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CLASS : M.Sc DATA SCIENCE

SEMESTER : I

SUBJECT : MATHEMATICAL FOUNDATION FOR DATA SCIENCE

SUBJECT CODE : MDS23012

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UNIT - I LINEAR PROGRAMMING PROBLEM

Definition of Operations Research,

- Operations Research is a scientific method of providing executive departments with a quantitative phases for decisions regarding the operations under their control.

- Operations Research is the art of giving bad answers to the problems to which otherwise worse answers are given.

- Operations Research is a scientific approach to problem solving for executive management.

Scope of OR :

There is a great scope for economics, statisticians, administrators and technicians working as a team to solve the problems of defence by using the OR approach. Besides this, OR is useful in various other important fields like agriculture, finance, industry, marketing personal management, production management, research and development, military operations.

Phases of Operations Research,

The procedure to be followed in the study of OR generally involve the following major phases,

- formulating the problem

- constructing a mathematical model
- deriving the solution from the model
- testing the model and its solution.
- ~~cot~~ controlling the solution.
- implementation.

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Models of OR,

(i) Classification by Structure

(ii) Classification by purpose

(iii) Classification by behaviour

(iv) Classification by Method & Solution

(v) Classification by use of digital computer

$= 5 + 10 - 5 + 50 \div 2 \times 5$

$= 50 \% 2 \times 5 + 10$

$= 25 \times 5 + 10$

$= 125 + 10 = 135$

Linear Programming Problem [LPP],

Linear Programming is a mathematical modeling technique designed to optimize the usage of limited resource such as Labour, material, machine, capital, energy to several computing activities such as product service, jobs, new equipments, projects, etc.

Successful applications of LP ~~exists~~ ~~exist~~ exist in the area of militry, industry, agriculture, transportation, economics, health systems and even behavioural and social sciences.

Formulation of LPP,

~~A~~ The LP Model includes 3 basic elements,

- decision variables
- objective function
- constraints

Steps of LP Model formulation,

Step 1 - identify the unknown decision variables to be determined and decide symbols to them.

Step 2 - formulate all the constraints imposed by the resource availability and express them as linear equality or inequality in terms of the decision variables.

Step 3 - define the objective function. That is determine whether the objective function is to be maximized or minimized, then express it as a linear function of decision variables multiplied by their profit or cost contributions.

The general linear programming problem with 'n' decision variables and 'm' constraints can be stated in the following form, optimize $Z = \sum_{j=1}^{n} c_j x_j$

Subject to linear constraints

$$\sum_{j=1}^{n} a_{ij} x_j (\leq, =, \geq) b_i \quad ; \; i = 1, 2, 3, \dots m$$

and

$$x_j \geq 0 \; ; \; j = 1, 2, \dots, n$$ [Non-negative condition]

Definition,

* feasible solution

- any solution that satisfies all the constraints of the model is the feasible solution.

* Optimum feasible solution ;

A feasible solution which optimize [maximize or minimize] the objective functions of the given LP model is called optimum feasible solution.

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Graphical Method

Problem

① Find the maximum value of $Z = 5x_1 + 3x_2$.

Subject to constraints,

$$3x_1 + 5x_2 \leq 15$$

$$5x_1 + 2x_2 \leq 10 \; ; \quad x_1, x_2 \geq 0.$$

Solution

$$\text{Max } Z = 5x_1 + 3x_2$$

Subject to constraints,

$$3x_1 + 5x_2 = 15 \quad - ①$$

$$5x_1 + 2x_2 = 10 \quad - ②$$

from ①,

let $x_1 = 0$,

$$3(0) + 5x_2 = 15$$

$$5x_2 = 15$$

$$x_2 = 3$$

$$A(x_1, x_2) = A(0, 3)$$

let $x_2 = 0$,

$$3x_1 + 5(0) = 15$$

$$3x_1 = 15$$

$$x_1 = 5$$

$$B(x_1, x_2) = B(5, 0)$$

from ②,

let $x_1 = 0$,

$$5(0) + 2x_2 = 10$$

$$2x_2 = 10$$

$$x_2 = 5$$

$$C(x_1, x_2) = C(0, 5)$$

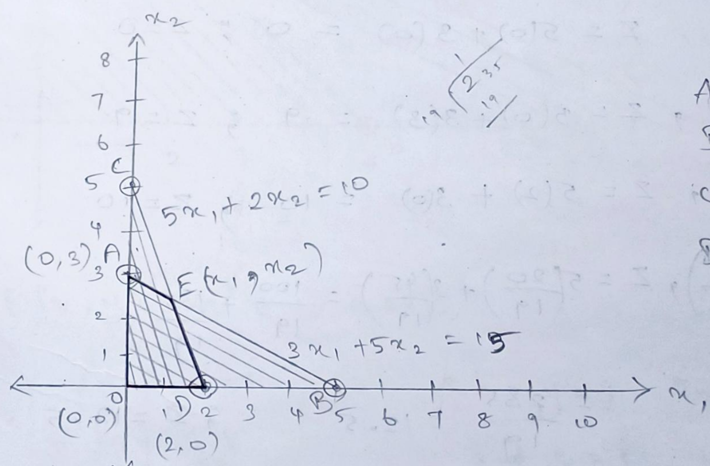
let $x_2 = 0$,

$$5(x_1) + 2(0) = 10$$

$$5x_1 = 10$$

$$x_1 = 2$$

$$D(x_1, x_2) = D(2, 0).$$



A (0, 3)
B (5, 0)
C (0, 5)
D (2, 0)

To find $E(x_1, x_2)$, solve ① and ②

Consider, $3x_1 + 5x_2 = 15$

$5x_1 + 2x_2 = 10$

$5 \times ① \Rightarrow 15x_1 + 25x_2 = 75$

$3 \times ② \Rightarrow 15x_1 + 6x_2 = 30$

$(-) \quad (-) \quad (-)$

$$19x_2 = 45$$

$$\boxed{x_2 = 45/19}$$

$$3x_1 + 5\left(\frac{45}{19}\right) = 15$$

$$3x_1 + \frac{225}{19} = 15$$

$$3x_1 = 15 - \frac{225}{19} \Rightarrow \frac{285-225}{19}$$

$$\cancel{3}x_1 = \frac{\overset{20}{\cancel{60}}}{19}$$

$$\boxed{x_1 = \frac{20}{19}}$$

$$\therefore E(x_1, x_2) = \left(\frac{20}{19}, \frac{45}{19}\right)$$

To find Max Z,

$$Z = 5x_1 + 3x_2$$

$O(0,0)$, $Z = 5(0) + 3(0) = 0$; $Z = 0$

$A(0,3)$, $Z = 5(0) + 3(3) = 9$; $Z = 9$

$D(2,0)$, $Z = 5(2) + 3(0) = 10$; $Z = 10$

$E\left(\frac{20}{19}, \frac{45}{19}\right)$, $Z = 5\left(\frac{20}{19}\right) + 3\left(\frac{45}{19}\right) = \frac{100}{19} + \frac{135}{19}$

$= \frac{235}{19} = 12.5$; $Z = 12.5$.

$\therefore$ Max $Z = 12.5$ at point, $E\left(\frac{20}{19}, \frac{45}{19}\right)$

---

② Find the minimum of the function. $Z = 2x - y$.

Subject to constraints,

$$x + y \leq 5$$

$$x + 2y \geq 8 \ ; \ x, y \geq 0.$$

Solution

To find Min $Z = 2x - y$.

