CORE COURSE – II MECHANICS   SUBJECT CODE: 16SCCPH2

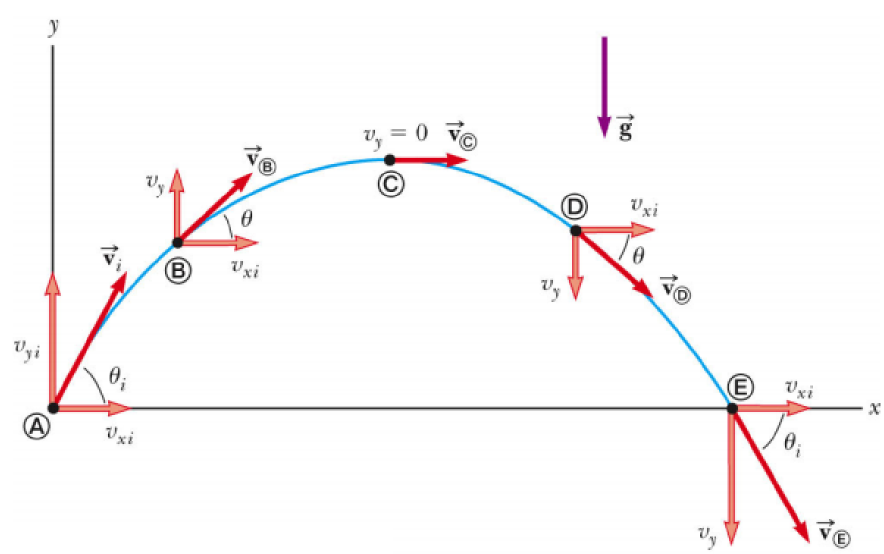
UNIT I   Projectile, Impulse and Impact

Projectile - particle projected in any direction - Path of a projectile is a parabola - Range of a projectile on plane inclined to the horizontal - Maximum range on the inclined plane - Impulse of a force - Laws of impact - Direct impact between two smooth spheres - oblique impact between two smooth spheres - Impact of a smooth sphere on a smooth fixed horizontal plane - Loss of KE due to direct impact - Oblique impact.

PROJECTILE MOTION

An object may move in both x and y directions simultaneously. The form of two – dimensional motion we will deal with is called projectile motion. The free fall acceleration is constant over the range of motion, it is directed downwards, it is reasonable as long as the range is small compared to the radius of the earth. The effect of air friction is negligible. These are few assumptions, an object in projectile motion will follow a parabolic path, this path is called the trajectory.

Projectile Motion Diagram



2.15. The Projectile—Motion of a Projectile in a non-resisting medium. Before Galileo's time, it was supposed that a body thrown horizontally, travelled in a straight line until it had exhausted its force and then fell vertically down. It was he who first showed that it must take a *parabolic path**, realising as he did, the physical independence of its horizontal and vertical motions, so that each could be considered separately.

Such a body, subjected simultaneously to a uniform horizontal motion and a uniform vertical acceleration, is called a **projectile**, and *the path it describes is called its* **trajectory.** Let us study its motion in some detail.

Let a body be projected upwards with a velocity u, at an angle θ with the horizontal. Then, resolving u into two rectangular components, along the vertical** and along the horizontal, we have (*i*) *the vertical component* (along OY) = $u \sin \theta$ and (*ii*) *the horizontal component* (along OX) = $u \cos \theta$. The latter component, being perpendicular to the direction of gravity, is not accelerated, and hence

$$dx/dt = u \cos \theta.$$

Since at $t = 0$, $x = 0$, we have $x = ut \cos \theta$ ...(*i*)

And, because the vertical component is subjected to a downward acceleration due to gravity, we have $d^2y/dt^2 = -g$,

integrating which, we have $dy/dt = -gt + C_1$,

where C_1 is another constant of integration.

Now, at $t = 0$, $dy/dt = u \sin \theta$; so that, $C_1 = u \sin \theta$.

$\therefore$ $dy/dt = u \sin \theta - gt.$

Integrating this again, we have $y = ut \sin \theta - \frac{1}{2} gt^2 + C_2$,

where C_2 is another constant of integration.

Since $y = 0$ at $t = 0$, we have $C_2 = 0$

So that, $y = ut \sin \theta - \frac{1}{2} gt^2$ ...(*ii*)

Now, from relation (*i*), we have $t = x/u \cos \theta$

Substituting this value of t in relation (*ii*), we have

$$y = u . \frac{x}{u \cos \theta} . \sin \theta - \frac{1}{2} g . \left(\frac{x}{u \cos \theta} \right)^2$$

Or $$y = x \tan\theta - \frac{1}{2} g . \frac{x^2}{u^2 \cos^2 \theta}$$

This is clearly an equation of the second degree in x and the first degree in y and thus represents a parabola, with its axis vertical. The trajectory of the particle is thus a *parabola*.

2.16. Horizontal Range of a Projectile. Clearly, the time taken by the body to reach the maximum height = $u \sin \theta/g^*$, its vertical velocity being $u \sin \theta$.

And, *since time of ascent is equal to time of descent*, the total time taken by the body for the whole flight = $2\,u \sin \theta/g$.

During this time, the *horizontal distance covered by the body*, with its uniform horizontal velocity $u \cos \theta$, is given by

$$u \cos \theta \cdot \frac{2u \sin \theta}{g} = \frac{2u^2 . \sin \theta . \cos\theta}{g} = \frac{u^2 \sin 2\theta}{g}. \qquad [\because 2 \sin \theta \cos \theta = \sin 2\theta.$$

This horizontal distance covered by a projectile is called its **horizontal range**, or, more usually, simply, its **range.** Denoting it by R, therefore, we have $R = u^2 \sin 2\theta/g$

2.17. Maximum Height attained by a Projectile. We have the kinematic relation, $v^2 - u^2 = 2aS$, where the symbols have their usual meanings.

Here, $a = -g$ (the body being projected upwards), and, at the highest point, obviously, $v = 0$. So that, if the *maximum height attained by the projectile be h*, (*i.e.*, $S = h$), we have

$$0 - (u \sin \theta)^2 = 2.(-g).h,$$

whence,

$$h = \frac{u^2 . \sin^2\theta}{2g} \qquad [\because \text{ the initial upward velocity here is } u \sin \theta \text{ and not } u.$$

2.18. Angle of Projection for Maximum Range. It is obvious that for a given initial velocity (u) of the body, its horizontal range (R) will depend upon its angle of projection (θ).

The horizontal range R, as we know, is given by $R = \dfrac{u^2 . \sin 2\theta}{g}$.

Putting x for R, we have $x = \dfrac{u^2 \sin 2\theta}{g}$

It is thus clear that the value of θ for maximum (horizontal) range of the projectile would be that for which $\sin 2\theta = 1$, *i.e.*, when $2\theta = 90°$, and, therefore, $\theta = 45°$.

Thus, for maximum range, the angle of projection should be 45°.

N.B. The following interesting result follows from the relation $R = u^2 \sin 2\theta/g$.

We know that the sine of an angle is the same as that of its supplement. And, therefore,

$$\sin 2\theta = \sin (180 - 2\theta)$$

from which it is clear that the *projectile will have the same range* (not the maximum), for the angles of projection θ and $(90-\theta)$, —the two paths taken being, however, different, and called the *high* (H) and the *low* (L) trajectories respectively, as shown in Fig. 2.13, in view of the different maximum heights attained by the body.

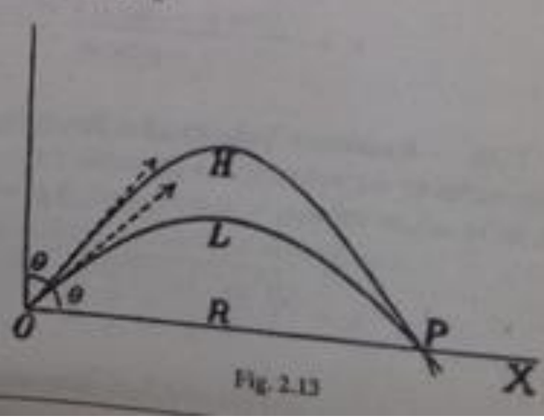


Fig. 2.13

2.19. Range on an Inclined Plane. We have seen above, (§ 2.15) that the equation to the trajectory of a projectile is

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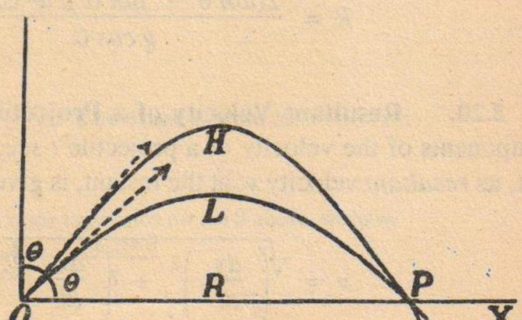


