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Programme : M.Sc. Mathematics
Course Title : Complex Analysis
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UNIT 2

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$f : U \to \mathbb{C}$ $\qquad U \overset{\text{open}}{\subseteq} \mathbb{C}$

$a \in U$

f is diff at a if $\forall \epsilon > 0 \left(\exists \delta > 0, \exists m \in \mathbb{C} \left(\forall z \in U \text{ with } 0 < |z-a| < \delta \right.\right.$

$\left.\left. \left(|f(z) - f(a) - m(z-a)| < \epsilon |z-a| \right) \right) \right)$

m is called as $f'(a)$

Thm $f : U \to \mathbb{C}$, $a \in U$ (U → open subset of $\mathbb{C}$)

f is diff at a iff $\exists$ fn $g : U \to \mathbb{C}$ s.t.

(i) $\forall z \in U$, $f(z) = f(a) + (z-a) g(z)$

(ii) g is conts at a

In this case $f'(a) = g(a)$ $\qquad \Big| \lim_{z \to a} \dfrac{f(z) - f(a)}{z - a} = f'(a)$

Proof ⟹

Suppose f is diff at a

Define $g : U \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & z \neq a \\ f'(a) & z = a \end{cases}$$

Now g is well defined.

& $\lim_{z \to a} g(z) = \lim_{z \to a} \dfrac{f(z) - f(a)}{z - a} = f'(a) = g(a)$

$\therefore$ g is conts at a.

Converse is an Exercise

A Thm or Boon '21310'

(i) $U = \mathbb{C}$, Fix $z_0 \in \mathbb{C}$

$f : U \to \mathbb{C}$

$f(z) = z_0$

Let $a \in U$

$$f(z) - f(a) = z_0 - z_0 = 0$$

$g : U \to \mathbb{C}$

$g(z) = \underline{0}$ $\qquad g(a) = 0 = f'(a)$

(ii) $f(z) = \alpha z$ (where $\alpha \in \mathbb{C}$ is fixed)

$$f(z) - f(a) = \alpha z - \alpha a$$
$$= \alpha(z - a)$$

$g : U \to \mathbb{C}$

$g(z) = \underline{\alpha}$

(ii) $f(z) = z^n$

$$f(z) - f(a) = z^n - a^n$$
$$= (z-a)\underbrace{(z^{n-1} + z^{n-2}a + \cdots + a^{n-1})}_{g(z)}$$

g is conts at a

$\underline{g(a)} = n a^{n-1} = f'(a)$

$e^{\pi} - e^{a}$

(iii) $f(z) = \exp(z)$

$$f(z) - f(a) = \exp(z) - \exp(a)$$

?

Ex// (iv) $f(z) = a^z$

$\underline{f'(z)} = a^z \log a$

$f(x) = a^x$

$f'(x) = a^x \underline{\log a}$

$\ln f(x) = x \ln g(x)$

$\frac{f'(x)}{f(x)} = \ln a$

$f'(x) = a^x \ln a$

Thm $f(z) = \sum a_n z^n$, $z \in B(0,R)$

Then f is diff at z & $f'(z) = \sum_n n a_n z^{n-1}$ on $B(0,R)$

$r < R$ $|z+h| \le |z| + |h| < r$

$\therefore |h| < r - |z|$

Proof

$z+h \in B(0,r)$

$$f(z+h) - f(z) = \sum_{n=1}^{\infty} a_n \left[(z+h)^n - z^n \right]$$

$$= \sum_{n=1}^{\infty} a_n \left[h \left((z+h)^{n-1} + (z+h)^{n-2} z + \dots + z^{n-1} \right) \right]$$

$$\frac{1}{h}\left(f(z+h) - f(z) \right) = \sum_{n=1}^{\infty} g_n(h)$$

$$g_n(h) = \begin{cases} a_n \left[(z+h)^{n-1} + (z+h)^{n-2} z + \dots + z^{n-1} \right], & h \neq 0 \\ n a_n z^{n-1}, & h = 0 \end{cases}$$

$$\lim_{h \to 0} g_n(h) = n a_n z^{n-1} \qquad |z| < r < R$$

g_n is conts at 0

$\therefore \forall h,$ $|h| < r - |z|$ $|g(h)| \le n |a_n| r^{n-1}$

$$h \neq 0, \ |g_n(h)| \le |a_n| \left[|z+h|^{n-1} + |z+h|^{n-2} |z| + \dots + |z|^{n-1} \right]$$

$$\le |a_n| \left[r^{n-1} + r^{n-2} r + \dots + r^{n-1} \right]$$

$$= |a_n| \, n r^{n-1}$$

$|z| < r$

$\therefore |g_n(h)| \le n |a_n| r^{n-1}$, $\forall h \in \mathbb{C}$, $|h| < r - |z|$

$$\phi(h) = \sum_{n=0}^{\infty} g_n(h)$$

$0 < r < 1$

(i) $r^n \to 0$

(ii) $n r^n \to 0$ (?)

