1)

Linear Algebra

Unit - 1

2m :-

1) A square matrix is symmetric if $A = A^T$

Let A be a symmetric matrix then

then $(i, j)^{th}$ of A

$= (i, j)^{th}$ entry of A

$= (j, i)^{th}$ entry of A^T

Hence $A = A^T$

Hence, A is symmetrix.

2) Let A be any square matrix $A + A^T$ is symmetric

Let A be any square matrix

$(A + A^T)^T = A^T + (A^T)^T$

$= A^T + A$

$= A + A^T$

Hence $A + A^T$ is symmetrix.

3) Let A and B be square matrix of the same order then (i) AB are Hermition

Let A and B. square matrix

$(\overline{A+B})^T = (\bar{A} + \bar{B})^T = \bar{A}^T + \bar{B}^T$

$= A + B.$

Since A and B are Hermition.

$\therefore$ A+B is Hermition.

4) AB are skew Hermition.

A+B is Hermition.

$(\overline{A+B})^T = (\bar{A}+\bar{B})^T = (-\bar{A}-\bar{B})^T$

$= -\bar{A}^T - \bar{B}^T$

$= -(A+B)$

since $A+B$ is skew Hermition.

5) A is Hermition, $AB+BA$ is Hermition.

$(\overline{AB}+\overline{BA})^T = (\overline{AB}+\overline{BA})^T$

$= (\overline{AB})^T + (\overline{BA})^T$

$= (\bar{A}\bar{B})^T + (\bar{B})^T\bar{A}^T$

$= \bar{A}^T\bar{B}^T + \bar{B}^T\bar{A}^T$

$= AB+BA$

$AB+BA$ is Hermition.

6) A is Hermition, A is Hermition.

$iA = -(-iA)^T$

$= iA^T$

$iA = \cancel{i}A^T$

$A = A^T$

A is a Hermition.

7) If $A = \begin{bmatrix} 4 & 6 & 9 \\ 3 & 5 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 & 1 \\ 4 & -7 & -8 \end{bmatrix}$ to find $A+B$ and $A-B$.

$A+B = \begin{bmatrix} 4 & 6 & 9 \\ 3 & 5 & 10 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 1 \\ 4 & -7 & -8 \end{bmatrix}$

$= \begin{bmatrix} 9 & 6 & 10 \\ 7 & -2 & 2 \end{bmatrix}$

$A-B = \begin{bmatrix} -1 & 6 & 8 \\ -1 & 12 & 18 \end{bmatrix}$ //

8) If $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ find adj A.

$\text{adj } A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ //

9) If $A = \begin{bmatrix} 5 & -2 \\ 1 & 7 \end{bmatrix}$ find adj A.

$\text{adj} A = \begin{bmatrix} -7 & -1 \\ 8 & 5 \end{bmatrix}$ //

10) If $A = \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix}$ find A^{-1}

$\text{adj} A = \begin{bmatrix} 3 & -4 \\ -1 & 5 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \text{adj} (A)$

$|A| = 15 - 4 = 11$

$= \frac{1}{|A|} \begin{bmatrix} 3 & -4 \\ -1 & 5 \end{bmatrix}$

$A^{-1} = \frac{1}{11} \begin{bmatrix} 3 & -4 \\ -1 & 5 \end{bmatrix}$

5 Mark :

1) Let A and B by symmetrix matrix of order

(i) A+B is symmetrix.

(ii) AB is symmetrix if AB = BA

(iii) AB + BA is symmetrix

(iv) kA is symmetrix.

(i) $(A+B)^T = A^T + B^T = A+B$

A and B are symmetrix

$\therefore$ A+B is symmetrix.

(ii) AB is symmetric.

$(AB)^T = AB$

$B^T A^T = BA$

$BA = AB$

(iii) $(AB + BA)^T = (AB)^T + (BA)^T$

$= A^T B^T + B^T A^T$

$= BA + AB$

since $AB + BA$ is symmetric

(iv) $(KA)^T = KA^T$

$= KA$

KA is symmetric //.