Fig. 2.13

2.19. Range on an Inclined Plane. We have seen above, (§ 2.15) that the equation to the trajectory of a projectile is

$$y = x \tan\theta - \frac{1}{2} g . \frac{x^2}{u^2 \cos^2\theta}$$

Let us consider a plane inclined to the horizontal at an angle α , (Fig. 2.14); so that,

$$y = x \tan \alpha$$

Now, to obtain the range on the inclined plane, we must determine the point where the trajectory of the projectile will meet the plane; and to do this, we must solve the above two equations. So that, substituting $y = x \tan \alpha$ in the equation for the trajectory, we have

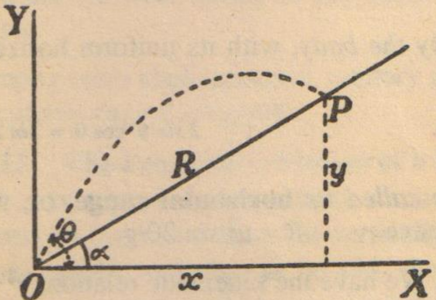


Fig. 2.14

$$x \tan \alpha = x \tan\theta - \frac{1}{2} . \frac{gx^2}{u^2 \cos^2 \theta}.$$

Or, $$\frac{gx}{2u^2 \cos^2\theta} = \tan\theta - \tan\alpha.$$

Or, $$x = \frac{2 (\tan\theta - \tan\alpha).u^2 \cos^2\theta}{g}$$

And, therefore, substituting *this* value of x in $y = x \tan \alpha$,

we have $$y = \frac{2 (\tan\theta - \tan\alpha).\, u^2 \cos^2\theta)}{g} . \tan \alpha.$$

So that, the range R is clearly given by the relation, $R^2 = x^2 + y^2$

$$= \left[\frac{2 (\tan\theta - \tan\alpha).\, u^2 \cos^2\theta}{g}\right]^2 + \left[\frac{2(\tan\theta - \tan\alpha) \tan\alpha . u^2 \cos^2\theta}{g}\right]$$

$$= \left[\frac{2 (\tan\theta - \tan\alpha).u^2 \cos^2\theta}{g}\right]^2 [1 + \tan^2\alpha]$$

Now, $$(1 + \tan^2\alpha) = \sec^2\alpha = 1/\cos^2\alpha.$$

$\therefore$ $$R^2 = \left[\frac{2(\tan\theta - \tan\alpha) . u^2 \cos^2\theta}{g}\right]^2 \Big/ \cos^2\alpha.$$

$$R = \frac{2(\tan\theta - \tan\alpha).\, u^2 \cos^2\theta}{g \cos\alpha}$$

IMPULSE OF A FORCE

Impact

12·1. Impulsive force. A large force which, acting on a body for an infinitesimally small period, produces a finite change of momentum in that interval, is called an *impulsive force*. The period in which the impulsive force acts should be so short that, during this period, the change of position of the point of application and the effects of the finite forces are negligible. The force experienced by a ball due to a hit by a bat is an example for an impulsive force.

12·2. Impulse. The effect of the action of an impulsive force is measured by the change in momentum produced by the force. This change is called the *impulse* of the impulsive action. So the defining equation of an impulse imparted to a particle of mass m is

$$\mathbf{I}=m\mathbf{v}'-m\mathbf{v}, \tag{12·2·1}$$

where $\mathbf{v}$ and $\mathbf{v}'$ are the velocities of the particle immediately before and immediately after the impulsive action. If τ is the short time during which the impulsive force acts, then

$$\mathbf{I}=m\mathbf{v}'-m\mathbf{v}=\Big[\,m\mathbf{v}\,\Big]_0^\tau=\int_0^\tau m\,d\mathbf{v}=\int_0^\tau m\frac{d\mathbf{v}}{dt}\,dt=\int_0^\tau m\mathbf{a}\,dt$$

$$=\int_0^\tau \mathbf{F}\,dt \tag{12·2·2}$$

Remark 1. The integral (12·2·2) may also be defined to be the impulse.

Remark 2. Care should be taken to differentiate impulse from an impulsive force, for impulsive force is a force whereas impulse is the change in momentum. Hence impulse is measured by the unit of momentum but not by the unit of force.

As the remark 1 and remark 2 says about the difference between impulse and impulsive force. Lets have a keen understanding on this.

Impulsive force ---- is a force

Impulse -------------- is the change in the momentum

LAWS OF IMPACT

They are the conservation of linear momentum and elasticity.

12·3. Conservation of linear momentum. From the equation of motion of a particle of mass m,

$$m\ddot{\mathbf{r}}=\mathbf{F}.$$

we see that, if $\mathbf{F}=0$, then

$$m\dot{\mathbf{r}} \quad \text{or} \quad m\mathbf{v}$$

is a constant vector ; that is, the linear momentum is conserved. Thus we obtain the *principle of conservation of momentum for a particle that, if the applied force on a particle is zero, then the linear momentum of the particle is conserved.*

From (7·2·2) it is evident that in the case of a rigid body or a system of particles, if the sum of the applied forces is zero, then

$$m_1\dot{\mathbf{r}}_1+m_2\dot{\mathbf{r}}_2+\ldots\ldots=\text{a constant vector.}$$

That is, the linear momentum of the system is conserved.

When the explosive charge in a gun forms a large amount of gas, the gun and the shot are subject to the action of a very high pressure. But this pressure is an internal force and not an external force. Therefore, when a gun is fired, the gun and the shot are not subject to any external force. So, by the principle of conservation of linear momentum, if the gun is at rest before firing, that is, if the momentum of the gun and the shot is zero before firing, then immediately after firing the sum of the momenta of the gun and the shot will also be zero. So, in firing a shot of mass m with a velocity $\mathbf{v}$, the gun of mass $\mathbf{M}$ gains a velocity $\mathbf{V}$ given by

$$mv+MV=0 \quad \text{as} \quad V=-\frac{m}{M}v.$$

As in explosion, the forces exerted in collision of two smooth bodies, are purely internal if the bodies are thought of to form a single unit. Hence *the momentum after collision is the same as the momentum before collision.* (12·3·1)

12·4. Elasticity. When two balls collide, during the time in which the balls are in contact, they become distorted losing their original shapes. Though this distortion is a complicated process, it was observed from experiments by Newton that the relative speed along the line of centres of the balls after collision, bears a constant ratio to the relative speed before collision. This constant which depends chiefly on the materials of which the balls are composed and not on the size, is called the coefficient of restitution and is usually denoted by e. Further, it was observed that $0<e<1$. In the ideal

cases, where $e=0$ and $e=1$, the ball are said to be *inelastic* and *perfectly elastic* respectively. Now Newton's experimental law may be stated as follows :

$$\left.\begin{array}{l}\textit{Component of the}\\ \textit{relative velocity after}\\ \textit{impact}\end{array}\right\}=-e\left\{\begin{array}{l}\textit{component of the}\\ \textit{relative velocity}\\ \textit{before impact}\end{array}\right. \qquad (12\cdot4\cdot1)$$

both the components being along the line of the balls.

DIRECT IMPACT BETWEEN TWO SMOOTH SPHERES, OBLIQUE IMPACT BETWEEN TWO SMOOTH SPHERES

12·5. Impact of two smooth spheres. Suppose that two smooth spheres collide with each other. In the infinitesimal interval of time during which the spheres are in contact, each is subject to the action of an impulsive force which is towards its centre. Thus there will be two impulsive forces acting on the spheres. By N. 3, they are equal in magnitude but opposite in direction. So also are the impulses imparted to the spheres.