$$\frac{1}{r} > 1$$
$$\frac{1}{r} = 1 + h$$
$$\frac{1}{r^n} = (1+h)^n > nh$$

$$0 < r^n < \frac{1}{nh} \to 0$$

$$\frac{1}{r^n} = (1+h)^n > {}^nC_2 h^2$$

$$0 < n r^n < \frac{1}{(n-1)h^2} \to 0$$

$n r^n \to 0$

$$|z| < r < s < R$$

$$\frac{r}{s} < 1$$

$$0 < t = \frac{r}{s} < 1$$

$n t^n \to 0$

$\therefore \exists N \in \mathbb{N}, \ \forall n \geq N, \quad n\left(\frac{r}{s}\right)^n < \epsilon = r$

$$n \frac{r^{n-1}}{s^n} < 1$$

$$n r^{n-1} < s^n$$

$\forall n \geq N$

$$|g_n(h)| \leq |a_n| \, n \, r^{n-1} \leq |a_n| \, s^n$$

By W-M-test

$\therefore \sum_{n \geq N} g_n(h)$ is uniformly conv

$\boxed{\phi(h) = \sum g_n(h)}$ is uniform conv

$\therefore \phi$ is conts. [Domain of ϕ: $D = \{h \in \mathbb{C} : |h| < r - |z|\}$]

$$\frac{1}{h}\left[f(z+h) - f(z)\right] = \phi(h) \quad \& \ \phi \text{ is conts at } h = 0$$

$$\lim_{h \to 0} \frac{1}{h}\left[f(z+h) - f(z)\right] = \lim_{h \to 0} \phi(h) = \phi(0)$$

by Cont

$$f'(z) = \phi(0) = \sum_n g_n(0) = \sum_n n a_n z^{n-1}$$

If you understand answer the following question

$\sum_{n=0}^{\infty} a_n z^n$ has R as radius

then $\sum_{n=1} n a_n z^{n-1}$ has R as radius

[Use the above proof $\quad |z| < r$

$$n|a_n| |z|^{n-1} < |a_n| n r^{n-1}$$

$$\leq |a_n| s^n \quad]$$

App

1) $f(z) = \exp(z)$

P.T $f'(z) = \exp(z)$

$$f(z) = \sum_n \frac{z^n}{n!}$$

$$f'(z) = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \exp(z) = f(z)$$

Exer

1. $f(z) = \exp(\alpha z)$

P.T $f'(z) = \alpha \exp(\alpha z)$

$$f : B(0, R) \to \mathbb{C}$$

$$f(z) = a_0 + a_1 z + \cdots$$

$$a_0 = f(0)$$

$$a_1 = f'(0)$$

$$a_2 = \frac{f^{(2)}(0)}{2}$$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$$f'(z) = a_1 + 2a_2 z^2 + \cdot$$

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) z^{n-k}$$

Complex Analysis, Sep 4, 2020

Ex 1. $f : U \to \mathbb{R}$ & f diff on U Then C-R equations ($f = u + iv$) $u_x = v_y$ & $u_y = -v_x$ are satis

Is the converse true?

ANS: FALSE

Counter Example $f(x+iy) = \sqrt{|xy|}$

f sat C-R equ at $z = 0$

f is not diff at $z = 0$

Ex 2. When the converse is true

State as a thm and prove it.

{idea If $f = u + iv$ is s.t u_x & u_y, v_x v_y are conts on U & f satisfies C-R eq. then f is diff on U]

(Geometric meaning of C-R equations will be given as a V. L.)

Integration:

$f : [a,b] \to \mathbb{C}$ $\qquad f = u + iv$

? $\int_a^b f(t)\,dt$ $\qquad$ $f(t) = \mathrm{Re} f(t) + i \,\mathrm{Im} f(t) = u(t) + i v(t)$

$$\int_a^b f(t)\,dt := \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt$$

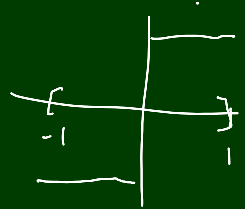
$f \in R[a,b] = \{ f : [a,b] \to \mathbb{C} : f \text{ is integrable} \}$

When $f \in R[a,b]$?