2) To find determinant $A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 7 & 9 \\ 1 & 6 & 4 \end{bmatrix}$

$B = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 2 & 5 \\ 6 & -2 & 7 \end{bmatrix} \Rightarrow 5(A+B) = 5A + 5B.$

L.H.S

$A+B = \begin{bmatrix} 5 & 4 & 7 \\ 8 & 9 & 14 \\ 7 & 4 & 11 \end{bmatrix}$

$5(A+B) = \begin{bmatrix} 25 & 20 & 37 \\ 40 & 45 & 70 \\ 35 & 20 & 55 \end{bmatrix}$

R.H.S

$5A = \begin{bmatrix} 10 & 15 & 25 \\ 20 & 35 & 45 \\ 5 & 30 & 20 \end{bmatrix}$

$5B = \begin{bmatrix} 15 & 5 & 10 \\ 20 & 10 & 25 \\ 30 & -10 & 35 \end{bmatrix}$

$$5A + 5B = \begin{bmatrix} 25 & 20 & 35 \\ 40 & 45 & 70 \\ 35 & 20 & 55 \end{bmatrix}$$

$$1 \cdot 4 \cdot 3 = 12 \cdot 4 \cdot 5$$ //

3) If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix}$ so find A^{-1}

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$$

$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{vmatrix}$$

$$= 1$$

$$\operatorname{adj}(A) = \begin{bmatrix} -26 & -7 & 2 \\ 11 & 3 & -5 \\ -5 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} -26 & -7 & 2 \\ 11 & 3 & -5 \\ -5 & -1 & 2 \end{bmatrix}$$ //

4) A square matrix A of the order n is non-singular if A is invertible

suppose A is invertible them there exists a matrix B.

$$AB = BA = I$$

Hence $|AB| = |I| = 1.$

$$|A|\,|B| = I$$

Hence, $|A| \neq 0$, so that A is non-singular

Conversely: Let A be non-singular.

Hence $|A| \neq 0$.

Now, consider the matrix.

$B = \frac{1}{|A|} \text{adj } A = A^{-1}$

Then $AB = BA = I$

A is invertible and B is inverse.

5) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 2 \\ 2 & -1 & 6 & 5 \end{bmatrix}$ find the Rank of matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 2 \\ 2 & -1 & 6 & 5 \end{bmatrix} \quad C_4 \leftrightarrow C_3$$

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 0 \\ 2 & -1 & 5 & 6 \end{bmatrix} \quad R_3 \to R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & -5 & -3 & 0 \end{bmatrix} \quad C_2 \leftrightarrow C_3$$

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & -3 & -5 & 0 \end{bmatrix} \quad R_3 \to R_3 + 5R_2$$

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 7 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$

10 Mark:

1) Let A and B skew symmetric matrix of order then, (i) A + B is skew symmetric matrix

(ii) kA is skew matrix $k \in F$

(iii) A^{2n} is a symmetric matrix and A^{2n+1} is a skew symmetric matrix where A is any +ve integer.

Proof: Let A and B be skew symmetric

(i) $(A+B)^T = A^T + B^T = -A - B = -(A+B)$

$(A+B)$ is a skew symmetric matrix.

(ii) $(kA)^T = kA^T$

$= -kA = -(kA)$.

kA is a skew symmetric matrix

(iii) Let m be any +ve integer.

$(A^m)^T = (A.A.A \ldots m \text{ times})^T$

$= (A^T A^T \ldots m \text{ times})$

$= (-A)(-A) \ldots m \text{ times}.$

$= -(A.A \ldots m \text{ times})$

$= (-1)^m (A^m)$

$$(A^m)^T = \begin{cases} A^m & \text{if } m \text{ is even} \\ -A^m & \text{if } m \text{ is odd.} \end{cases}$$

A^m is symmetric when m is even and

$-A^m$ is skew symmetric when m is odd.

8) Let A and B orthogonal matrix of the ... order then (i) A^T is orthogonal

(ii) AB is orthogonal.

Let A and B orthogonal matrix

(i) $A^T (A^T)^T = A^T \cdot A = I$

since A is orthogonal.

III^{ly} $(A^T)^T A^T = A \cdot A^T = I$

since A^T is orthogonal.

(ii) $(AB)(AB)^T = AB(B^T A^T)$

$= A(BB^T)A^T$

$= AIA^T = AA^T = I$

III^{ly} $(AB)^T AB = B^T A^T (AB)$

$= B^T (A^T A) B$

$= B^T I B$

$= B^T B$

$= I$

Hence AB is orthogonal

3) The characteristic roots of a hermition matrix are all real.

Let A be a hermition matrix

Since $A = A^{-1}$

Let λ be a characteristic root of A

and let x be a characteristic vectors.