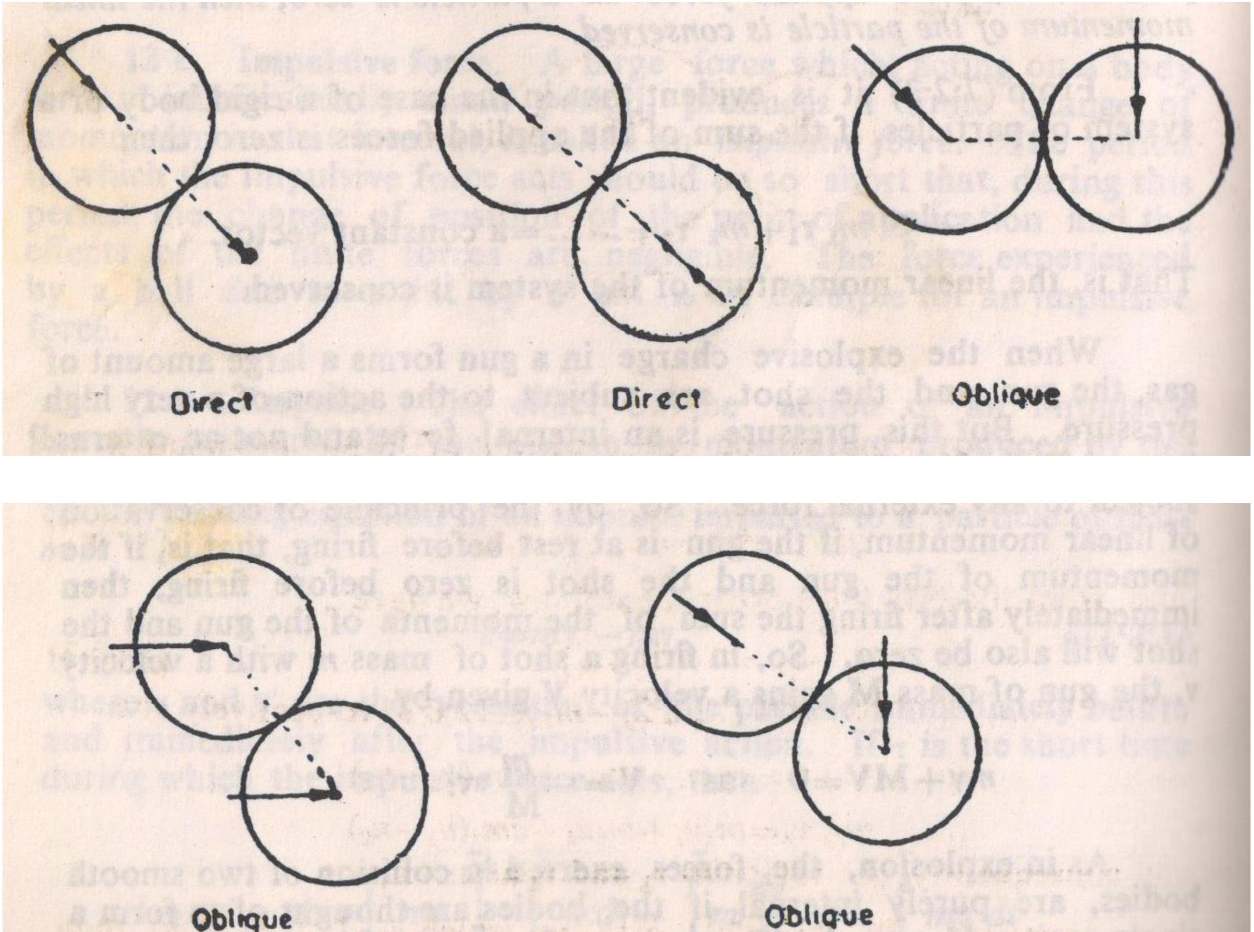
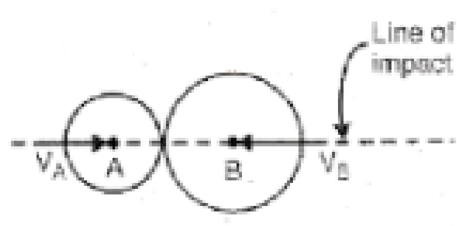
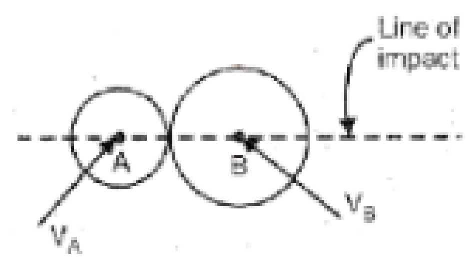


Fig. 12·5·1.

Impact of spheres can be classified into two groups, namely direct and oblique impacts. If C_1 and C_2 are the positions of the centres of the spheres at the time of impact and if the centres of the spheres had been moving before the impact along the straight line through C_1 and C_2, then the impact is said to be *direct* ; otherwise, it is said to be *oblique* (Fig. 12·5·1).



Direct Central Impact



Oblique Central Impact

IMPACT OF A SMOOTH SPHERE ON A SMOOTH FIXED HORIZONTAL PLANE – LOSS OF KINETIC ENERGY DUE TO DIRECT IMPACT – OBLIQUE IMPACT

12·6. Direct impact of two smooth spheres. In this section, given the motion before the impact of two smooth spheres, we obtain

(*i*) the motion after impact

(*ii*) the impulse imparted to each sphere due to impact and

(*iii*) the change in K.E. due to impact.

BOOK WORK 12·1. To find the velocities of two smooth spheres after a direct impact between them.

Let us have the following assumptions (Fig. 12·6·1) :

$$\left.\begin{array}{rl} m_1, m_2 : & \text{masses of the spheres} \\ u_1, u_2 : & \text{velocities of the spheres before impact } (u_1 > u_2). \\ e : & \text{coefficient of restitution} \\ v_1, v_2 : & \text{velocities of the spheres after impact} \end{array}\right\} \quad (12·6·1)$$

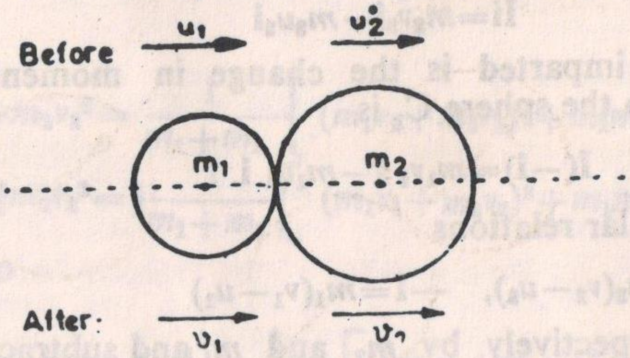


Fig. 12·6·1.

From the principle of conservation of linear momentum the momentum after impact equals the momentum before impact. So

$$m_1v_1+m_2v_2=m_1u_1+m_2u_2, \tag{12·6·2}$$

and, from the Newton's experimental law, we have

$$v_1-v_2=-e(u_1-u_2). \tag{12·6·3}$$

(12·6·2)$+m_2\times$(12·6·3) and (12·6·2)$-m_1\times$(12·6·3) respectively give

$$(m_1+m_2)\ v_1=m_1u_1+m_2u_2+em_2(u_2-u_1)$$
$$(m_1+m_2)\ v_2=m_1u_1+m_2u_2+em_1(u_1-u_2)$$

and consequently, the velocities v_1 and v_2 after impact.

Corollary. In the ideal case in which $e=1$, if $m_1=m_2$, then the equations (12·6·2) and (12·6·3) become

$$v_1+v_2=u_1+u_2,$$
$$v_1-v_2=-u_1+u_2$$

which, on solving, gives that $v_1=u_2$ and $v_2=u_1$ showing that the velocities of the spheres are interchanged by impact.

BOOK WORK 12·2. When two smooth spheres collide directly, to find the impulse imparted to each sphere and the change in the total kinetic energy of the spheres, (both in terms of the velocities before impact).

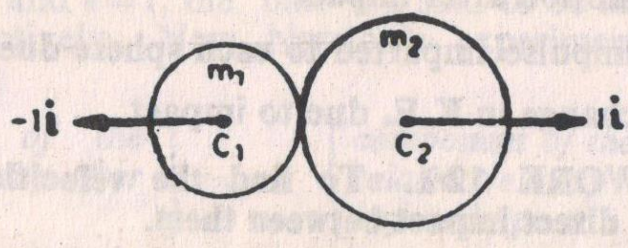


Fig. 12·6·2.

Let us make the same assumptions as in (12·6·1). Furthermore let C_1 and C_2 be the centres of the spheres at the time of impact and **i**, the unit vector in the direction of $\overline{C_1C_2}$. Then, if I is the magnitude of the impulse imparted to each of the spheres, then the impulse imparted to the sphere C_2, is

$$I\mathbf{i}=m_2v_2\mathbf{i}-m_2u_2\mathbf{i}$$

since the impulse imparted is the change in momentum. The impulse imparted to the sphere C_1 is

$$I(-\mathbf{i})=m_1v_1\,\mathbf{i}-m_1u_1\,\mathbf{i}.$$

Thus we get the scalar relations

$$I=m_2(v_2-u_2), \quad -I=m_1(v_1-u_1) \qquad (12\cdot6\cdot4)$$

Dividing them respectively by m_2 and m_1 and subtracting, we get

$$I\left(\frac{1}{m_1}+\frac{1}{m_2}\right)=(v_2-u_2)-(v_1-u_1)$$

$$= -(v_1 - v_2) + (u_1 - u_2)$$

$$= e(u_1 - u_2) + (u_1 - u_2) \text{ by experimental law}$$

$$= (1+e)(u_1 - u_2).$$

$$\therefore \quad I = \frac{m_1 m_2}{m_1 + m_2}(1+e)(u_1 - u_2) \qquad (12 \cdot 6 \cdot 5)$$

The total kinetic energies of the spheres after and before impact are

$$\tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2, \quad \tfrac{1}{2}m_1 u_1^2 + \tfrac{1}{2}m_2 u_2^2.$$

The increase in the kinetic energy due to impact is

$$\tfrac{1}{2}\{(m_1 v_1^2 + m_2 v_2^2) - (m_1 u_1^2 + m_2 u_2^2)\}$$

$$= \tfrac{1}{2}\{(m_1(v_1^2 - u_1^2) + m_2(v_2^2 - u_2^2)\}$$

$$= \tfrac{1}{2}\{m_1(v_1 - u_1)(v_1 + u_1) + m_2(v_2 - u_2)(v_2 + u_2)\}$$

$$= \tfrac{1}{2}\{(-I)(v_1 + u_1) + (I)(v_2 + u_2)\} \text{ by } (12 \cdot 6 \cdot 4)$$

$$= -\tfrac{1}{2} I\{(v_1 - u_1) - (v_2 + u_2)\}.$$

$$= -\tfrac{1}{2} I\{(v_1 - v_2) + (u_1 - u_2)\}$$

$$= -\tfrac{1}{2} I\{-e(u_1 - u_2) + (u_1 - u_2)\} \text{ by experimental law}$$

$$= -\tfrac{1}{2} I(1-e)(u_1 - u_2)$$

$$= -\tfrac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(1+e)(1-e)(u_1 - u_2)^2 \text{ by } (12 \cdot 6 \cdot 5)$$

$$= -\tfrac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(1-e^2)(u_1 - u_2)^2. \qquad (12 \cdot 6 \cdot 6)$$

Since it is a negative quantity, there is actually a decrease in kinetic energy, that is, a loss in kinetic energy due to impact.

