)$

$$\int_{OA} (x^2-y^2)\,dx + 2xy\,dy$$

$$= \int_0^a x^2\,dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

$x = 0$ to a
$y = 0;\ dy = 0$

Along AB $(x=a)$

$$\int_{AB}(x^2-y^2)dx+2xy\,dy = \int_0^b 2ay\,dy = \cancel{2}a\left[\frac{y^2}{\cancel{2}}\right]_0^b$$

$$= ab^2$$

$y=0$ to b
$x=0$; $dx=0$

Along BC $(y=b)$

$$\int_{BC}(x^2-y^2)dx+2xy\,dy = \int_a^0 (x^2-b^2)dx$$

$$= \left(\frac{x^3}{3}-b^2x\right)_a^0$$

$$= -\frac{a^3}{3}+ab^2$$

$x=a$ to 0
$y=b$; $dy=0$

Along CO $(x=0)$

$$\int_{CO}(x^2-y^2)dx+2xy\,dy = \int_{CO}(0+0) = 0$$

Hence

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA}+\int_{AB}+\int_{BC}+\int_{CO}$$

$$= \frac{a^3}{3}+ab^3-\frac{a^3}{3}+ab^2+0$$

$$= 2ab^2 \longrightarrow (1)$$

Evaluation of $\iint_S (\nabla\times\vec{f})\,\hat{n}\,.ds$

given $\vec{F} = (x^2-y^2)\vec{i}+2xy\vec{j}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1^2+n^2}(1\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1}{(1^2+n^2)\pi}\times(e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi)$$

$$= \frac{\cos n\pi\,(e^{\pi}-e^{-\pi})}{\pi(1^2+n^2)} = \frac{2(-1)^n \sinh\pi}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[-ne^{\pi}\cos n\pi + ne^{-\pi}\cos n\pi\right]$$

$$= \frac{n(-1)^n\left[e^{-\pi}-e^{\pi}\right]}{\pi(1+n^2)} = \frac{2n(-1)^{n+1}\sinh\pi}{\pi(1+n^2)}$$

Now we have

$$a_0 = \frac{2\sinh\pi}{\pi},\quad a_n = \frac{2(-1)^n\sinh\pi}{\pi(1+n^2)},$$

$$b_n = \frac{2n(-1)^{n+1}\sinh\pi}{\pi(1+n^2)}$$

$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(2y+2y) = 4y\vec{k}$

Here the surface S denotes the rectangle OABC and the unit outward normal vector is $\vec{k}$

$\hat{n} = \vec{k}$

$$\iint_S \nabla \cdot \vec{F}\,\hat{n}\,ds = \iint_S 4y\,\vec{k}\cdot\vec{k}\,dx\,dy$$

$$= \iint_S 4y\,dx\,dy$$

$$= 4\int_0^b\int_0^a y\,dx\,dy = 4\int_0^b y(x)_0^a\,dy$$

$$= 4\int_0^b ay\,dy$$

$$= \cancel{4}^{2} a\left[\frac{y^2}{\cancel{2}}\right]_0^b = 2ab^2$$

Hence the stoke's theorem is verified.

7) Find the fourier series for the function $f(x) = e^x$ defined in $(-\pi, \pi)$

Solution

The fourier series for the function $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

Here $f(x) = e^x$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx$$

$$= \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right]$$

$$= \frac{2}{\pi}\sin h\pi$$

Substituting these values in (1), we get

$$f(x) = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos nx - \sum_{n=1}^{\infty}\frac{n(-1)^n}{1+n^2}\sin nx\right]$$

8) Obtain the Fourier series to represent the function $f(x)=|x|$, $-\pi < x < \pi$ and deduce $\frac{1}{1^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{8}$

Solution

Given $f(x) = |x|$

$\therefore f(-x) = |-x| = |x|$

$f(x) = f(-x) = |x|$

The given function $f(x)=|x|$ is an even function

$\therefore$ The Fourier co-efficient $b_n = 0$

Hence the Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad [\because b_n = 0] \rightarrow (1)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}|x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi \quad [\because \text{In } (0,\pi)\ |x| = x] \rightarrow (2)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}|x|\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

(i.e) $a_n = 0$ if n is even

$= \frac{-4}{n^2\pi}$; if n is odd $\rightarrow (3)$

Substituting (2) & (3) in (1) we get

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi} \cos nx$$

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) \rightarrow (4).$$

Putting $x = 0$ in (4) we get

$$0 = \pi/2 - 4/\pi\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Here $x = 0$ is a point of continuity

$-\pi < x < \pi$.

9) Find the half-range cosine series for the function $f(x) = x^2,\ 0 \le x \le \pi$

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \cdots\cdots (1)$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$

10] Find a cosine series for the function

$$f(x) = \begin{cases} x & \text{in } 0 \le x \le \pi/2 \\ \pi - x & \text{in } \pi/2 \le x < \pi \end{cases}$$

Solution

The cosine series for the function f(x) in $(0,\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow (1)$$

Now,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right]$$

$$a_0 = \pi/2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\cancel{\frac{\pi/2 \sin \frac{n\pi}{2}}{n}} + \frac{\cos n\pi/2}{n^2} - \frac{1}{n^2} - \frac{\cos n\pi}{n^2} - \cancel{\frac{\pi/2 \sin\frac{n\pi}{2}}{n}} + \frac{\cos n\pi/2}{n^2}\right]$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{3\pi}\left[x^3\right]_0^{\pi} = \frac{2}{3\pi}\left[(\pi)^3 - 0\right] = \frac{2\pi^2}{3}$$

$$\therefore a_0 = \frac{2}{3}\pi^2 \quad \longrightarrow (2)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left\{\frac{-\cos nx}{n\times n}\right\} + (2)\left\{\frac{-\sin nx}{n\times n\times n}\right\}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2x\cos nx}{n^2}\right]_0^{\pi} \quad \left[\because (\sin\text{ terms})_0^{2\pi} = 0\right]$$

$$= \frac{4}{\pi n^2}\left[x\cos nx\right]_0^{\pi}$$

$$= \frac{4}{\pi n^2}\left[\pi\cos n\pi - 0\right] = \frac{4}{\pi n^2}\left[\pi(-1)^n\right] = \frac{4}{n^2}(-1)^n$$

$$\therefore a_n = \frac{4}{n^2}(-1)^n \quad \longrightarrow (3)$$

use (2) and (3) in (1) we get

$$f(x) = \frac{\frac{2}{3}\pi^2}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$