Subject to constraints,

$$x + y = 5 \quad -① $$
$$x + 2y = 8 \quad -② $$

Consider ①,

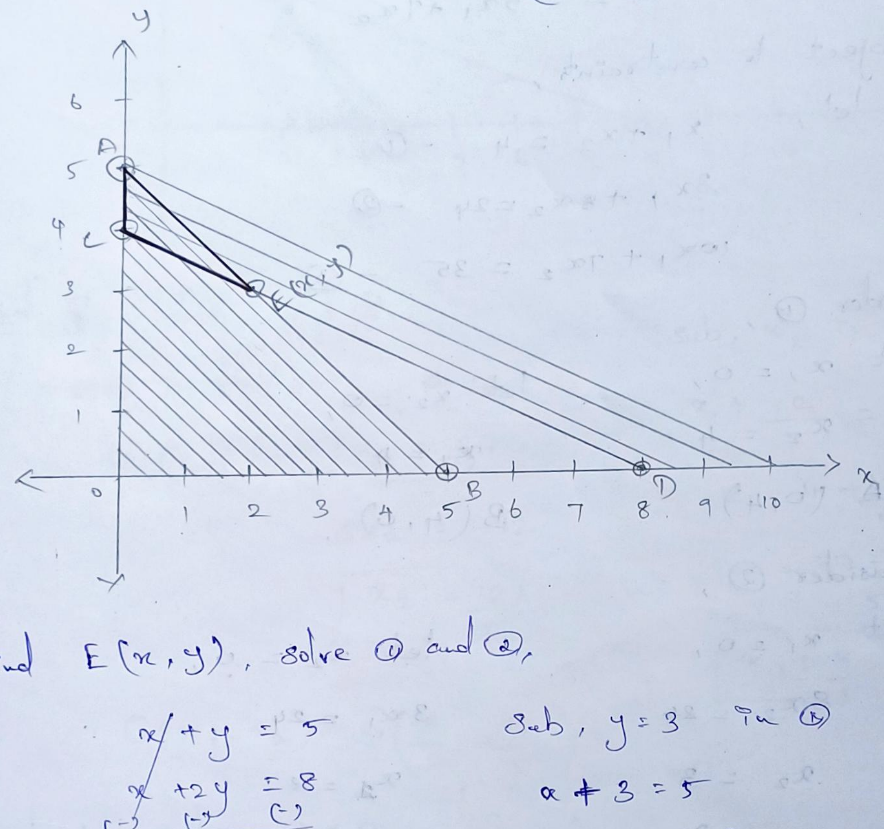
$x = 0$, $y = 5$ — $A(0,5)$

$y = 0$, $x = 5$ — $B(5,0)$

Consider ②,

$x = 0$, $2y = 8$, $y = 4$ — $C(0,4)$

$y = 0$, $x = 8$ — $D(8,0)$



To find $E(x,y)$, solve ① and ②,

$$\cancel{x} + y = 5$$
$$\cancel{x} + 2y = 8$$
$$(-) \quad (-) \quad (-)$$
$$-y = -3$$
$$\boxed{y = 3}$$

Sub, $y = 3$ in ①

$$x + 3 = 5$$
$$x = 5 - 3$$
$$\boxed{x = 2}$$

$$E(x,y) = E(2,3)$$

Min Z,

$$Z = 2x - y,$$

$A(0,5)$; $Z = 2(0) - 5 = -5$; $Z = -5$

$C(0,4)$; $Z = -4$; $Z = -4$

$E(2,3)$; $Z = 2(2) - 3 \Rightarrow 4 - 3 = 1$; $Z = 1$

Min $Z = -5$ at $A(0,5)$.

③ Find the maximum of the function $z = 5x_1 + 7x_2$.

Subject to constraints,

$$x_1 + x_2 \le 4$$

$$3x_1 + 8x_2 \le 24$$

$$10x_1 + 7x_2 \le 35 ; \quad x_1, x_2 \ge 0$$

Solution :-

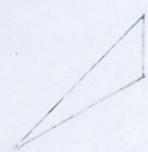
To find Max $Z = 5x_1 + 7x_2$.

Subject to constraints,

let, $x_1 + x_2 = 4$ — ①

$3x_1 + 8x_2 = 24$ — ②

$10x_1 + 7x_2 = 35$ — ③



Consider ①,

let $x_1 = 0$, $x_2 = 4$, A $(0,4)$

let $x_2 = 0$, $x_1 = 4$, B $(4,0)$.

Consider ②,

let $x_1 = 0$,

$8x_2 = 24$

$x_2 = 3$

C $(0,3)$

let $x_2 = 0$,

$3x_1 = 24$

$x_1 = 8$

D $(8,0)$.

Consider ③,

let $x_1 = 0$,

$7x_2 = 35$

$x_2 = 5$

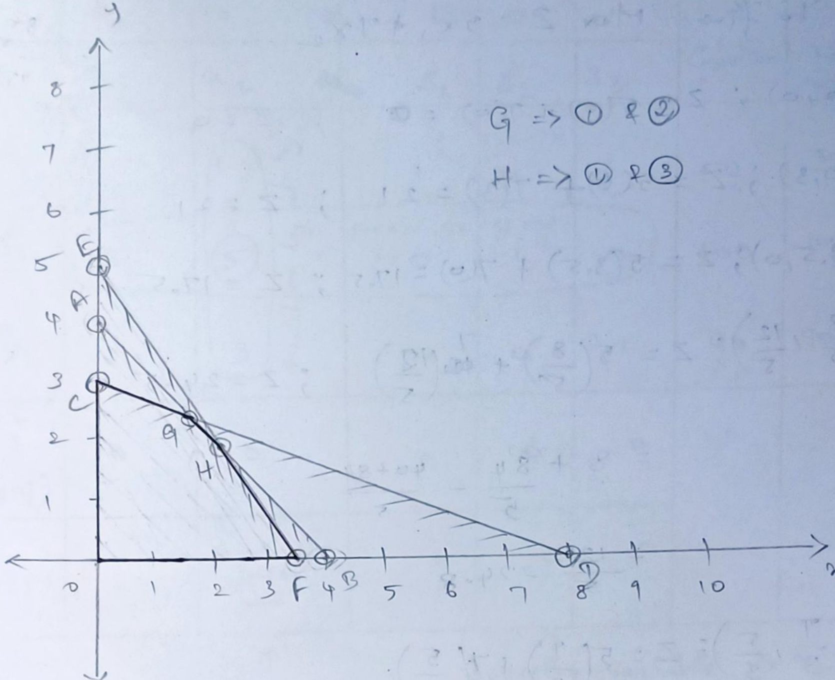
E $(0,5)$

let $x_2 = 0$,

$10x_1 = 35$

$x_1 = 3.5$

F $(3.5,0)$



G => ① & ②

H => ① & ③

To find G, solve ① & ②.

$$3 \times ① \Rightarrow 3x_1 + 3x_2 = 12$$

$$② \Rightarrow 3x_1 + 8x_2 = 24$$

$$(-)\quad (-)\quad (-)$$

$$-5x_2 = -12$$

$$\boxed{x_2 = \frac{12}{5}}$$

sub, x_2 in ①,

$$x_1 + \frac{12}{5} = 4$$

$$x_1 = 4 - \frac{12}{5}$$

$$= \frac{20-12}{5} = \frac{8}{5}$$

$$\boxed{x_1 = \frac{8}{5}}$$

$$G(x_1, x_2) = G\left(\frac{8}{5}, \frac{12}{5}\right)$$

To find H, solve ① & ③,

$$10 \times ① \Rightarrow 10x_1 + 10x_2 = 40$$

$$③ \Rightarrow 10x_1 + 7x_2 = 35$$

$$(-)\quad (-)\quad (-)$$

$$3x_2 = 5$$

$$x_2 = \frac{5}{3}$$

sub x_2 in ①,

$$x_1 + \frac{5}{3} = 4$$

$$x_1 = 4 - \frac{5}{3}$$

$$= \frac{12-5}{3}$$

$$x_1 = \frac{7}{3}$$

$$H(x_1, x_2) = \left(\frac{7}{3}, \frac{5}{3}\right)$$

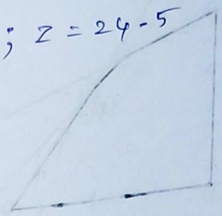
To find Max $Z = 5x_1 + 7x_2$

$O(0,0)$; $Z = 5(0) + 7(0) = 0$; $Z = 0$

$C(0,3)$; $Z = 5(0) + 7(3) = 21$; $Z = 21$

$F(3.5,0)$; $Z = 5(3.5) + 7(0) = 17.5$; $Z = 17.5$

$G\left(\frac{8}{5}, \frac{12}{5}\right)$; $Z = 5\left(\frac{8}{5}\right) + 7\left(\frac{12}{5}\right)$; $Z = 24.5$



$= 8 + \frac{84}{5} = \frac{40+84}{5}$

$= \frac{124}{5} = 24.8$

$H\left(\frac{7}{3}, \frac{5}{3}\right)$; $Z = 5\left(\frac{7}{3}\right) + 7\left(\frac{5}{3}\right)$

$= \frac{35}{3} + \frac{35}{3}$

$= \frac{70}{3}$

$= 23.33$; $Z = 23.3$

Max $Z = 24.8$ at point $G\left(\frac{8}{5}, \frac{12}{5}\right)$.