Remark 1. The loss in kinetic energy can also be obtained by using the identity

$$(m_1 + m_2)(m_1 v_1^2 + m_2 v_2^2) = (m_1 v_1 + m_2 v_2)^2 + m_1 m_2 (v_1 - v_2)^2.$$

The truth of this identity can easily be verified by simplifying the left and right hand expressions separately. This identity may be rewritten as

$$m_1v_1^2+m_2v_2^2=\frac{1}{m_1+m_2}\left\{(m_1v_1+m_2v_2)^2+m_1m_2(v_1-v_2)^2\right\}$$

or $$\tfrac{1}{2}m_1v_1^2+\tfrac{1}{2}m_2v_2^2=\tfrac{1}{2}\frac{1}{m_1+m_2}\left[(m_1v_1+m_2v_2)^2+m_1m_2(v_1-v_2)^2\right]$$

So also

$$\tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2=\tfrac{1}{2}\frac{1}{m_1+m_2}\left[(m_1u_1+m_2u_2)^2+m_1m_2(u_1-u_2)^2\right].$$

Subtraction of the second from the first gives the change in kinetic energy as

$$\tfrac{1}{2}\frac{1}{m_1+m_2}\{m_1m_2(v_1-v_2)^2-m_1m_2(u_1-u_2)^2.\}$$

since $m_1v_1+m_2v_2=m_1u_1+m_2u_2$

$$=\tfrac{1}{2}\frac{1}{m_1+m_2}\{m_1m_2e^2(u_1-u_2)^2-m_1m_2(u_1-u_2)^2\}$$

by Newton's experimental law

$$=-\tfrac{1}{2}\frac{m_1m_2}{m_1+m_2}(1-e^2)(u_1-u_2)^2.$$

Remark 2. Only in the ideal case when $e=1$, the loss in kinetic energy is zero.

12·7. Impact of a smooth sphere on a fixed smooth plane. If a sphere collides with a plane with its centre moving along a normal to the plane, then the collision is said to be *direct* ; otherwise, it is said to be *oblique*.

UNIT II Motion on a plane curve

Centripetal and centrifugal forces - Hodograph - Expression for normal acceleration - Motion of a cyclist along a curved path - Motion of a railway carriage round a curved track- upsetting of a carriage - Motion of a carriage on a banked up curve - Effect of earth's rotation on the value of the acceleration due to gravity - Variation of 'g' with altitude, latitude and depth.

CENTRIPETAL FORCE AND CENTRIFUGAL FORCE

2.10. Centripetal Force. According to Newton's first law of motion, a body must continue to move with a uniform velocity in a straight line, unless acted upon by a force. It follows, therefore, that when a body moves along a circle, some force is acting upon it, which continually deflects it from its straight or linear path; and since the body has an acceleration towards the centre, it is obvious that the force must also be acting in the direction of this acceleration, *i.e.*, along the radius, or towards the centre of its circular path. It is called the *centripetal force*, and its value is given by the product of the mass of the body and its centripetal acceleration. Thus, if m be the mass of the body, we have

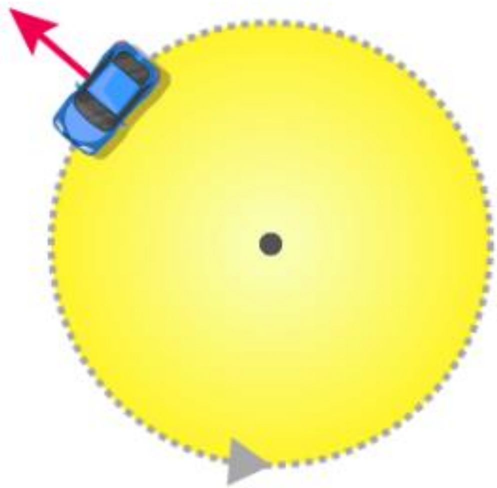
$$\text{centripetal force} = mv\omega = mv^2/r = mr\omega^2 = 4\pi^2 n^2 mr,$$

in *dynes* if m be in *gm.*, v in *cm/sec* and r in *cm* and in *newtons* if m be in *kg*, v in *m/ sec* and r in *metres*.

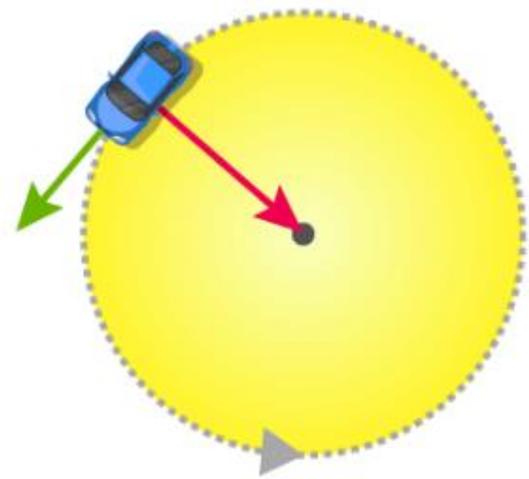
Numerous examples of centripetal force are met with in daily life. Thus, (*i*) in the case of a stone, whirled round at the end of a string whose other end is held in the hand, the centripetal force is supplied by the tension of the string ; (*ii*) in the case of a motor car or a railway train, negotiating a curve, it is supplied by the push due to the rails on the wheel of the train and (*iii*) in the case of (*a*) the planets revolving round the sun, or (*b*) the moon revolving round the earth, by the gravitational attraction between them.

2.11. Centrifugal Force. The equal and opposite *reaction* to the centripetal force is called *centrifugal force*, because it tends to take the body away from the centre, (from '*fugo*' — *I flee*). Centripetal force and centrifugal force being just *action* and *reaction* in the sense of Newton's third law of motion, the numerical values of the two are the same, *viz.*, $mv^2/r = mr\omega^2 = 4\pi^2 n^2 mr$ *dynes* or *newtons* according as the *C.G.S.* system or the *M.K.S* (or the *S.I.*) system is used.

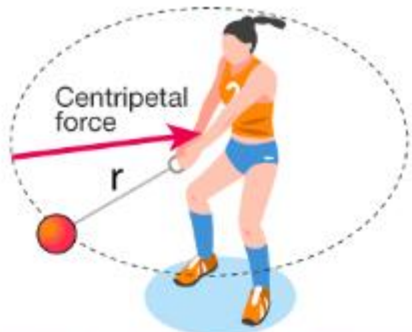
Thus, in the case of a stone, whirled round at the end of a string, not only is the stone acted upon by a force, (this centripetal force), along the string towards the centre, but the stone also exerts an equal and opposite force, (the centrifugal force), on the hand, away from the centre, also along the string, the two balancing each other and keeping the stone in dynamic equilibrium, constraining it to move with a constant speed along the tangent to its circular path at every point. In the event of the string getting snapped or getting loosened , this pair of constraining forces disappears and the stone flies of tangentially to the circular path at the point where the string snaps.



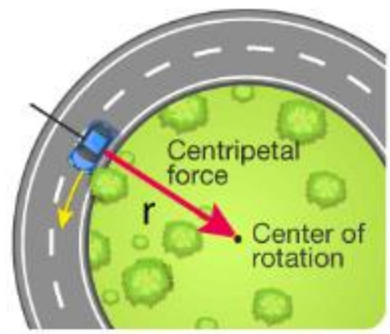
Centrifugal force



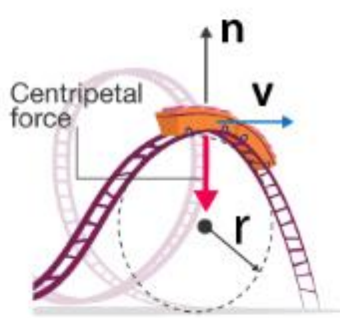
Centripetal force



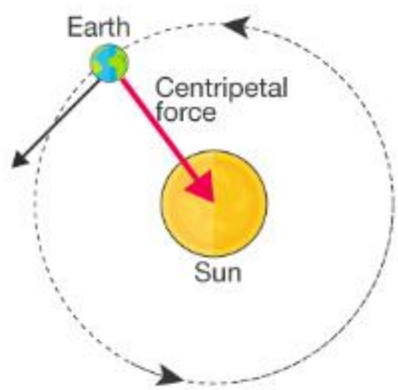
(a) Spinning a ball on a string or twirling a lasso



(b) Turning a car



(c) Going through a loop on a roller coaster



(d) Planets orbiting around the Sun

Centripetal Force Vs Centrifugal Force

Check the table below to learn the detailed **comparison between Centripetal and Centrifugal Force**

Differences Between Centripetal And Centrifugal Force	
Centrifugal Force	**Centripetal Force**
If an object moving in a circle and experiences an outward force than this force is called the centrifugal force	If the object travels in a uniform speed in a circular path is called centripetal force.
The object has the direction along the centre of the circle from the centre approaching the object	The object has the direction along the centre of the circle from the object approaching the centre.
Mud flying of a tire is one example of the centrifugal force.	A satellite orbiting a planet is an example of the centripetal force.