$Ax = \lambda x$ —(1)

Now

$A\bar{x} = \bar{\lambda}\bar{x}$

$\bar{x}^T A x = \bar{x}^T \lambda x$

$\Rightarrow x^T A^T (\bar{x}^T)^T = \lambda \bar{x}^T x$

$\Rightarrow x^T A^T \bar{x} = \lambda \bar{x}^T x$

$\Rightarrow \overline{x^T A^T \bar{x}} = \overline{\lambda \bar{x}^T x}$

$x \bar{x}^T \bar{A}^T = \bar{\lambda}\, x^T \bar{x}$

$\Rightarrow \lambda(\bar{x}^T x) = \bar{\lambda}(x^T \bar{x})$ —(3)

Now, $\bar{x}^T x = \bar{x}_1 x_1 + \bar{x}_2 x_2 + \dots \bar{x}_n x_n$

$= |x_1|^2 + |x_2|^2 + |x_3|^2 \dots |x_n|^2$

$\neq 0$.

from (3) we get, $\lambda = \bar{\lambda}$

Hence λ is real.

4) $(AB)C = A(BC)$ $\Rightarrow A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix}$

L.H.S :- $AB = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 6 & 3 \\ 2 & 12 & 9 \end{bmatrix}$

$(AB)C = \begin{bmatrix} 23 & 37 & 56 \\ 31 & 47 & 54 \end{bmatrix}$

R.H.S

$$A(BC) = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 19 & 14 \\ 5 & -1 & 2 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 9 & 19 & 14 \\ 5 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 37 & 30 \\ 31 & 77 & 54 \end{bmatrix}$$

$$(AB)C = A(BC)$$ //

5) find A^{-1} of $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix} = 35.$$

$$\text{adj} A = \begin{bmatrix} +(2-6) & -(4-3) & +(8-2) \\ -(3-14) & +(1-7) & -(2-3) \\ +(9-14) & -(3-28) & +(2-12) \end{bmatrix}^T$$

$$= \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 5 \\ 6 & 1 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj} A.$$

$$A^{-1} = \frac{1}{35} \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 5 \\ 6 & 1 & -10 \end{bmatrix}$$ //

Unit - 5

1) 2 Mark

1) Define characteristic equation?

If any matrix A. $|A-\lambda I|=0$ is called the characteristic equation of matrix

2) Find the characteristic equation of $A=\begin{bmatrix}1 & 2\\ 0 & 2\end{bmatrix}$

$$|A-\lambda I| = \left|\begin{pmatrix}1 & 2\\ 0 & 2\end{pmatrix} - \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right|$$

$$= \begin{vmatrix}1-\lambda & 2\\ 0 & 2-\lambda\end{vmatrix}$$

$$(1-\lambda)(2-\lambda)-0=0.$$

$$2-\lambda-2\lambda+\lambda^2=0$$

$$\lambda^2-3\lambda+2=0$$

3) Find the characteristic equation of $\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}$

$$|A-\lambda I| = \begin{vmatrix}1-\lambda & -1\\ 2 & 3-\lambda\end{vmatrix}$$

$$= 3-3\lambda-\lambda+\lambda^2+2=$$

$$\lambda^2-4\lambda+5=0$$

4) Find the characteristic eqn of $\begin{bmatrix}1 & 3\\ 4 & -2\end{bmatrix}$

$$|A-\lambda I| = \begin{vmatrix}1-\lambda & 3\\ 4 & -2-\lambda\end{vmatrix}$$

$$(1-\lambda)(-2-\lambda)-12=0.$$

$$-2+2\lambda-\lambda+\lambda^2-12=0.$$

$$\lambda^2+\lambda-14=0.$$

5) S.T the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ satisfies its characteristic equation.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$|A - \lambda I| = 0.$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 1-\lambda \end{vmatrix}$$

$$= \lambda^2 - 2\lambda - 5 = 0.$$

$$A^2 - 2A - 5I = \begin{pmatrix} 7 & 4 \\ 6 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 6 & 2 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} //$$

6) Define eigen vectors?

The column matrix which satisfies the equation $AX = \lambda X$ for each corresponding value of λ is called eigen vectors.

7) Define eigen values?

Let A be an $n \times n$ matrix given a matrix. The roots of the characteristic equation $|A - \lambda I| = 0$ is called characteristic root.