$\frac{35}{2}$ = 17.5

5) 124 (24.8
10
24
20
40

3) 70 (23
6
10

3rd

SIMPLEX METHOD

1) Solve the LPP using simplex method.

Max $Z = 4x_1 + 10x_2$

Subject to constraints,

$2x_1 + x_2 \leq 50$

$2x_1 + 5x_2 \leq 100$

$2x_1 + 3x_2 \leq 90$

Solution

Max $Z = 4x_1 + 10x_2 + 0S_1 + 0S_2 + 0S_3$

Sub. to constraints,

$2x_1 + x_2 + S_1 = 50$ — (1)

$2x_1 + 5x_2 + S_1 = 100$ — (2)

$2x_1 + 3x_2 + S_3 = 90$ — (3)

iteration-1

C_B	C_j	4	10	0	0	0	Solution	ratio
	B	x_1	x_2	S_1	S_2	S_3		
0	S_1	2	1	1	0	0	50	$\frac{50}{1} = 50$
0	S_2	2	5	0	1	0	100	$100/5 = 20$ ←
0	S_3	2	3	0	0	1	90	$90/3 = 30$
	Z_j $(\Sigma C_B a_j)$	0	0	0	0	0		
	$Z_j - C_j$	-4	-10 ↑	0	0	0		

pivotal or key column

pivotal or key row.

pivotal or key element

Entering variable → x_2

Leaving variable → S_2

$S_1 \to$ 2 1 1 0 0 50

$x_2 \to$ 2/5 1 0 1/5 0 20 (−)

8/5 0 1 −1/5 0 30

$S_3 \to$ 2 3 0 0 1 90

$x_2 \to$ 6/5 3 0 3/5 0 60 (−)

4/5 0 0 −3/5 1 30

iteration-2.

C_B	C_j	4	10	0	0	0	Solution	ratio
	B	x_1	x_2	S_1	S_2	S_3		
0	S_1	8/5	0	1	−1/5	0	30	
10	x_2	2/5	1	0	1/5	0	20	
0	S_3	4/5	0	0	−3/5	1	30	
	Z_j $[\Sigma C_B a_j]$	4	10	0	2	0	~~200~~	
	$Z_j - C_j$	0	0	0	2	0		

$\therefore$ $Z_j - C_j$ all are positive,

$x_1 = 0, \; x_2 = 20$

Max $Z = 4(x_1) + 10x_2 \Rightarrow 4(0) + 10(20)$

$= 0 + 200$

Max $Z = 200$.

② Using Simplex method to Maximize $z = 5x_1 + 4x_2$.
Subject to constraints,

$$4x_1 + 5x_2 \le 10$$
$$3x_1 + 2x_2 \le 9$$
$$8x_1 + 3x_2 \le 12 \;;\quad x_1, x_2 \ge 0$$

Solution:

$$\text{Max } Z = 5x_1 + 4x_2 + 0S_1 + 0S_2 + 0S_3$$

Sub. to constraints,

$$4x_1 + 5x_2 + S_1 = 10 \quad -①$$
$$3x_1 + 2x_2 + S_2 = 9 \quad -②$$
$$8x_1 + 3x_2 + S_3 = 12 \quad -③$$

iteration 1

	C_j	5	4	0	0	0		
C_B	B	x_1	x_2	S_1	S_2	S_3	Solution	ratio
0	S_1	4	5	1	0	0	10	$10/4 = 2.5$
0	S_2	3	2	0	1	0	9	$9/3 = 3$
0	S_3	8	3	0	0	1	12	$12/8 = 3/2 = 1.5 \rightarrow$
	Z_j $(\Sigma C_B a_i)$	0	0	0	0	0		
	$Z_j - C_j$	-5	-4	0	0	0		
		$\uparrow$						

Iteration 2

$r_1 \rightarrow$ 4 5 1 0 0 10

$4 \times r_3 \rightarrow$ (−) 4 3/2 0 0 1/2 6

$5 - \frac{3}{2}$

0 7/2 1 0 −1/2 4

$r_2 \rightarrow$ 3 2 0 1 0 9

$3 \times r_3 \rightarrow$ (−) 3 9/8 0 0 3/8 9/2

0 7/8 0 1 −3/8 9/2

$2 - \frac{9}{8}$

$9 - \frac{9}{2}$

	C_j	5	4	0	0	0		
C_B	B	x_1	x_2	S_1	S_2	S_3	Solution	ratio
0	S_1	0	7/2	1	0	-1/2	4	$4 \times \frac{2}{7} = \frac{8}{7} = 1.14$ →
0	S_2	0	7/8	0	1	-3/8	9/2	$\frac{9}{2} \times \frac{8}{7} = \frac{36}{7} = 5.1$
5	x_1	1	3/8	0	0	1/8	3/2	$\frac{3}{2} \times \frac{8}{3} = 4$
	Z_j	5	15/8	0	0	5/8		
	$Z_j - C_j$	0	-17/8 ↑	0	0	5/8		

$\frac{8}{7} \times \frac{3}{8}$ $\quad \frac{8}{7} \times \frac{7}{8}$ $\quad -\frac{1}{7} \times \frac{3}{8}$ $\quad \frac{2}{7} \times \frac{7}{8}$ $\quad -\frac{1}{7} \times \frac{7}{8}$ $\quad \frac{1}{2} \times \frac{2}{7} \times \frac{3}{8}$ $\quad -\frac{1}{2} \times \frac{2}{7}$ $\quad 4 \times \frac{2}{7}$

iteration 3

	C_j	5	4	0	0	0		
C_B	B	x_1	x_2	S_1	S_2	S_3	Solution	ratio
4	x_2	0	1	2/7	0	-1/7	8/7	
0	S_2	0	0	-1/4	1	-1/4	7/2	
5	x_1	1	0	-3/28	0	5/28	15/14	
	Z_j	5	4	17/28	0	9/28		
	$Z_j - C_j$	0	0	17/28	0	9/28		

$r_2 \to$	0	7/8	0	1	-3/8	9/2
$\frac{7}{8} \times r_1 \to$	0	7/8	1/4	0	-1/8	1
(-)						
	0	0	-1/4	1	-1/4	7/2

$r_3 \to$	1	3/8	0	0	1/8	3/2
$\frac{3}{8} \times r_1 \to$	0	3/8	3/28	0	-3/56	3/7
(-)						
	1	0	-3/28	0	5/28	15/14

$0 - \frac{1}{4}$

$-\frac{3}{8} + \frac{1}{8}$

$\frac{9}{2} - 1 \quad \frac{9-2}{2}$

$\frac{1}{8} + \frac{3}{56} \quad \frac{3}{2} - \frac{3}{7}$

$\frac{10}{56} = \frac{5}{28} \quad \frac{21-6}{14}$

∴ $Z_j - C_j$ all are positive.

$x_1 = \frac{15}{14}, \quad x_2 = \frac{8}{7}$

$Z = 5x_1 + 4x_2 \Rightarrow 5\left(\frac{15}{14}\right) + 4\left(\frac{8}{7}\right)$

$Z = \frac{139}{14}$

Homework

Using simplex method to Max $Z = 3x_1 + 5x_2$.

Subject to constraints,

$3x_1 + 2x_2 \leq 18$ $x_1 = 2,\ x_2 = 6$

$x_1 \leq 4$

$x_2 \leq 6$; $x_1, x_2 \geq 0$.