The Centripetal Force Formula is given as the product of mass (in kg) and tangential velocity (in meters per second) squared, divided by the radius (in meters). Which implies that on doubling the tangential velocity, the centripetal force will be quadrupled. Mathematically it is written as:

$$F = ma_c = \frac{mv^2}{r}$$

Where,

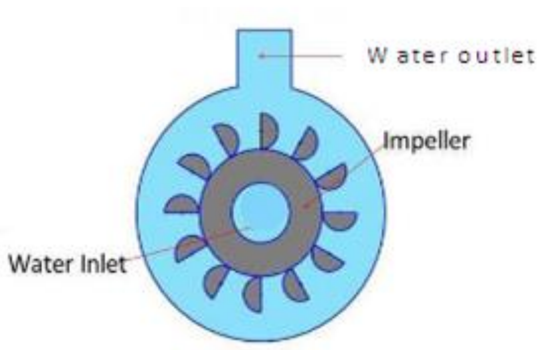
- ***F*** is the Centripetal force.
- a_c is the Centripetal acceleration.
- ***m*** is the mass of the object.
- ***v*** is the speed or velocity of the object.
- ***r*** is the radius.

Centrifugal Force Formula is given as the negative product of mass (in kg) and tangential velocity (in meters per second) squared, divided by the radius (in meters). Which implies that On doubling the tangential velocity, the centripetal force will be quadrupled. Mathematically it is written as:

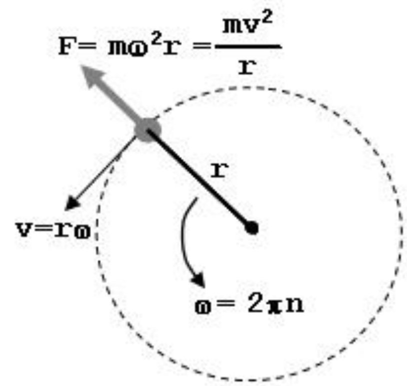
$$F_c = -\frac{mv^2}{r}$$

Where,

- F_c is the Centrifugal force
- ***m*** is the mass of the object
- ***v*** is the velocity or speed of the object.
- ***r*** is the radius.



Centrifugal Pump



HODOGRAPH

The Hodograph may be defined as an auxiliary curve obtained by joining the free ends of a moving vector representing the velocity of a moving particle along any path.

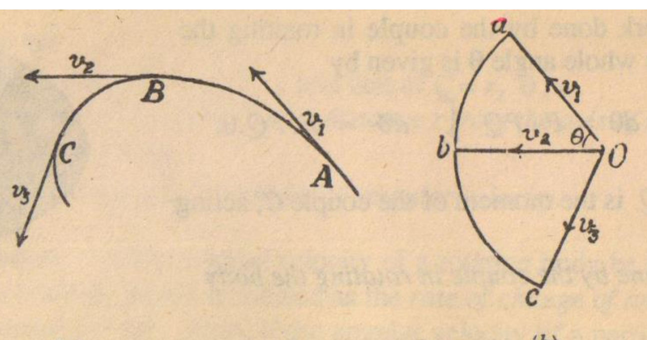
2.7. **The Hodograph.** When a body describes a ***curvilinear path,*** so that its motion is accelerated and also changes in direction, its acceleration and its path may easily be determined by means of what is called the *hodograph* of its motion.

The hodograph may be defined as an auxiliary curve, obtained by joining the free ends of a moving vector representing the velocity of a moving particle along any path.

For instance, if a point *P* moves along a curve *ABC*, [Fig. 2.4 (*a*)] such that its velocities are v_1, v_2, and v_3.... respectively at *A*, *B* and *C* etc., then, if we take any point *O* and draw straight lines, *i.e.*, *vectors*, *Oa*, *Ob* and *Oc*, [Fig. 2.4(*b*)], representing the velocities of *P* at *A*, *B* and *C*, in magnitude as well as in direction, the curve passing through *a*, *b* and *c* is the ***hodograph*** of the motion of *P*.

Blurred on the left edge are – accelerated, means, moving vector, v1,v2 and v3, lines ,magnitude, motion

A **hodograph** is a diagram that gives a vectorial visual representation of the movement of a body or a fluid. It is the locus of one end of a variable vector, with the other end fixed. The position of any plotted data on such a diagram is proportional to the velocity of the moving particle.



(a) (b)

Fig. 2.4

Now, different cases arise :

(*i*) If point *P* be moving with a *uniform velocity along the same direction*, points *a*, *b*, *c* etc. will all lie in the same place and the hodograph will therefore, be a *single point*.

(*ii*) If point *P* be moving with a *variable velocity*, but *in the same direction*, the hodograph will be a *straight line*, passing through *O*. For example, in the case of a body falling freely under the action of gravity, the hodograph will be a *vertical line*, passing through *O*.

(*iii*) If *P* be projected with a horizontal velocity, the path described will be a *parabola*, (see §2.14), and both the direction and the magnitude of the velocity will change. The horizontal velocity will throughout remain constant and equal to the initial horizontal velocity, because the acceleration due to gravity acts vertically downwards. The points *a*, *b*, *c*, etc. will, therefore, always be at the same horizontal distance from *O*, and the hodograph, in this case, will thus be a *vertical line, not passing through O*.

(*iv*) If the path of *P* be a *closed curve*, the hodograph will also be a *closed curve*. For example, if *P* moves in a circle with a uniform speed v, the hodograph will also be a circle of radius v, because all the lines, *Oa*, *Ob*, *Oc*, etc. will be of the same length v. If, on the other hand, it moves in a circle with a *variable speed*, the hodograph might be an *oval curve* about the point *O*.

2.8. Velocity in the Hodograph. An important property of the hodograph is that the acceleration of *P* at any point on the curve *ABC* is represented, in magnitude as well as direction, by the velocity of the corresponding point on the hodograph, as can be seen from the following :

Let *A* and *B* be two points, close together, [Fig. 2.4(*a*)], and let *P* move from *A* to *B* in time t, such that its velocity v_1 at *A* is changed to v_2 at B.

Further, let another point *p* describe the hodogrpah *abc*, [Fig. 2.4(*b*)], while *P* describes the curve *ABC*.

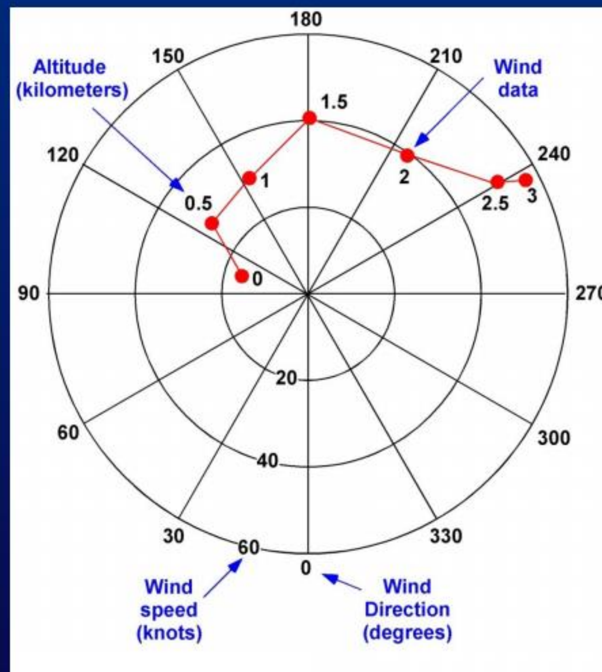
Then, clearly, the point *p* moves from *a* to *b* in time t, and its velocity is, therefore, equal to ab/t.

But, since *oa* represents the velocity of *P* at *A* and *ob*, that at *B*, *ab* represents, in accordance with the law of triangle of velocities, the change in velocity of *P* in time t, and, therefore, *the rate of change of velocity*, or the *acceleration* of *P*, is represented by ab/t, *i.e.*, by the *velocity* of *p* in the hodograph.