8) Find the eigen values of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$|A - \lambda I| = 0.$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix}$$

$$(1-\lambda^2) = 0$$

$$\lambda = \pm 1 //$$

5 Mark

) ✓ 1) Find the characteristic eqn of $\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(-1-\lambda)(1-\lambda)-0\right] - 2(2-2\lambda-0)+0$$

$$= (1-\lambda)\left[(-1-\lambda)(1-\lambda)\right] - 4 + 4\lambda$$

$$= (1-\lambda)\left[-1-\lambda+\lambda+\lambda^2\right] - 4 + 4\lambda$$

$$= \lambda^3 - \lambda^2 - 5\lambda + 5 = 0 //$$

2) find the characteristic eqn of $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 3 & 1 \\ 0 & 5-\lambda & 2 \\ 1 & 0 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(5-\lambda)(3-\lambda)-0\right] - 3\left[0-2\right] + \left[5-\lambda\right]$$

$$= 2-\lambda\left[15-5\lambda-3\lambda+\lambda^2\right] + 6 - 5 + \lambda.$$

$$= 30 - 10\lambda - 6\lambda + 2\lambda^2 - 15\lambda + 5\lambda^2 + 3\lambda^2 - \lambda^3 + 1 + \lambda$$

$$= -\lambda^3 + 10\lambda^2 - 30\lambda + 31$$

$$= \lambda^3 - 10\lambda^2 + 30\lambda - 31 = 0 //$$

3) cayley Hamiltons theorem ?

<u>Statement</u> : Every square matrix satisfies its own characteristic equation $|A-\lambda I| = 0$.

Proof:- Every square matrix A.

The characteristic equation $|A-\lambda I| =$

If $a_0 + a_1 x + a_2 x^2 + \dots a_n x^n = 0$.

Then its said to be cayly hamilton theorem

Hence proved//.

4) Find the characteristic equation of $\begin{pmatrix} 5 & 4 \\ 1 & 3 \end{pmatrix}$ and deduce A^{-1}

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & 4 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (5-\lambda)(3-\lambda) - 4$$

$$= \lambda^2 - 8\lambda + 11.$$

$$A^2 - 8A + 11I = 0.$$

$$A^2 \cdot A^{-1} - 8A \cdot A^{-1} + 11 I A^{-1} = 0.$$

$$A - 8I + 11A^{-1} = 0.$$

$$11A^{-1} = -A + 8I$$

$$A^{-1} = \frac{1}{11}(-A - 8I)$$

$$= -\frac{1}{11}\left[\begin{pmatrix} 5 & 4 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}\right]$$

$$= -\frac{1}{11}\begin{pmatrix} -3 & 4 \\ 1 & -5 \end{pmatrix}$$ //.

5) Find the eigen values of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= 2-\lambda\left[6 - 3\lambda - 2\lambda + \lambda^2 - 2\right] - 2\left[2-\lambda-1\right]$$

$$+ 1\left[2 - 3 + \lambda\right]$$

$$= (2-\lambda)\left[\lambda^2 - 5\lambda + 4\right] - 2 + 2\lambda - 1 + \lambda$$

$$= 2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda + 3\lambda - 3 = 0.$$

$$-\lambda^3 + 7\lambda^2 - 3\lambda + 5 = 0.$$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0.$$

$$\begin{array}{r|rrrr} & 1 & -7 & 11 & -5 \\ 1 & 0 & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & \boxed{0} \end{array}$$

$\boxed{\lambda = 1}$. $\lambda^2 - 6\lambda + 5 = 0.$

$$(\lambda - 1)(\lambda - 5) = 0.$$

$$\boxed{\lambda = 1, 5}$$

6) Determine Eigen values & Eigen vectors of the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$$|A - \lambda I| = 0.$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix}$$

$= (5-\lambda)(2-\lambda)-4$

$= 10-2\lambda+\lambda^2-4$

$= \lambda^2-2\lambda+6.$

$(\lambda-1)(\lambda-6)=0.$

$\boxed{\lambda=1,6}$

$|A-\lambda I|x=0.$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\boxed{\text{put } \lambda=1}$

$$\begin{vmatrix} 4 & 4 \\ 1 & 1 \end{vmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$4x_1+4x_2=0.$

$x_1=-x_2$

$\boxed{x_1=-1}$ $\boxed{x_2=1}$.

$\boxed{\text{put } \lambda=6}$.

$$\begin{vmatrix} -1 & 4 \\ 1 & -4 \end{vmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1=4x_2$

$x_1/4 = x_2/1$

$\boxed{x_1=4,\ x_2=1}$.