Solution

Max $Z = 3x_1 + 5x_2 + 0S_1 + 0S_2 + 0S_3$

Sub. of constraints,

$3x_1 + 2x_2 + S_1 = 18$ — (1)

$x_1 + S_2 = 4$ — (2)

$x_2 + S_3 = 6$ — (3)

iteration 1

C_B	C_j	3	5	0	0	0	Solution	ratio
	B	x_1	x_2	S_1	S_2	S_3		
0	S_1	3	2	1	0	0	18	$\frac{18}{2} = 9$
0	S_2	1	0	0	1	0	4	∞
0	S_3	0	1	0	0	1	6	6 →
	Z_j	0	0	0	0	0		
	$Z_j - C_j$	-3	-5 ↑	0	0	0		

iteration 2

$r_1 \rightarrow$ 3 2 1 0 0 18

$2 \times r_3 \rightarrow$ 0 2 0 0 2 12
(in iteration 1)

3 0 1 0 -2 6

C_B	C_j / B	3 / x_1	5 / x_2	0 / S_1	0 / S_2	0 / S_3	Solution	ratio
0	S_1	3	0	1	0	-2	6	2
0	S_2	1	0	0	1	0	4	4
5	x_2	0	1	0	0	1	6	∞
	Z_j	0	5	0	0	5		
	$Z_j - C_j$	-3	0	0	0	5		

iteration 3

$r_2 \rightarrow$ 1 0 0 1 0 4

$r_1 \rightarrow$ 1 0 1/3 0 -2/3 2

(-) 0 0 -1/3 1 2/3 2

C_B	C_j / B	3 / x_1	5 / x_2	0 / S_1	0 / S_2	0 / S_3	Solution	ratio
3	x_1	1	0	1/3	0	-2/3	2	
0	S_2	0	0	-1/3	1	2/3	2	
5	x_2	0	1	0	0	1	6	
	Z_j	3	5	1	0	3		
	$Z_j - C_j$	0	0	1	0	3		

$\because Z_j - C_j$ all are positive

$x_1 = 2 ; x_2 = 6$

$\therefore$ Max $Z = 3x_1 + 5x_2 \Rightarrow 3(2) + 5(6)$

$= 6 + 30$

Max $Z = 36$

5/8/24.

Big-M Method or Penalty Method

① Use penalty (Big-M) method to Max $Z = 3x_1 - x_2$

Subject to constraints,
$2x_1 + x_2 \geq 2$
$x_1 + 3x_2 \leq 3$
$x_2 \leq 4$; $x_1, x_2 \geq 0$.

Solution

Max $Z = 3x_1 - x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1$

$\geq \Rightarrow -S, +A$
$\leq \Rightarrow +S$
$= \Rightarrow +A$

Subject to constraints,

$2x_1 + x_2 - S_1 + A_1 = 2$

$x_1 + 3x_2 + S_2 = 3$

$x_2 + S_3 = 4$

iteration - 1.

C_B	C_j / B	3 / x_1	-1 / x_2	0 / S_1	0 / S_2	0 / S_3	-M / A_1	Solution	Ratio
-M	A_1	2	1	-1	0	0	1	2	1 ←
0	S_2	1	3	0	1	0	0	3	3
0	S_3	0	1	0	0	1	0	4	0
	Z_j	-2M	-M	M	0	0	-M		
	$Z_j - C_j$	-2M-3	-M+1	M	0	0	0		

↑
Entering col.

iteration - 2.

$r_2 \Rightarrow$ 1 3 0 1 0 0 3

new, $r_1 \Rightarrow$ 1 1/2 -1/2 0 0 1/2 1
(−)

0 5/2 1/2 1 0 -1/2 2

C_B	C_j / B	3 / x_1	-1 / x_2	0 / S_1	0 / S_2	0 / S_3	-M / A_1	Solution	Ratio
3	x_1	1	$1/2$	$-1/2$	0	0	$1/2$	1	-2
0	S_2	0	$5/2$	$1/2$	1	0	$-1/2$	2	4 ←
0	S_3	0	1	0	0	1	0	4	0
	Z_j	3	$3/2$	$-3/2$	0	0	$3/2$		
	$Z_j - C_j$	0	$5/2$	$-3/2$ ↑	0	0	$+\frac{3}{2}+M$		

Iteration 3

$x_1 \rightarrow$ 1 $1/2$ $-1/2$ 0   0 $1/2$ 1

new $x_2 \rightarrow$ 0 $5/2$ $1/2$ 1   0 $-1/2$ 2 +

1   3   0   1   0   0   3

C_B	C_j / B	3 / x_1	-1 / x_2	0 / S_1	0 / S_2	0 / S_3	-M / A_1	Solution	Ratio
3	x_1	1	3	0	1	0	0	3	
0	S_1	0	5	1	2	0	-1	4	
0	S_3	0	1	0	0	1	0	4	
	Z_j	3	9	0	3	0	0		
	$Z_j - C_j$	0	10	0	3	0	M		

$Z_j - C_j$ all are positive,

$x_1 = 3$, $x_2 = 0$,

$\text{Max } Z = 3x_1 - x_2 \Rightarrow 3(3) - 0$

$\text{Max } Z = 9.$

② Use Big-M method to Max $Z = 6x_1 + 4x_2$

Sub. to constraints, $2x_1 + 3x_2 \le 30$,

$3x_1 + 2x_2 \le 24$,

$x_1 + x_2 \ge 3$; $x_1, x_2 \ge 0$

Solution

Max $Z = 6x_1 + 4x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1$

Sub. to constraints,

$2x_1 + 3x_2 + S_1 = 30$

$3x_1 + 2x_2 + S_2 = 24$

$x_1 + x_2 - S_3 + A_1 = 3$

iteration 1

	c_j	6	4	0	0	0	-M		
C_B	B	x_1	x_2	S_1	S_2	S_3	A_1	Solution	Ratio
0	S_1	2	3	1	0	0	0	30	15
0	S_2	3	2	0	1	0	0	24	8
-M	A_1	1	1	0	0	-1	1	3	3 ←
	Z_j	-M	-M	0	0	M	-M		
	$Z_j - C_j$	-M-6	-M-4	0	0	M	0		

↑

iteration 2

$r_1 \to$ 2 3 1 0 0 0 30

new $r_3 \to$ ×2 (−) 2 2 0 0 -2 2 6

0 1 1 0 2 -2 24

$r_2 \to$ 3 2 0 1 0 0 24

new $r_3 \to$ ×3 3 3 0 0 -3 3 9

0 -1 0 1 3 -3 15

C_B	C_j / B	6 / x_1	4 / x_2	0 / S_1	0 / S_2	0 / S_3	$-M$ / A_1	Solution	Ratio
0	S_1	0	1	1	0	2	-2	24	12
0	S_2	0	-1	0	1	3	-3	15	5 →
6	x_1	1	1	0	0	-1	1	3	-3
Z_j		6	6	0	0	-6	6		
$Z_j - C_j$		0	2	0	0	-6	$6+M$		

↑

iteration 3

$r_1 \to 0 \quad 1 \quad 1 \quad 0 \quad 2 \quad -2 \quad 24$

new $r_2 \to 0 \quad -2/3 \quad 0 \quad -2/3 \quad -2 \quad +2 \quad -10$

$x_2 \quad 0 \quad 5/3 \quad 1 \quad -2/3 \quad 0 \quad 0 \quad 14$

$r_3 \to 1 \quad 1 \quad 0 \quad 0 \quad -1 \quad 1 \quad 3$

new $r_2 \to 0 \quad -1/3 \quad 0 \quad 1/3 \quad 1 \quad -1 \quad 5$

$1 \quad 2/3 \quad 0 \quad 1/3 \quad 0 \quad 0 \quad 8$

C_B	C_j / B	6 / x_1	4 / x_2	0 / S_1	0 / S_2	0 / S_3	$-M$ / A_1	Solution	Ratio
0	S_1	0	5/3	1	$-2/3$	0	0	14	
0	S_3	0	$-1/3$	0	1/3	1	-1	5	
6	x_1	1	2/3	0	1/3	0	0	8	
Z_j		6	4	0	2	0	0		
$Z_j - C_j$		0	0	0	2	0	M		

$Z_j - C_j$ all are positive,

$x_1 = 8$; $x_2 = 0$

$\text{Max } Z = 6x_1 + 4x_2 \Rightarrow 6(8) + 4(0)$

$\text{Max } Z = 48$.