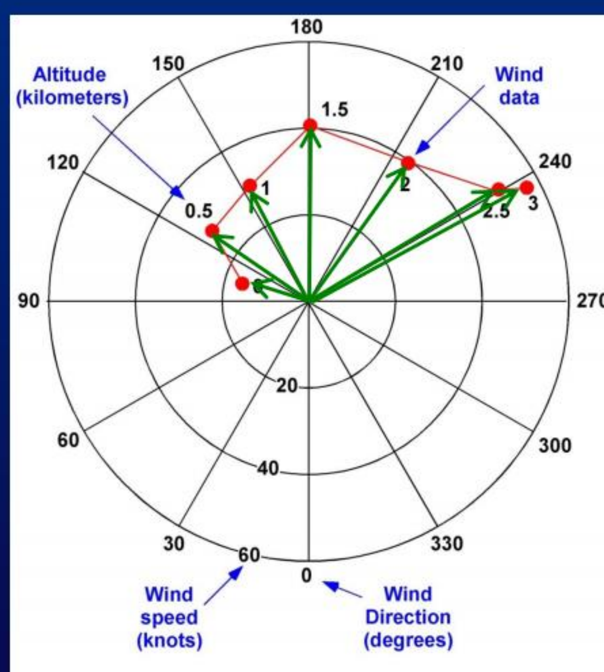
We thus see that, *at any instant, the acceleration of P is given by the velocity of p in the hodograph of its motion.*

Hodograph

- A hodograph is a line connecting the tips of wind vectors between two arbitrary heights in the atmosphere
- Each point on a hodograph represents a measured wind direction and speed at a certain level from RAOB data (or forecast data from a model)
- A hodograph is a plot of vertical wind shear from one level to another
- The points are then connected to form the hodograph line (red)



- Green arrows drawn from the origin allows one to better assess (visualize) wind field and wind shear
- Length of red line between 2 points shows amount of *speed* shear if line is *parallel* to radial, amount of *directional* shear if line is *normal* to radial, and amount of *speed and directional* shear if line is at *angle* to radial
- Total vertical shear = speed and directional
- Green lines represent "*ground-relative*" winds, i.e., the actual wind at various levels
- To determine total shear (in kts), lay out length of line along a radial
- Common layers assessed for severe weather – 0-1 km, 0-3 km, 0-6 km



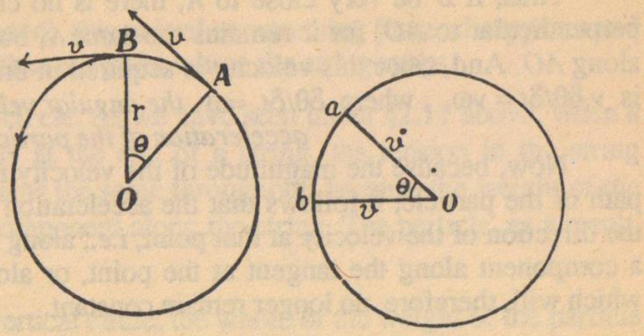
EXPRESSION FOR NORMAL ACCELERATION

2.9. Uniform Circular Motion. The above affords us a very simple method of determining the acceleration of a body or a particle, moving with a constant speed along a circle. Such a circular motion is essentially a translatory motion, with only the path of the body or the particle

lying along the circumference of a circle and the force acting on it directed towards the centre of the circle, as will be clear from the following.

Let a particle P move in a circle, with centre O and radius r, with a uniform speed v, [Fig. 2.5(a)]. Then, the hodograph is also a circle, of radius r, [Fig. 2.5(b)].

Now, the velocity of P at any instant is *at right angles* to the radius of its circular path, passing through P. Therefore, *oa is perpendicular to OA* and *ob is perpendicular to OB* and $\angle AOB = \angle aob = \theta$ (in circular measure).



(a) (b)

Fig. 2.5

If P takes time t to describe the arc AB, its velocity $v = AB/t = r\theta/t$, whence, $\theta = vt/r$.

And, the velocity of the corresponding point p, in the hodograph, is $ab/t = v\,\theta/t$

Since the velocity of p in the hodograph gives the acceleration of P in its actual path, we have

$$\textit{acceleration of} \quad P = \frac{v\,\theta}{t} = \frac{v}{t} \times \frac{vt}{r} = \frac{v^2}{r}$$

And, since ab is small, it is, *in the limit, perpendicular to oa*, or *parallel to AO*.

Thus, the *acceleration* of P is v^2/r and is *directed along the radius or towards the centre of the circular path* in which it is moving.

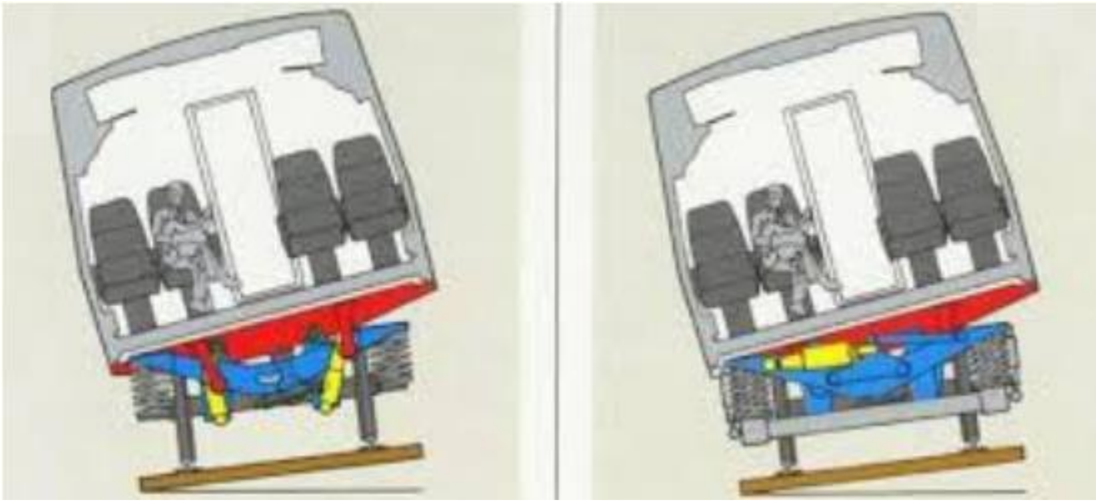
Further, since $v = r.\,\omega$ (where ω is the *angular velocity* of P), we have

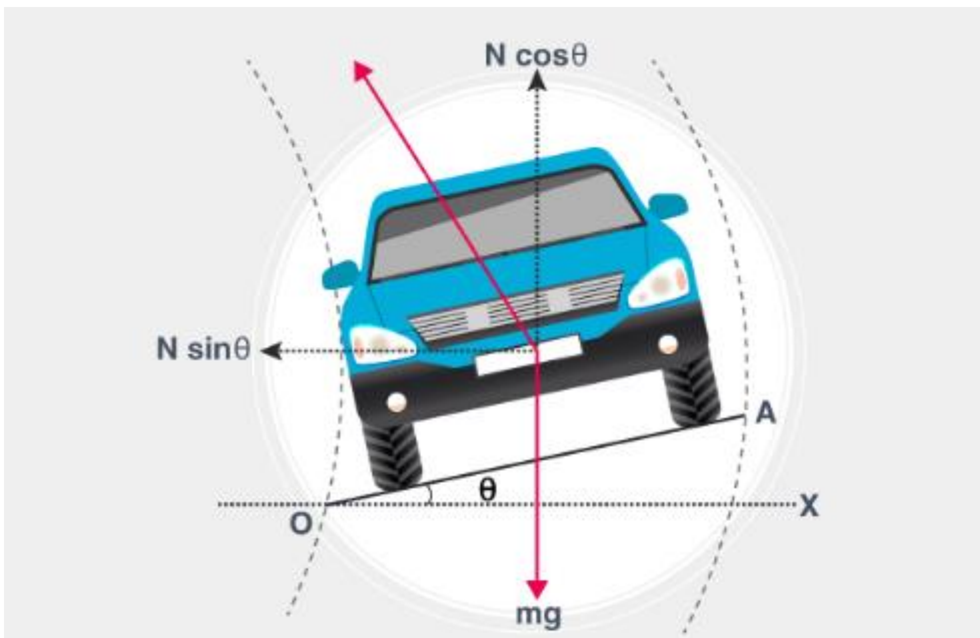
$$\textit{acceleration of } P\text{, also} = r^2.\,\omega^2/r = r\omega^2.$$

MOTION OF A RAILWAY CARRIAGE ROUND A CURVED TRACK , UPSETTING OF A CARRIAGE, MOTION OF A CARRIAGE ON A BANKED UP CURVE , MOTION OF A CYCLIST ALONG A CURVED PATH

BANKING OF RAILWAY LINES AND ROADS

When a railway train goes round a level curve on a railway track the necessary centripetal force is provided only by the force between the flanges or the rims of the wheels and the rails, the normal reaction of the ground or the track acting vertically upwards and supporting its weight. This result in a grinding action between the wheels and the rails, resulting in their wear and tear.




N cosθ
N sinθ
A
θ
X
O
mg

and the rails, resulting in their wear and tear. Not only that, it may also prove dangerous in the sense that it may bring about a displacement of the rails and hence a derailment of the train.

To avoid these eventualities, *the level of the outside rail is raised a little above that of the inside one*. This is known as the **banking** of railway lines, and the angle that the track makes with the horizontal is called the **angle of banking.**

With the track thus *banked, i.e.*, with the outer rail thus raised above the level of the inner one, the reaction R acts perpendicularly to the track, as before, but is now inclined to the vertical at an angle equal to the angle of banking and *its horizontal component* (and not the lateral thrust of the wheel flanges on the outside rail) *now supplies the necessary centripetal force to keep the train moving along the curve*, thereby eliminating all unnecessary wear and tear.