10 Mark

1) Verify the cayley hamilton theorem, $\begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$
and hence find inverse of the matrix.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & -1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(-1-\lambda) - 2(-\lambda) - 0] - 0[0-2]$$
$$+ 3[0 - (-1-\lambda)]$$

$$= (\lambda^2 - \lambda - 2) - \lambda^3 + \lambda^2 + 2\lambda + 4 + 3 + 3\lambda$$

$$= \lambda^2 - \lambda - 2 - \lambda^3 + \lambda^2 + 2\lambda + 4 + 3\lambda$$

$$= 2\lambda^2 - \lambda^3 + 4\lambda + 5.$$

$$= \lambda^3 - 2\lambda^2 - 4\lambda - 5 = 0.$$

$$A^3 - 2A^2 - 4A - 5I = 0.$$

$$A^2 = \begin{pmatrix} 17 & 8 & 38 \\ 4 & 3 & 12 \\ 0 & 4 & 27 \end{pmatrix}$$

$$A^3 - 2A^2 - 4A - 5I = 0.$$

$$\begin{pmatrix} 17 & 8 & 38 \\ 4 & 3 & 12 \\ 10 & 4 & 27 \end{pmatrix} - \begin{pmatrix} 8 & 0 & 26 \\ 4 & 2 & 4 \\ 6 & 4 & 14 \end{pmatrix} - \begin{pmatrix} 4 & 8 & 12 \\ 0 & -4 & 1 \\ 4 & 0 & 1 \end{pmatrix}$$
$$- \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = 0.$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

$\times A^{-1} \Rightarrow A^3 \times A^{-1} - 3A^2 \times A^{-1} - 4A \times A^{-1} - 5I \times A^{-1} = 0.$

$$A^2 - 3A - 4I - 5A^{-1} = 0.$$

$$-5A^{-1} = -A^2 + 3A + 4I$$

$$5A^{-1} = A^2 - 3A - 4I$$

$$A^{-1} = \frac{1}{5}\left[\begin{matrix} 4 & 0 & 13 \\ 2 & 1 & 2 \\ 5 & 2 & 7 \end{matrix}\right] - \left[\begin{matrix} 3 & 4 & 6 \\ 0 & -1 & 4 \\ 3 & 0 & 4 \end{matrix}\right] - \left[\begin{matrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{matrix}\right]$$

$$= \frac{1}{5}\left[\begin{matrix} -3 & -4 & 7 \\ 3 & -1 & -3 \\ 1 & 0 & -1 \end{matrix}\right]$$ //.

3) Verify the cayley hamilton theorem and $\left[\begin{matrix} 2 & 3 & 0 \\ 2 & 1 & 1 \\ -7 & 3 & -3 \end{matrix}\right]$ and hence find inverse of matrix.

$$|A - \lambda I| = \left|\begin{matrix} 2-\lambda & 3 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{matrix}\right|$$

$$= (2-\lambda)\left[(1-\lambda)(-3-\lambda) - 1\right] - 3\left[2(-3-\lambda) + 7\right] + 0\left[4 + 7(1-\lambda)\right]$$

$$= (2-\lambda)(\lambda^2 + 2\lambda - 5) - 3(-1 - 2\lambda)$$

$$= (2-\lambda)(\lambda^2 + 2\lambda - 5) + 3 + 4\lambda$$

$$= -\lambda^3 + 13\lambda - 12 = 0.$$

$$= A^3 - 13A + 12 = 0$$

$$A^2 = \begin{bmatrix} 8 & 6 & 2 \\ -1 & 7 & -2 \\ 11 & -18 & 11 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 26 & 0 \\ 26 & 1 & 13 \\ 91 & 26 & -15 \end{bmatrix}$$

$$A^3 - 13A + 12I = 0$$

$$\begin{bmatrix} 14 & 26 & 0 \\ 26 & 1 & 13 \\ 41 & 26 & -51 \end{bmatrix} - \begin{bmatrix} 26 & 26 & 0 \\ 26 & 13 & 13 \\ -91 & 26 & -39 \end{bmatrix} + \begin{bmatrix} 12 & 0 & \\ 0 & 1 & \\ 0 & 0 & \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^3 - 13A + 12I = 0$$

$$A^3 \cdot A^{-1} - 13A \cdot A^{-1} + 12A^{-1} = 0$$

$$A^{-1} = -\tfrac{1}{12}\left[A^2 + 13I\right]$$

$$A^{-1} = -\tfrac{1}{12}\begin{bmatrix} 8 & 6 & 2 \\ -1 & 7 & -2 \\ 11 & -18 & 11 \end{bmatrix} - \begin{bmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{bmatrix}$$

$$= -\tfrac{1}{12}\begin{bmatrix} -5 & 6 & 2 \\ -1 & -6 & -2 \\ 11 & -18 & -2 \end{bmatrix} //$$

3) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ find the eigen values and eigen vectors.

$$|A - \lambda I| = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix}$$

$$= \cos^2\theta + \lambda^2 - \lambda\cos\theta - \lambda\cos\theta - \sin^2\theta.$$

$$= \lambda^2 + \cos^2\theta - \sin^2\theta - 2\lambda\cos\theta.$$

$a = 1, \quad b = -2\cos\theta \quad c = \cos^2\theta - \sin^2\theta.$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4(\cos^2\theta - \sin^2\theta)}}{2(1)}$$

$$= \frac{2\cos\theta \pm \sqrt{4\sin^2\theta}}{2}$$

$$= \frac{2\cos\theta \pm 2\sin\theta}{2}$$

$$= \cos\theta + \sin\theta = 0 \; //$$

4) Determine Eigen values & Eigen vectors of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$|A - \lambda I| x = 0.$

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)(\lambda^2 - 6\lambda + 4) + 2 + \lambda - 42 + 9\lambda$

$= 7\lambda^2 - \lambda^3 - 10\lambda + 4 + \lambda - 42 + 9\lambda$

$= \lambda^3 - 7\lambda^2 + 36 = 0.$

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$\boxed{\lambda = -2}$ $\therefore$ $\lambda^2 - 9\lambda + 18 = 0$

$(\lambda - 3)(\lambda - 6) = 0.$

$\boxed{\lambda = 3, 6, -2}$

$(A - \lambda I) x = 0.$

$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\boxed{\text{put } \lambda = -2}$

$$\begin{vmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{vmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$3x_1 + x_2 + 3x_3 = 0$, $x_1 + 7x_2 + x_3 = 0$,

$3x_1 + x_2 + 3x_3 = 0.$

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & 3 & & 1 \\ 7 & & 1 & & 1 & & 7 \end{array}$$

$$\frac{x_1}{|1-21|} = \frac{x_2}{|3-3|} = \frac{x_3}{|21-1|}$$

$x_1 = -20$, $x_2 = 0$, $x_3 = 20$

$\boxed{\text{put } \lambda = 3}$

$$\begin{vmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{vmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2x_1 + x_2 + 3x_3 = 0$, $x_1 + 2x_2 + x_3 = 0$, $3x_1 + x_2 - 2x_3 = 0.$

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & -2 & & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{|1-6|} = \frac{x_2}{|3+2|} = \frac{x_3}{|-4-1|}$$

$x_1 = -5, \quad x_2 = 5, \quad x_3 = -5$

$\boxed{\text{put } \lambda = 6}$

$$\begin{vmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{vmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-5x_1 + x_2 + 3x_3 = 0, \quad x_1 - x_2 + x_3 = 0,$

$3x_1 + x_2 - 5x_3 = 0.$

$$\begin{array}{cccc} x_1 & x_2 & & x_3 \\ & & -5 & 1 \\ 1 & 3 & & \\ -1 & 1 & 1 & -1 \end{array}$$

$$\frac{x_1}{|1+3|} = \frac{x_2}{|3+5|} = \frac{x_3}{|5-1|}$$

$x_1 = 4, \quad x_2 = 8, \quad x_3 = 4$ //.

Unit - 2

2 Mark:

1) Define finite dimension?

Let V be a vector space over field F. Let V is said to finite dimensional if there exist a finite subset S of V, such that $L(S) = V$.

2) Define linear independent?

Let V be a vector space over a field F a finite set of vectors $v_1, v_2, \ldots v_n$ in V. In said to be linear independent

$$\text{If } \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n = 0.$$

$$\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = 0.$$

If $v_1, v_2, \ldots v_n$ are not linear independent is said to be linear dependent.

3) In $V_3(R)$ the vector $(1,2,1), (2,1,0), (1,-1,2)$ are linearly independent.

$$\alpha_1 (1,2,1) \; \alpha_2 (2,1,0) \; \alpha_3 (1,-1,2)$$

$$\alpha_1 + 2\alpha_2 + \alpha_3 = 0. \quad \text{—— (1)}$$

$$2\alpha_1 + \alpha_2 - \alpha_3 = 0 \quad \text{—— (2)}$$

$$\alpha_1 + 2\alpha_3 = 0 \quad \text{—— (3)}$$

(1) + (2) $\Rightarrow 3\alpha_1 + 3\alpha_2 = 0$ —— (4)

$8 \times 3 \rightarrow \alpha_1 + \alpha_2 = 0.$

	α_1	α_2	α_3
2	1	1	2
1	-1	2	1

$$\frac{\alpha_1}{1-4} = \frac{\alpha_2}{-3} = \frac{\alpha_3}{-1} = 0.$$

$\alpha_1 = 0, \ \alpha_2 = 0, \ \alpha_3 = 0.$

It is linearly independent.