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Two - Phase Method

① Use two-phase simplex method to, Min $Z = x_1 + x_2$

Sub. to constraints, $2x_1 + x_2 \geq 4$

~~$x_1 \geq 4$~~

$x_1 + 7x_2 \geq 7$; $x_1, x_2 \geq 0$.

Solution

to convert Min z to Max z, multiply by (-).

Max $Z = -x_1 - x_2 + 0S_1 + 0S_2 - A_1 - A_2$

Sub. to constraints;

$2x_1 + x_2 - S_1 + A_1 = 4$

$x_1 + 7x_2 - S_2 + A_2 = 7$

Phase I

iteration consider,

Max $Z = 0x_1 + 0x_2 + 0S_1 + 0S_2 - A_1 - A_2$

Sub. to constraints,

$2x_1 + x_2 - S_1 + A_1 = 4$

$x_1 + 7x_2 - S_2 + A_2 = 7$

	C_j	0	0	0	0	-1	-1		
C_B	B	x_1	x_2	S_1	S_2	A_1	A_2	Solution	ratio
-1	A_1	2	1	-1	0	1	0	4	$4/1 = 4$
-1	A_2	1	$\boxed{7}$	0	-1	0	1	7	$7/7 = 1$ →
	Z_j	-3	-8	1	1	-1	-1		
	$Z_j - C_j$	-3	-8 ↑	1	1	0	0		

iteration 2

$r_1 \Rightarrow$ 2 1 -1 0 1 0 4

new $r_2 \Rightarrow$ $1/7$ 1 0 $-1/7$ 0 $1/7$ 1

(-)

$13/7$ 0 -1 $1/7$ 1 $-1/7$ 3

	C_j	0	0	0	0	-1	-1		
C_B	B	x_1	x_2	S_1	S_2	A_1	A_2	Solution	ratio
-1	A_1	$\boxed{13/7}$	0	-1	1/7	1	-1/7	3	$21/13 = 1 \cdot \frac{13}{7}$ →
0	x_2	1/7	1	0	-1/7	0	1/7	1	7
	Z_j	-13/7	0	1	-1/7	-1	1/7		
	$Z_j - C_j$	-13/7 ↑	0	1	-1/7	0	8/7		

iteration 3

$r_2 \rightarrow$ 1/7 1 0 -1/7 0 1/7 1

new $r_1 \rightarrow$ (−) 1/7 0 −1/13 1/91 1/13 −1/91 3/13

$\times \frac{1}{7}$

0 1 1/13 −14/91 −1/13 14/91 10/13

	C_j	0	0	0	0	-1	-1	
C_B	B	x_1	x_2	S_1	S_2	A_1	A_2	Solution ratio
0	x_1	1	0	-7/13	1/13	7/13	-1/13	21/13
0	x_2	0	1	1/13	-14/91	-1/13	14/91	10/13
	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$	0	0	0	0	1	1	

$Z_j - C_j \geq 0$

Phase II

iteration 1

	C_j	-1	-1	0	0			
C_B	B	x_1	x_2	S_1	S_2	$\cancel{A_1}$	$\cancel{A_2}$	Solution ratio
-1	x_1	1	0	-7/13	1/13			21/13
-1	x_2	0	1	1/13	-14/91			10/13
	Z_j	-1	-1	6/13	7/91			
	$Z_j - C_j$	0	0	6/13	7/91			

$$\text{Min } Z = \frac{21}{13} + \frac{10}{13} = \frac{31}{13}.$$

$Z_j - C_j \geq 0$; $x_1 = \frac{21}{13}$; $x_2 = \frac{10}{13}$

13)21(1.6
13
80
78
20

② Use two-phase simplex method. Min $Z = x_1 + x_2 + x_3$

Sub. to constraints, $x_1 - 3x_2 + 4x_3 = 5$

$$x_1 - 2x_2 \leq 3$$

$$2x_2 - x_3 \geq 4 \quad ; \quad x_1, x_2, x_3 \geq 0.$$

Solution

To convert Min z to Max z, multiple objective func. with (-)

$$\text{Max } Z = -x_1 - x_2 - x_3 + 0S_1 + 0S_2 - A_1 - A_2$$

Sub. to constraints,

$$x_1 - 3x_2 + 4x_3 + A_1 = 5$$

$$x_1 - 2x_2 - S_1 = 3$$

$$2x_2 - x_3 - S_2 + A_2 = 4$$

Phase I

iteration 1

consider, Max $Z = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 - A_1 - A_2$

C_B	C_j	0	0	0	0	0	-1	-1		
	B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	Solution	ratio
-1	A_1	1	-3	$\boxed{4}$	0	0	1	0	5	$5/4 = 1.25$ →
0	S_1	1	-2	0	-1	0	0	0	3	0
-1	A_2	0	2	-1	0	-1	0	1	4	-4
Z_j		-1	1	-3	0	1	-1	-1		
$Z_j - C_j$		-1	1	-3	0	1	0	0		

↑

iteration 2

$r_3 \rightarrow$ 0 2 -1 0 -1 0 1 4

new $r_1 \rightarrow$ $1/4$ $-3/4$ 1 0 0 $1/4$ 0 $5/4$

$1/4$ $5/4$ 0 0 -1 $1/4$ 1 $\frac{21}{4}$

C_B	C_j / B	0 x_1	0 x_2	0 x_3	0 S_1	0 S_2	-1 A_1	-1 A_2	Sol	ratio
0	x_3	1/4	-3/4	1	0	0	1/4	0	5/4	-5/3
0	S_1	1	-2	0	-1	0	0	0	3	-3/2
-1	A_2	1/4	5/4	0	0	-1	1/4	1	21/4	21/5 →
	Z_j	-1/4	-5/4	0	0	+1	-1/4	-1		
	$Z_j - C_j$	-1/4	-5/4 ↑	0	0	1	3/4	0		

iteration 3

$r_1 \to$ 1/4 -3/4 1 0 0 1/4 0 5/4

new $r_3 \to$ 3/20 3/4 0 0 -3/5 3/20 3/5 62/20

8/20 = 2/5 0 1 0 -3/5 2/5 3/5 88/20 = 22/5

$r_2 \to$ 1 -2 0 -1 0 0 0 3

new $r_3 \to$ 2/5 2 0 0 -8/5 2/5 8/5 42/5

7/5 0 0 -1 -8/5 2/5 8/5

C_B	C_j / B	0 x_1	0 x_2	0 x_3	0 S_1	0 S_2	-1 A_1	-1 A_2	Sol	ratio
0	x_3	2/5	0	1	0	-3/5	2/5	3/5	22/5	
0	S_1	7/5	0	0	-1	-8/5	2/5	8/5	57/5	
0	x_2	1/5	1	0	0	-4/5	1/5	4/5	21/5	
	Z_j	0	0	0	0	0	0	0		
	$Z_j - C_j$	0	0	0	0	0	1	1		

$Z_j - C_j \geq 0.$

Phase II

C_B	C_j / B	-1 x_1	-1 x_2	-1 x_3	0 s_1	0 s_2	Sol	ratio
-1	x_3	2/5	0	1	0	-3/5	22/5	
0	s_1	7/5	0	0	-1	-8/5	57/5	
-1	x_2	1/5	1	0	0	-4/5	21/5	
	Z_j	-3/5	-1	-1	0	7/5		
	$Z_j - C_j$	2/5	0	0	0	7/5		

$Z_j - C_j \geq 0.$

$x_1 = 0,$

$x_2 = 21/5$

$x_3 = 22/5$

$$\text{Min } Z = x_1 + x_2 + x_3$$

$$= 0 + \frac{21}{5} + \frac{22}{5}$$

$$= \frac{43}{5}.$$

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UNIT - 3 GEOMETRY LINEAR EQUATIONS AND VECTOR SPACES

MATRIX DEFINITION:

An $m \times n$ matrix A is an array of 'mn' numbers a_{ij}, where '$1 \le i \le m$', $1 \le j \le n$. Array arranged in m rows and n columns, as follows,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

If $m = n$, then A is called as Square Matrix with order n or m.