$u, v : [a,b] \to \mathbb{R}$

$u, v \in R[a,b]$ ✓

$u(t) = \begin{cases} -1 & \text{if } t \le 0 \\ 1 & \text{if } t > 0 \end{cases}$



1. Every monotonic fn on $[a,b]$ is Riemann integr.

2. Every Conts fn is Riemann integr.

$$C[a,b] \subsetneq R[a,b]$$

$$\|f\|_1 = \int_a^b |f(t)|\,dt$$

1) $f : [a,b] \to \mathbb{C}$ is conts

then we define

$$\int_a^b f(t)\,dt := \int_a^b \mathrm{Re} f(t)\,dt + i\int_a^b \mathrm{Im} f(t)\,dt$$

Note that $\because f$ is conts, $\mathrm{Re} f$ & $\mathrm{Im} f$ are conts & hence the Riemann integral exists.

2) $f : [0, 2\pi] \to \mathbb{C}$. Fix $n \in \mathbb{Z}$

$$f(t) = e^{int} = \cos nt + i \sin nt$$

$$g(t) = e^{int}$$
$$g'(t) = in\,e^{int}$$

Case(i)

$n \neq 0$

$$\int_0^{2\pi} f(t)\,dt = \int_0^{2\pi} \cos nt + i\int_0^{2\pi} \sin nt$$

$$= \left.\frac{\sin nt}{n}\right|_0^{2\pi} + i\left(\frac{-\cos nt}{n}\right)_0^{2\pi}$$

$$= \frac{-i}{n}[1-1] = 0$$

$n = 0$

$$\int_0^{2\pi} f(t)\,dt = \int_0^{2\pi} dt = 2\pi$$

2) State Fundamental thm of I.C

Let $f : [a,b] \to \mathbb{C}$ be conts.

$$F(x) = \int_a^x f(t)\,dt$$

Then F is diff &

$$F'(x) = f(x)$$

2.) Let $G : [a,b] \to \mathbb{C}$ be dif fn s.t

$\therefore G'(t) = g(t)$ & G' is conts

Then:
$$\int_a^b g(t)\,dt = G(b) - G(a)$$

Proof

Apply F.T.C real version to Ref & Imf

Again do problem. 1

$$f(t) = e^{int}, \quad n \in \mathbb{Z}$$

Case(i) $n \neq 0$

$$G(t) = \frac{e^{int}}{in}$$

$$G'(t) = e^{int} = f(t)$$

$$\int_0^{2\pi} f(t)\,dt = G(2\pi) - G(0) = 0$$

$$e^{i2n\pi} - e^{0} = 1 - 1 = 0,$$

2)

Let $\lambda = a + ib \in \mathbb{C}^*$

Evaluate $\int_0^t e^{\lambda s}\,ds$ (& separate real & Img)

$G(s) = \frac{e^{\lambda s}}{\lambda}$

$$\int_0^t e^{\lambda s}\,ds = \int_0^t G'(s)\,ds$$

$$= G(t) - G(0)$$

$$= \frac{e^{\lambda t}}{\lambda} - \frac{1}{\lambda}$$

$$\int_0^t e^{(a+ib)s}\,ds = \frac{e^{(a+ib)t}}{\lambda} - \frac{1}{\lambda}$$

$$= \frac{e^{at} \cdot e^{ibt} - 1}{a+ib}$$

$$= \frac{(a - ib)}{a^2 + b^2}\left[e^{at} \cdot e^{ibt} - 1\right]$$

$$= \frac{(a-ib)}{a^2+b^2}\left(e^{at}(\cos bt + i\sin bt) - 1\right)$$

Real part of R.H.S

$$= \frac{(a-ib)}{(a^2+b^2)}\left[(e^{at}\cos bt - 1) + i\, e^{at}\sin bt\right]$$

$$= \frac{e^{at}a\cos bt - a}{a^2+b^2} + \frac{e^{at}\, b\sin bt}{a^2+b^2}$$

$$= \frac{e^{at}(a\cos bt + b\sin bt) - a}{a^2+b^2}$$

$$L.H.S = \int_0^t e^{(a+ib)s}\, ds$$

$$= \int_0^t e^{as}(\cos bs + i\sin bs)\, ds$$

$$= \int_0^t e^{as}\cos bs\, ds + i\int_0^t e^{as}\sin bs\, ds$$

S.T.

$$(a^2+b^2)\int_0^t e^{as}\cos bs\, ds = e^{at}(a\cos bt + b\sin bt) - a$$

Ex. Let $\gamma : [a,b] \to \mathbb{C}$ be conts diff map s.t $\gamma(t) \neq z_0$ $\forall\, t \in [a,b]$

$$\text{Let } g(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds$$

S.T $h(t) = e^{-g(t)}[\gamma(t) - z_0]$ is constant.

$$\therefore \exp g(t) = \frac{\gamma(t) - z_0}{\gamma(a) - z_0} \quad \text{for } t \in [a,b]$$

Why this exc.
Try some examples.

If $\gamma(a) = \gamma(b)$ what can you say about g?.

Look for some examples & do.

We will comback later about g.