4) $S = \{(1,0,0), (1,1,0)\}$ is linearly independent but not basis for $V_3(R)$.

$$\alpha(1,0,0) + \beta(1,1,0) = (0,0,0)$$

$\alpha + \beta = 0.$

$\boxed{\beta = 0}$.

$\boxed{\alpha = 0}$

$\alpha = \beta = 0$

Hence S is linearly indepenent

Also, $L(S) = \{(a,b,0) / a,b \in R\} \neq V_3(R)$

$\therefore$ S is not basis.

5) Define dimension?

Let V be a finite dimensional vector space over a field F. The number of element in any basis of V is called dimensional of V is denote by $\dim V$.

6) $S = \{(1,0,0), (0,1,0), (1,1,1)\}$ is a basis for $V_3(R)$.

We shall show that any element (a,b,c) of $V_3(R)$ can be expressed as a linear combination of the vector of S.

$$(a,b,c) = \alpha(1,0,0) + \beta(0,1,0) + \gamma(1,1,1)$$

$\alpha + \gamma = a$	$\alpha + c = a$	$\beta + c = b$
$\beta + \gamma = b$	$\alpha = c - a$	$\beta = b - c$
$\gamma = c$		

$$(a,b,c) = (a-c)(1,0,0) + (b-c)(0,1,0) + c(1,1,1)$$

$\therefore$ S is a basis of $V_3(R)$.

7) Define Maximum linearly independent?

Let V be a vector space and $S = \{v_1, v_2 \ldots v_n\}$ be a set of indepenet vectors in V. then S is called a maximum linearly independent set if for every $v \in V - S$, the set.

$\{v_1, v_2 \ldots v_n\}$ is linearly dependent.

8) Define Minimal generating?

Let $S = \{v_1, v_2 \ldots v_n\}$ be a set of vectors in V and let $L(S) = V$. then S is called a minim generating set for any $v \in S$.

$$L(S - \{v_i\}) \neq V.$$

9) Define Rank and Nullity?

Let $T : V \to W$ be a linear transformation then the dimension of $T(V)$ is called the Rank of T. The dimension of ker T is called the nullity of T

10) Define non-singular & singular?

A linear transformation $T : V \to W$ is called a non-singular. If T is one to one. Otherwise T is called singular.

5 Mark:

1) Let V denote my set of all polynomials of degree $\leq n$ in $R[x]$

Let $T : V \to W$ be defined by $T(f) = df/dx$

w.k.T T is linear transformation. since $df/dx = 0 \Leftrightarrow$ f is constant ker T consists of all constant polynomials.

The dimension of this subspace of V is 1

Hence, nullity T is 1. since,

$\dim V = n+1$

$\text{rank } T = n$//.

8) Let V be a finite dimensional vector space over a field F. A be a subspace of V then there exists a subspace B of V such that $V = A \oplus B$.

proof: Let $S = \{v_1, v_2 \ldots v_n\}$ be a basis of we can find $\omega_1, \omega_2 \cdots \omega_s \in V$.

such that $S_1 = \{v_1, v_2 \cdots v_r, \omega_1, \omega_2 \cdots \omega_s\}$

Is a basis of V.

Now, let $B = L\{\omega_1, \omega_2 \cdots \omega_s\}$

we claim that:

$A \cap B = \{0\}$ and $V = A + B$

Now, let $v \in A \cap B$ then $v \in A$ and $v \in B$.

Hence $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$

$= \beta_1 \omega_1 + \cdots + \beta_s \omega_s$

$\therefore \alpha_1 v_1 + \cdots + \alpha_n v_n - \beta_1 \omega_1 + \cdots \beta_s \omega_s = 0$.

Then $v = \{\alpha_1 v_1 + \cdots \alpha_n v_n\} + \{\beta_1 \omega_1 + \cdots + \beta_s \omega_s\} \in A + B$.

Hence $A + B = V$ so that $V = A \oplus B$.

(3) Any two basis of a finite dimensional vector space V. Have the same numbers of elements.