Thus, if θ be the *angle of banking* [Fig. 2.9(*a*)] and R, the *normal reaction* acting perpendicular to it, we have

$$\text{vertical component of } R = R \cos\theta$$

and

$$\text{horizontal component of } R = R \sin\theta$$

The former component balances the weight mg of the train and the latter supplies the required centripetal force mv^2/r, where v is the speed of the train and r, the radius of the curve it negotiates. So that,

$$R \sin\theta = mv^2/r$$

and $$R \cos\theta = mg.$$

$$\therefore \quad \frac{R \sin\theta}{R \cos\theta} = \frac{mv^2/r}{mg}.$$

Or, $$\tan\theta = v^2/rg \qquad \ldots(i)$$

Or, $$\theta = \tan^{-1}(v^2/rg).$$

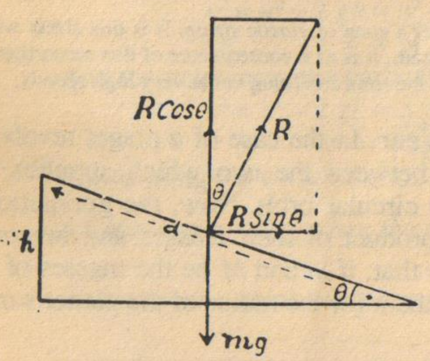


Fig. 2.9(*a*)

The angle of banking thus depends upon the *speed* (v) *of the train* and the *radius* (r) *of the curve* of the track. Obviously, therefore, a track can be *banked* correctly only for a particular speed of the train, – in practice, naturally for its *average* speed, given by $v = \sqrt{rg \tan\theta}$ from relation (i) above. At higher or lower speeds than this, there is again a lateral thrust due to the wheel flanges on the outer or the inner rail of the track respectively.

Clearly, the angle that the track makes with the horizontal is equal to θ, *i.e.*, equal to the angle of inclination of the train with the vertical, (Fig. 2.8).

Further, it will be readily seen that if the distance between the rails be d and the height of the outer rail above the inner one be h, we also have $\sin\theta = h/d$. Or, *sine of the angle of banking*

$$= \frac{\text{height of the outer rail over the inner one}}{\text{distance between the rails}}$$

In the above discussion, we have neglected the frictional force between the wheels and the rails. If this too be taken into account, the centripetal force will be supplied not only be the horizontal component $R \sin\theta$ of the normal reaction R but also by that of the frictional force F, as shown in Fig. 2.9 (*b*). So that, we now have

Since the maximum value of F is μR, we have

$$R \sin \theta + \mu R \cos \theta = mv^2/r$$

and $$R \cos\theta + \mu R \sin \theta = mg$$

So that,

$$\frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta} = \frac{\tan \theta + \mu}{1 - \mu \tan \theta} = \frac{v^2}{rg},$$

Whence $$v = \sqrt{rg. \frac{\tan \theta + \mu}{1 - \mu \tan \theta}},$$

which gives the maximum speed of the train while negotiating the curve.

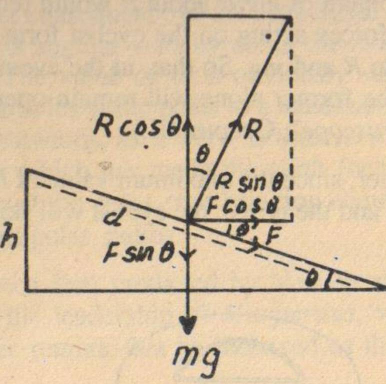


Fig. 2.9 (*b*)

Similarly, in the case of a car moving round a level corner, the centrifugal force is largely provided by the friction between the road and the tyres of the wheel. That is why, when the road is slippery and the frictional force not enough, the car begins to slide or skid. Here, too, therefore, the roads are '*banked*', the slope being generally steeper outwards, — more or less like a saucer — *the outer parts being meant to be used at higher speeds and the inner ones, at lower speeds.*

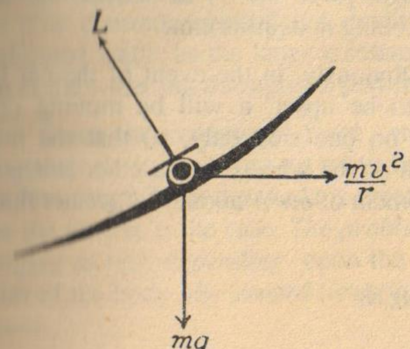


Fig. 2.10

Again, an aeroplane, in order to turn, must also bank, the centripetal force here being supplied by the horizontal component of the lift L, (Fig. 2.10)

The same applies to a cyclist, when negotiating a curve or a corner, and he has to lean *inwards*, (*i.e.*, towards the centre of the curve), by an angle $\theta = \tan^{-1} v^2/rg$; so that the faster his speed and the sharper the curve, the more must he lean over. This will be clear from the following :

Let Fig. 2.11 represent a cyclist turning to the left in a circle of radius r, at a speed v. Then, the ***normal reaction R*** of the ground acts vertically upwards, with the *force of friction F* between the ground and the tyres and the *centrifugal force* mv^2/r in the direction shown, where

$$R = mg \text{ and } F = mv^2/r.$$

Then, for equilibrium, clearly, we have moment of mg about P *equal and opposite* to moment of mv^2/r about P.

Or, $$mg \times PQ = (mv^2/r) \times GQ.$$

Or, $$mg \times PG.\sin \theta = (mv^2/r).PG.\cos \theta$$

whence, $$\sin \theta / \cos \theta = \tan \theta = v^2/rg.$$

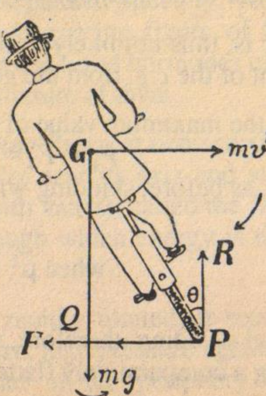


Fig. 2.11

In other words, in order to keep himself in equilibrium, the cyclist must lean inwards from the vertical at an angle $\theta = tan^{-1}(v^2 / rg)$.

If he were to remain vertical, his weight would act through P, having no moment about it, so that the moment of mv^2/r about P would remain unbalanced. In fact it will be readily seen that the system of forces acting on the cyclist form two pairs of couples, one due to F and mv^2/r and the other due to R and mg. So that, in the event of the latter couple vanishing (*i.e.*, if the cyclist were vertical), the former alone will remain operative, resulting in the cyclist toppling over. (*See* also under "*Gyroscope*", Chapter III.)

Further, since the maximum value of $F = \mu mg$, (where μ is the coefficient of friction between the ground and the tyres), the cyclist will skid when $mv^2/r > \mu mg$, or when $v^2 > \mu rg$.

Thus, *skidding will occur (i) if v is large, i.e., if the speed of the cyclist is large; (ii) if μ is small, i.e., if the road is slippery and (iii) if r is small, i.e., if the curve is sharp.*

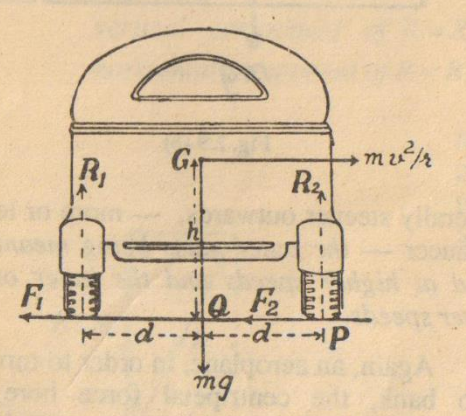


Fig. 2.12

Similar conditions apply in the case of a motor car or any other vehicle. For, here too, if we imagine it to be turning to the left (Fig. 2.12), the various forces acting on it are the *normal reactions* R_1 and R_2, the *frictional forces* F_1 and F_2, its *weight mg* and the *centrifugal force* mv^2/r, as shown, the whole system being in equilibrium.

Obviously, in the event of the car being about to be upset, it will be moving on the wheels on one side only, so that the normal reaction on the wheels on the other side will be zero; say $R_1 = 0$. So that, it will overturn as soon as the moment of mv^2/r about P is greater than the opposing moment of mg about P, *i.e.*,

$$mv^2/r \,.\, GQ > mg.\, PQ. \text{ Or, } (mv^2/r).h > mg.d,$$

where h is the height of the *c.g.*, G, of the car above the ground and $2d$, the distance between the two wheels.

For the car to be upset, therefore, we have $v^2 > d.rg/h$.

The car is, thus not likely to be upset if $2d$, the distance between the two wheels is large and if h, the height of the *c.g.* from the ground is small.

Again, the maximum value of the total frictional force

$$F_1 + F_2 = F = \mu.mg.$$

So that, as before, skidding will occur when $mv^2/r > \mu mg$.