Complex Analysis, Sep 1, 2020

Thm Let $f(z) = \sum_n a_n (z - z_0)^n \quad z \in B(z_0, R)$

Then f is analytic on $B(z_0, R)$

In fact For $a \in B(z_0, R)$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (z-a)^n, \quad z \in B(a, R - |a|)$$

Proof $z_0 = 0$

$$\therefore f(z) = \sum_n a_n z^n, \quad z \in B(0, R)$$

Then For any $a \in B(0, R)$

we have to s.t $f(z) = \sum_n \frac{f^{(n)}(a)}{n!} (z-a)^n, \quad z \in B(a, R - |a|)$

That is f admits power series expansion

Recall

$$f(z) = \sum_n a_n z^n, \quad z \in B(0, R)$$

$f' = f, \; f(0) = 1$

Then $a_0 = f(0) = 1$

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$$

$$f(z) = \sum_n \frac{z^n}{n!} \; ; \; z \in B(0, R)$$

$$\therefore g(z) = \exp z \quad \text{on } B(0,R) \quad (\text{why?})$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$f = g$ Because of unique powerseries (Cor.).

$(a+b)^n = a^n$

$$f(z) = \sum_n a_n z^n$$

$$= \sum_n a_n ((z-a) + a)^n$$

$$= \sum_n a_n \left(\sum_{k=0}^{n} {}^nC_k (z-a)^k a^{n-k} \right)$$

$$= \sum_{n=0}^{\infty} a_n \left[\sum_{k=0}^{n} {}^nC_k \; a^{n-k} (z-a)^k \right]$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_n \, {}^nC_k \, a^{n-k} \; (z-a)^k \right)$$

$$f(z) = \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} a_n \, {}^nC_k \, a^{n-k} \right] (z-a)^k \quad (?)$$

$$f(z) = \sum_n a_n z^n$$

$$f^{(k)}(z) = \sum_{n=k}^{\infty} a_n \, n(n-1) \cdots (n-k+1) \cdot z^{n-k}$$

$$(n-k+1) \, a^{n-k}$$

$$f^{(k)}(a) = \sum_{n=k}^{\infty} a_n \, n \cdot (n-1) \cdot$$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^k(a)}{k!} (z-a)^k$$

By the uniqueness of power series

$$\sum_{j=0}^{\infty} \sum_{i=0}^{j} a_{ij}$$

$$a_{00} + (a_{01} + a_{11}) + (a_{02} + a_{12} + a_{22}) + (a_{03} + a_{13} + a_{23} + a_{33})$$

1st

$$(a_{00} + a_{01} + a_{02} + a_{03} + a_{04} + \cdots)$$

$$+ (a_{11} + a_{12} + a_{13} + \cdots)$$

$$+ (a_{22} + a_{23} + \cdots)$$

$$\mathbb{Z}_+ \times \mathbb{Z}_+$$

$$\sum_{i=0}^{\infty} \sum_{j=i}^{\infty} a_{ij}$$

Supp $a : \mathbb{Z}_+ \times \mathbb{Z}_+$ $\quad a(i,j) = a_{ij}$

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) \overset{?}{=} \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right)$$

Thm

& Suppose$_{i \in \mathbb{N}}$ $\sum_{j=1}^{\infty} |a_{ij}| = b_i$

& $\sum b_i$ convergent ✓

Then $\sum_i \sum_j a_{ij} = \sum_j \sum_i b_{ij}$

Do it Exercise

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= \sum_{n=0}^{\infty} a_n (a + z - a)^n$$

$$= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} {}^nC_k (z-a)^k \cdot a^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_n \, {}^nC_k (z-a)^k a^{n-k}|$$

$$\sum_{n=0}^{\infty} |a_n| \left(|(z-a) + (a)| \right)^n$$

$$\leq \sum_{n=0}^{\infty} |a_n| \left(|z-a| + |a| \right)^n$$

$$\sum_{n=0}^{\infty} |a_n| \left(|z-a| + |a| \right)$$

$$|z-a| + |a| < R$$

$$|z-a| < R - |a|$$

Going rather
Tracing back to holomorphic fns.

We prove chain rule

Thm chain-rule

1. Let $\gamma : [a,b] \to U$ is diff and $f : U \to \mathbb{C}$ is diff,

$g = (f \circ \gamma) : [a,b] \to \mathbb{C}$ is diff,

$g'(t) = (f \circ \gamma)'(t) = f'(\gamma(t)) \cdot \gamma'(t)$

2. Let U & V be open sets in $\mathbb{C}$

$f : U \to V$ is holom at a, $g : V \to W$ is holomorphic at $b = f(a)$

Then $g \circ f$ is diff at a &

$(g \circ f)'(a) = g'(b) \cdot f'(a)$,

Give a proof

make-it ready.

Bye

References

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