DEFINITION:

Two Matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be equal if A and B here the same number of rows and columns and the corresponding entries in the two matrices are same.

Addition of two Matrices:

Addition of 2, $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ then, $A + B = (a_{ij} + b_{ij})$

Example,

if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 9 & 5 \end{bmatrix}_{3\times 2}$, $B = \begin{bmatrix} 0 & 4 \\ 2 & 1 \\ -1 & 0 \end{bmatrix}_{3\times 2}$

$$A + B = \begin{bmatrix} 1+0 & 2+4 \\ 3+2 & 4+1 \\ 9-1 & 5+0 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1 & 6 \\ 5 & 5 \\ 8 & 5 \end{bmatrix}_{3\times 2}$$

MATRIX MULTIPLICATION:

Let $A = (a_{ij})$ ~~be~~ be an $m \times n$ matrix and $B = (b_{ij})$ be an $n \times p$. We define the product, AB as the $m \times p$ matrix C_{ij} where the ~~i~~ $(ij)^{th}$ entry C_{ij} is given by $C_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$

Example,

Let $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{3\times4}$, $B = \begin{bmatrix} 4 & 1 \\ 1 & 5 \\ 3 & 2 \\ 1 & 0 \end{bmatrix}_{4\times2}$

The ~~no~~ number of rows of A is equal to the number of columns in the B, therefore it can be multiplied and the order of the result matrix is 3×2.

$$AB = \begin{bmatrix} 1+0+6+3 & 1+0+4+0 \\ 0+2+3+1 & 0+10+2+0 \\ 1+0+0+1 & 1+0+0+0 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 6 & 12 \\ 2 & 1 \end{bmatrix}_{3\times2}$$

INVERSE OF THE MATRIX $[A^{-1}]$

A 2×2 matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse, if and only if, $|A| = ad - bc \neq 0$ and the inverse of A is given by $\frac{1}{|A|}(\text{adj } A) \Rightarrow \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example,

$A = \begin{pmatrix} 10 & 2 \\ 4 & 1 \end{pmatrix}$, find A^{-1}.

$A^{-1} = \frac{1}{|A|} \text{adj } A$

$|A| = 10 - 8 = 2 \neq 0$

$\text{adj } A = \begin{bmatrix} 1 & -2 \\ -4 & 10 \end{bmatrix}$

$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -2 \\ -4 & 10 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & -2/2 \\ -4/2 & 10/2 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1/2 & -1 \\ -2 & 5 \end{bmatrix}$

SINGULAR AND NON-SINGULAR MATRIX.

A square matrix 'A' is said to be singular, if $|A| = 0$.

'A' is called a non-singular matrix if $|A| \neq 0$.

Example,

Compute the inverse of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{vmatrix} = 2\left[6(2) - (-5)(-2)\right] - (-1)\left[-15(2) - 5(-5)\right] + 1\left[-15(-2) - 6(5)\right]$$

$$= 2\left[12 - (10)\right] + (-30 + 25) + (30 - 30)$$

$$= 2(2) - 5 = 4 - 5$$

$|A| = -1 \neq 0$. $\therefore A^{-1}$ exists.

$$\operatorname{adj} A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} 12-10 & -(-2+2) & (5-6) \\ -(-30+25) & (4-5) & -(-10+15) \\ 30-30 & -(-4+5) & 12-15 \end{bmatrix}$$

$$a_{11} = \left[6(2) - (-5)(-2)\right]$$

$$= \begin{bmatrix} 2 & 0 & -1 \\ 5 & -1 & -5 \\ 0 & -1 & -3 \end{bmatrix}$$

$$a_{21} = -\left[-15(2) - 5(-5)\right]$$

a_{21}

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 2 & 0 & -1 \\ 5 & -1 & -5 \\ 0 & -1 & -3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ -5 & 1 & 5 \\ 0 & 1 & 3 \end{bmatrix}$$

~~TRANSVERSE~~ TRANSPOSE OF MATRIX:

Let $A = (a_{ij})$ be an $m \times n$ matrix, then the $n \times m$ matrix $B = (b_{ij})$ where $b_{ij} = a_{ji}$ is called the ~~transpose~~ transpose of the matrix A and it is denoted by A^T.

Example

let $A = \begin{bmatrix} 1 & 5 & 2 & 3 \\ 2 & 5 & 0 & -10 \\ 1 & 5 & 12 & 8 \end{bmatrix}$ find A^T.

$$A^T = \begin{bmatrix} 1 & 2 & 1 \\ 5 & 5 & 5 \\ 2 & 0 & 12 \\ 3 & -10 & 8 \end{bmatrix}$$

[Note $\Rightarrow (A^T)^T = A$]

Example 2.

Let $A = \begin{bmatrix} 3 & 4 & 6 \\ -1 & 7 & 2 \\ 4 & 3 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 0 & 0 \\ 3 & 4 & 1 \end{bmatrix}$

Find, A^T, B^T, $(A+B)^T$, $(AB)^T$, $B^T A^T$.

Solution

(i) $A^T = \begin{bmatrix} 3 & -1 & 4 \\ 4 & 7 & 3 \\ 6 & 2 & 0 \end{bmatrix}$

(ii) $B^T = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 0 & 4 \\ 2 & 0 & 1 \end{bmatrix}$

(iii) $(A+B)^T$

$A + B = \begin{bmatrix} 3 & 5 & 8 \\ -3 & 7 & 2 \\ 7 & 7 & 1 \end{bmatrix}$ $(A+B)^T = \begin{bmatrix} 3 & -3 & 7 \\ 5 & 7 & 7 \\ 8 & 2 & 1 \end{bmatrix}$

(iv) $(AB)^T$

$$AB = \begin{bmatrix} 0-8+18 & 3+0+24 & 6+0+6 \\ 0-14+6 & -1+0+8 & -2+0+2 \\ 0-6+0 & 4+0+0 & 8+0+0 \end{bmatrix} = \begin{bmatrix} 10 & 27 & 12 \\ -8 & 7 & 0 \\ -6 & 4 & 8 \end{bmatrix}^T$$

$$(AB)^T = \begin{bmatrix} 10 & -8 & -6 \\ 27 & 7 & 4 \\ 12 & 0 & 8 \end{bmatrix}$$

(v): $B^T A^T$

$$B^T = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 0 & 4 \\ 2 & 0 & 1 \end{bmatrix}; \quad A^T = \begin{bmatrix} 3 & -1 & 4 \\ 4 & 7 & 3 \\ 6 & 2 & 0 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 0-8+18 & 0-14+6 & 0-6+0 \\ 3+0+24 & -1+0+8 & 4+0+0 \\ 6+0+6 & -2+0+2 & 8+0+0 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 10 & -8 & -6 \\ 27 & 7 & 4 \\ 12 & 0 & 8 \end{bmatrix}$$

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VECTOR SPACE

A non-~~ent~~ empty set V is said to be a vector space over a field F. If,

① V is an abelian group under an operation called addition which we denoted by +. [a+b = b+a]

② For every ~~alpha~~ $\alpha \in F$ and $v \in V$ there is defined an element αv in V, subject to the following condition,

(a) $\alpha(u+v) = \alpha u + \alpha v$, for all $u, v \in V$ & $\alpha \in F$

(b) $(\alpha+\beta)v = \alpha v + \beta v$, for all $v \in V$ & $\alpha, \beta \in F$

(c) $\alpha(\beta u) = \alpha\beta(u)$, for all $\alpha, \beta \in F$ & $u \in V$