Proof: Let V is finite dimensional. If has a basis $S = \{v_1, v_2 \cdots v_n\}$ let $S' = \{\omega_1, \omega_2 \cdots \omega_m\}$ be any other basis for V. now $L(S) = V$ and is a set of m. linearly independent vectors.

Hence $m \leq n$.

also, since $L(S') = V$ and S is a set of linearly dependent vectors $n \leq m$. Therefore hence

$m = n$ //.

4) Let $S = \{v_1, v_2, \ldots v_n\}$ be a linearly independent set of vector in a vector space V over a field F. The every element of $L(S)$ can be uniquelly can be returned in the form $\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n$ where $\alpha_i \in F$.

proof: By definition every element of $L(S)$ is of the form $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ now,

Let $\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \ldots \beta_n v_n$

$$(\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \ldots (\alpha_n - \beta_n)v_n = 0.$$

$$\alpha_1 - \beta_1 = 0$$

$$\alpha_1 = \beta_1$$

since S is linearly independent set $\alpha_i = \beta_i$

$$\alpha_i - \beta_i = 0.$$

$$\alpha_i = \beta_i$$

where $i = 1, 2, 3 \ldots n.$

$$\boxed{\alpha = \beta}.$$

5) Let V be a vector space of dimension then

(i) any set of n vectors where $m > n$ is linearly dependent

(ii) Any set of m vectors where $m < n$ is cannot spa

proof: (i) Let $S = \{v_1, v_2, \ldots v_n\}$ be a basis for V. Hence $L(S) = V$. Let S' be any set consisting of m vectors where $m > n$ subspace S' is linearly independent.

since, S span V by ...
contradiction. Hence S' is linearly dependent.

(ii) Let S' be a set consisting of m vectors where $m < n$ suppose $L(S') = V$ now, $S = \{v_1, v_2 \dots v_n\}$ is a basis for V. and hence linearly independent. Hence, $n \leq m$ which is contradiction.

Hence S' cannot span V.

<u>10 Mark</u>:

1) Let V be a finite dimensional vectors space over a field F. any linearly independent set of vectors in V is part of a basis.

Let $S = \{v_1, v_2 \dots v_r\}$ be a linearly independent set of vectors if $L(S) = V$ then S itself is a basis if $L(S) \neq V$.

Choose an element $v_{r+1} \in V - L(S)$

Now, consider $S_1 = \{v_1, v_2 \dots v_r, v_{r+1}\}$

we shall prove that S_1 is a linearly independent by showing that no vectors in S_1 is a linear combination of the proceeding vectors.

since $\{v_1, v_2 \dots v_r\}$ is linear independent v_i where $1 \leq i \leq r$ is not a linear combination of the proceeding vectors also, v_{r+1} $L(S)$ an hence v_{r+1} is not a linear combination of $v_1, v_2 \cdots \cdots v_n$.

Hence S_1 is a linearly independent if $L(S_r) = V_1$, then

S_1 is a basis for V. if not take an element $v_{r+2} \in V - L(S_1)$ and proceed a before since the dimension of V is finite. this process stop at a certain strange giving the required basis containing.

8) Any vector space of dimensional over a field F. is isomorphic to $V_n(F)$.

Let V be a vector space of dimensional n. Let $\{v_1, v_2 \cdots v_n\}$ be a basis for V. then we know that if $v \in V$.

we can be uniquely $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n$

where $\alpha_i \in F$.

Now, considered the map $f: V \to V_n(F)$

given by, $f(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$

$= (\alpha_1 + \alpha_2 + \cdots \cdots \alpha_n)$

clearly f is 1 to 1 and onto

Let $v, w \in V$

then $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

$w = \beta_1 v_1 + \beta_2 v_2 + \cdots \beta_n v_n$

$F(v+w) = F(\alpha_1 v_1 + \dots + \alpha_n v_n + \beta_1 v_1 + \dots \beta_n v_n)$

$= (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) + \dots + (\alpha_n + \beta_n)$

$= (\alpha_1 + \alpha_2 \dots \alpha_n) + (\beta_1, \beta_2 \dots \beta_n)$

$= F(v) + F(w).$

Also, $F(\alpha v) = F(\alpha\alpha_1 v + \dots \alpha\alpha_n v_n).$

$= (\alpha\alpha_1 + \alpha\alpha_2 + \dots \alpha\alpha_n)$

$= \alpha(\alpha_1 + \alpha_2 + \dots + \alpha_n)$

$= \alpha F(v)$

Hence F is an isomorphism of V to $V_n(F)$.