Or, when $\mu^2 > \mu rg$.

To avoid skidding, therefore, while taking a turn at a fast speed, the corner must be cut so as to move along a comparatively flatter curve than that of the actual turning.

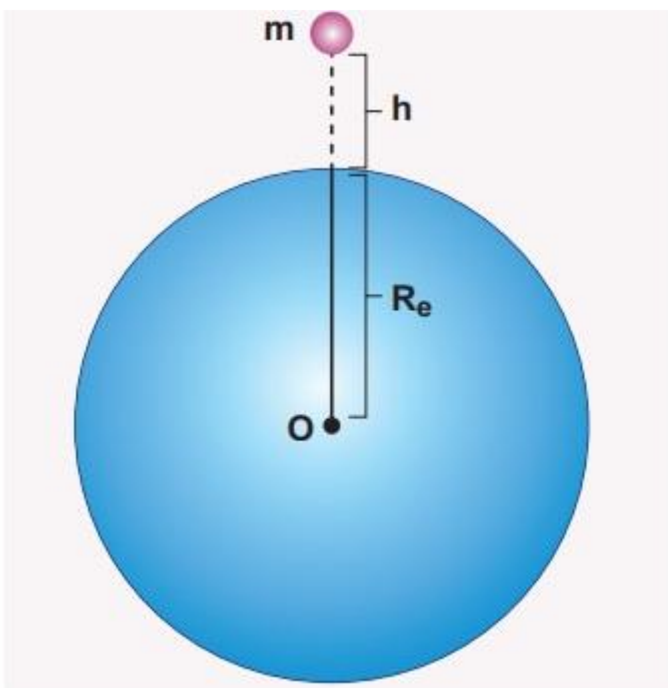
EFFECT OF EARTH'S ROTATION ON THE VALUE OF THE ACCELERATION DUE TO GRAVITY

1. **Rotation of the Earth — Its Effect.** As we know already, the earth rotates or spins about its axis once during a day. It is this rotation of it which is responsible for its getting flattened at the poles and its bulging out at the equator—a direct consequence of the centrifugal force $m\omega^2 R$ acting on each particle of mass m of it, where ω is its angular velocity about the axis of rotation and R, the distance of the particle from this axis. The value of ω is obviously *same* for each particle, but the distance R increases from *zero* for particles at the poles to a *maximum* for those at the equator. The centrifugal force pulling the earth outwards, as it were, is thus *zero at the poles* and *the maximum at the equator* and it is this force which has made the earth (behaving like a plastic body) to bulge out at the equator and to flatten at the poles, thus bringing about an increase of about 13 miles in its equatorial, as compared with its polar radius.

This effect of the rotation of the earth had been first predicted by *Newton* and was duly verified by a French expedition to Lapland under the leadership of *Maupertius*, whose undue pomposity provoked *Voltaire* into making the caustic remark that he 'behaved as though he had flattened the poles himself'.

VARIATION OF 'g' WITH ALTITUDE , LATITUDE AND DEPTH

Consider an object of mass m at a height h from the surface of the earth. Acceleration experienced by the object due to earth is



Mass at a height h from the center of the earth

$$g' = \frac{GM}{(R_e + h)^2}$$

$$g' = \frac{GM}{R_e^{\,2}\left(1 + \frac{h}{R_e}\right)^2}$$

$$g' = \frac{GM}{R_e^{\,2}}\left(1 + \frac{h}{R_e}\right)^{-2}$$

If $h << R_e$

We can use Binomial expansion. Taking the terms upto first order

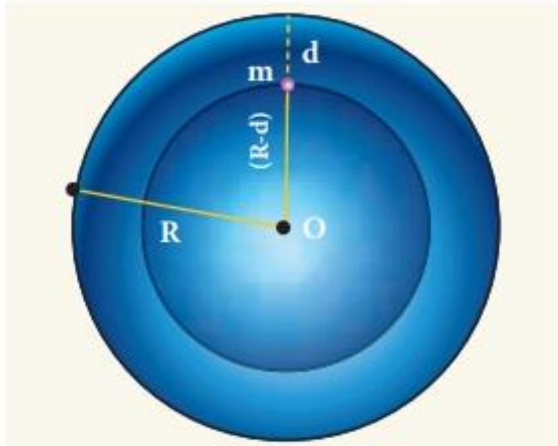
$$g' = \frac{GM}{R_e^{\,2}}\left(1 - 2\frac{h}{R_e}\right)$$

$$g' = g\left(1 - 2\frac{h}{R_e}\right)$$

We find that $g' < g$. This means that as altitude h increases the acceleration due to gravity g decreases.

Variation of g with depth:

Consider a particle of mass m which is in a deep mine on the Earth. (Example: coal mines in Neyveli). Assume the depth of the mine as d. To calculate g' at a depth d, consider the following points.



PARTICLE IN A MINE

The part of the Earth which is above the radius $(R_e - d)$ do not contribute to the acceleration. The result is proved earlier and is given as

$$g' = \frac{GM'}{(R_e - d)^2}$$

Here M' is the mass of the Earth of radius $(R_e\text{-}d)$

Assuming the density of Earth ρ to be constant,

$$\rho = \frac{M}{V}$$

where M is the mass of the Earth and V its volume, Thus,

$$\rho = \frac{M'}{V'}$$

$$\frac{M'}{V'} = \frac{M}{V} \text{ and } M' = \frac{M}{V}V'$$

$$M' = \left(\frac{M}{\frac{4}{3}\pi R_e^{\,3}}\right)\left(\frac{4}{3}\pi (R_e - d)^3\right)$$

$$M' = \frac{M}{R_e^{\,3}}(R_e - d)^3$$

$$g' = G\frac{M}{R_e^{\,3}}(R_e - d)^3 . \frac{1}{(R_e - d)^2}$$

$$g' = GM\frac{R_e\left(1 - \frac{d}{R_e}\right)}{R_e^{\,3}}$$

$$g' = GM\frac{\left(1 - \frac{d}{R_e}\right)}{R_e^{\,2}}$$

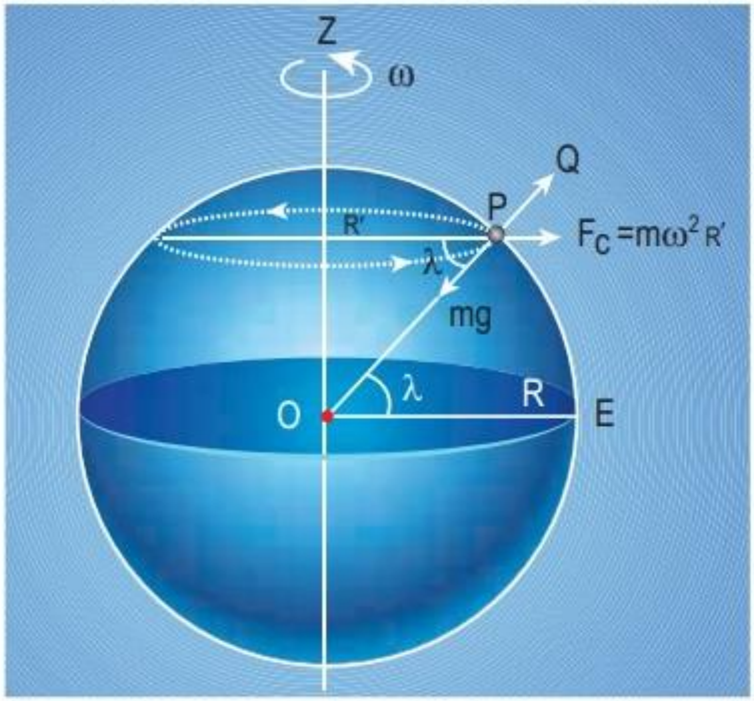
Thus

$$g' = g\left(1 - \frac{d}{R_e}\right)$$

Here also $g' < g$. As depth increases, g' decreases. It is very interesting to know that acceleration due to gravity is maximum on the surface of the Earth but decreases when we go either upward or downward.

Variation of g with latitude:

Whenever we analyze the motion of objects in rotating frames [explained in chapter 3] we must take into account the centrifugal force. Even though we treat the Earth as an inertial frame, it is not exactly correct because the Earth spins about its own axis. So when an object is on the surface of the Earth, it experiences a centrifugal force that depends on the latitude of the object on Earth. If the Earth were not spinning, the force on the object would have been mg. However, the object experiences an additional centrifugal force due to spinning of the Earth.



Variation of g with latitude

This centrifugal force is given by $m\omega^2 R'$.

$$R' = R\cos\lambda$$

where λ is the latitude. The component of centrifugal acceleration experienced by the object in the direction opposite to g is

$$a_{PQ} = \omega^2 R' \cos\lambda = \omega^2 R \cos^2\lambda$$

$$since\ R' = R\cos\lambda$$

Therefore,

$$g' = g - \omega^2 R \cos^2\lambda$$

From the expression (6.52), we can infer that at equator, $\lambda = 0$; $g' = g - \omega^2 R$. The acceleration due to gravity is minimum. At poles $\lambda = 90$; $g' = g$, it is maximum. At the equator, g' is minimum.