(d) $1u = u$

THEOREM

Let V be a vector space over a field F then,

① $\alpha 0 = 0$ for all $\alpha \in F$

② $0v = 0$ for all $v \in V$

③ $(-\alpha)v = \alpha(-v) = -\alpha v$ for all $\alpha \in F$ & $v \in V$

④ $\alpha v = 0 \Rightarrow \alpha = 0$ or $v = 0$

Solution

① $\alpha 0 = 0$ for all $\alpha \in F$

Consider,

$$\alpha 0 = \alpha(0+0)$$

$$\alpha 0 = \alpha 0 + \alpha 0$$

$$\alpha 0 - \alpha 0 = \alpha 0$$

$$0 = \alpha 0$$

$$\therefore \alpha 0 = 0 \ ; \alpha \in F$$

② $0v = 0$

$$0v = (0+0)v$$

$$0v = 0v + 0v$$

$$0v - 0v = 0v$$

$$\therefore 0v = 0$$

③ Consider, $0 = [\alpha + (-\alpha)]v$

$$= \alpha v + ((-\alpha)v)$$

$$-\alpha v = (-\alpha)v \quad -①$$

Similarly,

$$-\alpha v = (-\alpha)v = \alpha(-v) \quad -②$$

④ $\alpha v = 0$

Let $\alpha = 0$,

$\alpha v = 0$ is nothing prove. $-①$

If $\alpha \neq 0$, then,

α^{-1} exists.

$$v = (\alpha\alpha^{-1})v$$

$$v = \alpha v \alpha^{-1}$$

$$= (\alpha v)\alpha^{-1} \text{ from ①}$$

$$= 0(\alpha^{-1})$$

$$v = 0.$$

DEFINITION

Let V be a vector space over a field F. A non-empty subset W(V) is called a ~~surface~~ subspace of V. If W itself is a vector space over F under the operations of V.

Note:

① Let V be a vector space over F. A non-empty subset W of V is a subspace, if and only iff if W is closed with respect to vector addition $(\alpha + v \in W)$ and scalar multiplication $(\alpha u \in W)$ in V.

② Let V be a vector space over a field F. A non-empty subset W of V is subspace, if and only if $u, v \in W$ and $\alpha, \beta \in F$ that implies $\alpha u + \beta v \in W$

DIRECT SUM:

Let A and B be ~~surfa~~ subspaces of a vector space V. Then V is called the direct sum of A and B, if,

① $A + B = V$

② $A \cap B = \{0\}$ element should 0

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SPAN

Let V be a vector space over a field F, let $v_1, v_2, \ldots, v_n$ belongs to V. then an element of the form $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ where $\alpha_i \in F$, this called a linear combination of the vectors $v_1, v_2, \ldots, v_n$.

Let S be a non-empty subset of a vector space V. Then the set of all linear combinations of finite sets of elements of S is called the Linear span of S and is denoted by L(S).

Let V be a vector space over a field F. A finite set of vectors $v_1, v_2, \dots, v_n$ in V is said to be linearly independent if $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$ that implies $\alpha_1 = \alpha_2 = \alpha_3 = \dots = \alpha_n = 0$.

if $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \neq 0$, therefore it said to be linear dependent.

A linearly independent subset S of a vector space V which spans the whole space V is called a bases of a vector space.

Let V be a finite dimensional vector space over F. The number of elements in any bases of V is called the dimensions of V, and is denoted by 'dim V'.

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UNIT - IV

EIGEN VALUES AND EIGEN VECTORS.

(1) Find the eigen values and eigen vector of a matric.

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution Method - 1

The characteristic equation, $\lambda^2 - S_1\lambda + S_2 = 0$.

S_1 - Sum of the main diagonal ; S_2 - $|A|$

$S_1 = 2+3 = 5$ $\quad S_2 = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 6 - 2 = 4$

$\lambda^2 - 5\lambda + 4 = 0$

$(\lambda - 4)(\lambda - 1) = 0$ $\quad$ 4: $-4, -1$; -5

$\lambda - 4 = 0$ $\quad \lambda - 1 = 0$

$\lambda = 4$ $\quad \lambda = 1$ $\Rightarrow$ Eigen Values.

Method - 2

$|A - \lambda I| = 0$

$$\left| \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(3-\lambda) - 2 = 0 \Rightarrow 6 - 2\lambda - 3\lambda + \lambda^2 - 2 = 0$

$\lambda^2 - 5\lambda + 4 = 0$

$\lambda = 4, \lambda = 1.$

Eigen vectors.

$$(A - \lambda I)X = 0$$

$$\left[\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

if $\lambda = 1$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \begin{array}{l} x_1 + 2x_2 = 0, \\ x_1 + 2x_2 = 0 \end{array}$$

Consider,

$$x_1 + 2x_2 = 0$$

$$1x_1 = -2x_2$$

$$\frac{x_1}{-2} = \frac{x_2}{1} \Rightarrow \frac{x_1}{2} = \frac{x_2}{-1}$$

$\therefore$ Eigen vector, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

if $\lambda = 4$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$-2x_1 + 2x_2 = 0 \quad ; \quad x_1 - x_2 = 0.$$

Consider, $x_1 - x_2 = 0$

$$1x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{1}$$

$\therefore$ Eigen ~~value~~ vectors, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

2) Find the eigen values and eigen vectors of the matrix,

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Solution

The characteristic equation, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 2 + 3 + 2 = 7$

$$S_2 = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$$

$= (6-2) + (4-1) + (6-2)$

$= 4 + 3 + 4$

$S_2 = 11$

$$S_3 = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 2(6-2) - 2(2-1) + 1(2-3)$$

$= 2(4) - 2(1) + 1(-1) = 8 - 2 - 1$

$S_3 = 5$

$\therefore \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0.$

$$\begin{array}{r|rrrr} \lambda = 1 & 1 & -7 & 11 & -5 \\ & 0 & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & \boxed{0} \end{array}$$

$\lambda^2 - 6\lambda + 5 = 0$

$(\lambda - 1)(\lambda - 5) = 0$

$\therefore$ Eigen values ; $\lambda = 1$, $\lambda = 5$, $\lambda = 1$.

Eigen vectors,

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

If $\lambda = 1$,

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + 2x_2 + x_3 = 0$; $x_1 + 2x_2 + x_3 = 0$; $x_1 + 2x_2 + x_3 = 0$.

x_1		x_2		x_3	
2	1		1	2	
2	1		1	2	

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{0}$$ $\therefore$ Eigen vectors, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

If $\lambda = 5$,

$$\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-3x_1 + 2x_2 + x_3 = 0$; $x_1 - 2x_2 + x_3 = 0$; $x_1 + 2x_2 - 3x_3 = 0$

x_1		x_2		x_3	
2	1		-3	2	
-2	1		1	-2	

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$\times 4 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

$\therefore$ Eigen vector, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

H.W

(3) $A = \begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}$ find : eigen values and vectors.

Solution.

Characteristic equation, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

$S_1 = 3$

$S_2 = \begin{vmatrix} -2 & -5 \\ -4 & -6 \end{vmatrix} + \begin{vmatrix} 11 & -7 \\ 10 & -6 \end{vmatrix} + \begin{vmatrix} 11 & -4 \\ 7 & -2 \end{vmatrix}$

$= (12 - 20) + (-66 + 70) + (-22 + 28)$

$= -8 + 4 + 6$

$S_2 = 2$

$S_3 = 11(12 - 20) - (-4)(-42 + 50) + (-7)(-28 + 20)$

$= 11(-8) + 4(8) - 7(-8)$

$= -88 + 32 + 56 = -88 + 88$

$S_3 = 0$

$\lambda^3 - 3\lambda^2 + 2\lambda - 0 = 0$.

1	1	-3	2	0
	0	1	-2	0
	1	-2	0	0

$\lambda = 1$.

$\lambda^2 - 2\lambda + 0 = 0$.

$\lambda(\lambda - 2) = 0$

$\lambda = 0 \quad \lambda - 2 = 0$

$\lambda = 2$

$\therefore$ Eigen values, $\lambda = 0, \lambda = 1, \lambda = 2$.

Eigen vectors

$$(A - \lambda I)x = 0$$

$$\left[\begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{bmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

if $\lambda = 0$,

$$\begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$11x_1 - 4x_2 - 7x_3 = 0$; $7x_1 - 2x_2 - 5x_3 = 0$; $10x_1 - 4x_2 - 6x_3 = 0$.

$$\begin{array}{ccccc} & x_1 & & x_2 & & x_3 \\ -4 & & -7 & & 11 & & -4 \\ -2 & & -5 & & 7 & & -2 \end{array}$$

$$\frac{x_1}{6} = \frac{x_2}{6} = \frac{x_3}{6}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} \quad \therefore \text{ Eigen vectors } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

if $\lambda = 1$

$$\begin{pmatrix} 10 & -4 & -7 \\ 7 & -3 & -5 \\ 10 & -4 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$10x_1 - 4x_2 - 7x_3 = 0$; $7x_1 - 3x_2 - 5x_3 = 0$; $10x_1 - 4x_2 - 7x_3 = 0$.

$$\begin{array}{ccccc} & x_1 & & x_2 & & x_3 \\ -4 & & -7 & & 10 & & -4 \\ -3 & & -5 & & 7 & & -3 \end{array}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-2} \Rightarrow (-)\times \Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{2}$$

Eigen vectors $\begin{pmatrix}1\\-1\\2\end{pmatrix}$

If $\lambda = 2$

$$\begin{pmatrix}9 & -4 & -7\\7 & -4 & -5\\10 & -4 & -8\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 0$$

$9x_1 - 4x_2 - 7x_3 = 0$; $7x_1 - 4x_2 - 5x_3 = 0$; $10x_1 - 4x_2 - 8x_3 = 0$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -4 & -7 & 9 & -4 \\ -4 & -5 & 7 & -4 \end{array}$$

$$\frac{x_1}{-8} = \frac{x_2}{-4} = \frac{x_3}{-8} \Rightarrow \times(-4)$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{2}$$

Eigen vectors $\begin{pmatrix}2\\1\\2\end{pmatrix}$

1/10/24

UNIT - 3

GAUSSIAN ELIMINATION METHOD

1) Solve the equation using Gaussian Elimination method.

$$x + 2y + 3z = 6$$
$$2x + 4y + z = 7$$
$$3x + 2y + 9z = 14$$

Solution

from given,

$$A \quad X = B$$

$$\begin{pmatrix}1 & 2 & 3\\2 & 4 & 1\\3 & 2 & 9\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}6\\7\\14\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3 & 6\\2 & 4 & 1 & 7\\3 & 2 & 9 & 14\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Eigen vectors $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

If $\lambda = 2$

$$\begin{pmatrix} 9 & -4 & -7 \\ 7 & -4 & -5 \\ 10 & -4 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$9x_1 - 4x_2 - 7x_3 = 0$; $7x_1 - 4x_2 - 5x_3 = 0$; $10x_1 - 4x_2 - 8x_3 = 0$

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ -4 & & -7 & & 9 & & -4 \\ -4 & \times & -5 & \times & 7 & \times & -4 \end{array}$$

$$\frac{x_1}{-8} = \frac{x_2}{-4} = \frac{x_3}{-8} \Rightarrow \times (-4)$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{2}$$

Eigen vectors $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

1/10/24.

UNIT - 3

GAUSSIAN ELIMINATION METHOD

1) Solve the equation using Gaussian Elimination method.

$$x + 2y + 3z = 6$$
$$2x + 4y + z = 7$$
$$3x + 2y + 9z = 14$$

Solution

from given, $A \quad X = B$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 14 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 2 & 4 & 1 & 7 \\ 3 & 2 & 9 & 14 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & -5 & -5 \\ 0 & -4 & 0 & -4 \end{pmatrix}$$

$$\begin{array}{lrrrr} R_2 \rightarrow & 2 & 4 & 1 & 7 \\ 2\times R_1 \rightarrow & \underset{(-)}{2} & \underset{(-)}{4} & \underset{(-)}{6} & \underset{(-)}{12} \\ \hline & 0 & 0 & -5 & -5 \end{array}$$

$$\begin{array}{lrrrr} R_3 \rightarrow & 3 & 2 & 9 & 14 \\ 3\times R_1 \rightarrow & \underset{(-)}{3} & \underset{(-)}{6} & \underset{(-)}{9} & \underset{(-)}{18} \\ \hline & 0 & -4 & 0 & -4 \end{array}$$

$R_2 \rightleftarrows R_3$

$$= \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & -4 & 0 & -4 \\ 0 & 0 & -5 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$x + 2y + 3z = 6 \quad -①$

$-4y = -4$

$-5z = -5$

$\therefore z = 1, \ y = 1$ sub in ①

$x + 2(1) + 3(1) = 6$

$x + 2 + 3 = 6$

$x + 5 = 6$

$\boxed{x = 1}$.

$\therefore x = 1, \ y = 1, \ z = 1$.

---

2) Solve using Gaussian Elimination method.

$x + y + z = 9, \ 2x - 3y + 4z = 13; \ 3x + 4y + 5z = 40$.

Solution

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right)$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{pmatrix}$$

$$\begin{array}{lrrrr} R_2 \rightarrow & 2 & -3 & 4 & 13 \\ 2\times R_1 \rightarrow & \underset{(-)}{2} & \underset{(-)}{2} & \underset{(-)}{2} & \underset{(-)}{18} \\ \hline & 0 & -5 & 2 & -5 \end{array}$$

$$\begin{array}{lrrrr} R_3 \rightarrow & 3 & 4 & 5 & 40 \\ 3\times R_1 \rightarrow & \underset{(-)}{3} & \underset{(-)}{3} & \underset{(-)}{3} & \underset{(-)}{27} \\ \hline & 0 & 1 & 2 & 13 \end{array}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 12 & 60 \end{pmatrix}$$

$$\begin{array}{lrrrr} R_2 \rightarrow & 0 & -5 & 2 & -5 \\ 5\times R_3 \rightarrow & \underset{(+)}{0} & \underset{(+)}{5} & 10 & 65 \\ \hline & 0 & 0 & 12 & 60 \end{array}$$

$x + y + z = 9$

$-5y + 2z = -5$

$12z = 60$

$z = 60/12$

$\boxed{z = 5}$

$-5y + 2(5) = -5$

$-5y = -5 - 10$

$y = -15/-5$

$\boxed{y = 3}$

$x + 3 + 5 = 9$

$x = 9 - 8$

$\boxed{x = 1}$

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(3) Solve the equation using Gaussian Elimination Method.

$x + y + z = 9;\quad 2x + 5y + 7z = 52;\quad 2x + y - z = 0.$

Solution

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{array}\right)$$

$$\begin{array}{lcccc} R_2 \to & 2 & 5 & 7 & 52 \\ 2 \times R_1 \to & 2 & 2 & 2 & 18 \\ & (-) & (-) & (-) & (-) \\ \hline & 0 & 3 & 5 & 34 \end{array}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -1 & -3 & -18 \end{pmatrix}$$

$$\begin{array}{lcccc} R_3 \to & 2 & 1 & -1 & 0 \\ 2 \times R_1 \to & 2 & 2 & 2 & 18 \\ & (-) & (-) & (-) & (-) \\ \hline & 0 & -1 & -3 & -18 \end{array}$$

$34 \times 3 = 102$

$18 \times 3 = 54$; $54 - 34 = 20$

$$= \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & -4 & -20 \end{pmatrix}$$

$$\begin{array}{lcccc} \cancel{R_3 \to} & \cancel{0} & \cancel{-1} & \cancel{-3} & \cancel{-18} \\ \cancel{3 \times R_2 \to} & \cancel{0} & \cancel{3} & \cancel{15} & \cancel{102} \end{array}$$

$$\begin{array}{lcccc} R_2 \to & 0 & 3 & 5 & 34 \\ 3 \times R_3 \to & 0 & -3 & -9 & -54 \\ \hline & 0 & 0 & -4 & -20 \end{array}$$

$x + y + z = 9$

$3y + 5z = 34$

$-4z = -20$

$\boxed{z = 5}$

$3y + 5(5) = 34$

$\begin{array}{r} \overset{2}{\cancel{3}}\overset{1}{4} \\ 25 \\ \hline 9 \end{array}$

$3y + 25 = 34$

$3y = 34 - 25 \Rightarrow 9$

$\boxed{y = 3}$

$x + 3 + 5 = 9$

$x + 8 = 9$

$\boxed{x = 1}$