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Programme : M.Sc. Mathematics
Course Title : Complex Analysis
Course code : 24S3M09CC

UNIT 1

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Complex Analysis, Aug 9, 2021.
Variables

$\mathbb{R}$ is a ring. $\mathbb{R}[x]$ - Set of all poly. with coeff in $\mathbb{R}$

FTA: Any polynomial $f(x) \in \mathbb{R}[x]$ ($\mathbb{C}$) of deg ≥ 1 has a root in $\mathbb{C}$.

Easy proof is done in first course on CA.

$\mathbb{C}$

$z \in \mathbb{C}$, $z = x + iy$ where $x, y \in \mathbb{R}$

$\boxed{i := \sqrt{-1}}$

$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$

$\operatorname{Re} z := x$

$\operatorname{Im} z := y$

$\boxed{i^2 = -1}$

$\mathbb{R} \subseteq \mathbb{C}$

$x_1 + iy_1$

$x_2 + iy_2$

$x_1x_2 + i y_1 y_2$

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$z_1 + z_2 := (x_1 + x_2) + i(y_1 + y_2)$

$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1x_2 - y_1y_2 + i(y_1x_2 + x_1y_2)$.

$(\mathbb{C}, +, \cdot)$ is field

Yes Dictionary.

Is there an order in $\mathbb{C}$?

There is no order $>$ in $\mathbb{C}$ which make $(\mathbb{C}, >)$ a complete ordered field.

Suppose there exist an order $>$

i. 0

By law of trichotomy Exactly one of the
(a) $i = 0$, (b) $i > 0$ (c) $i < 0$

$\because i^2 = -1$

If $i = 0$, $i^2 = 0 \cdot 0 = 0$

$\Rightarrow \Leftarrow$

$\therefore i \neq 0$

$\mathbb{R} \subseteq \mathbb{C}$ ✓

Sup $i > 0$

Then $i > 0$

$\therefore i > 0 \cdot i$

$\therefore i^2 > 0$

$-1 > 0$

Also $1 > 0$

$-1 > 0$

$\therefore 1 > 0$

$-1 + 1 > 0$

$\therefore 0 > 0$

$\Rightarrow \Leftarrow$.

Illy $i < 0$

$-i > 0$

$(-i)\cdot(-i) > 0$

$i^2 > 0$

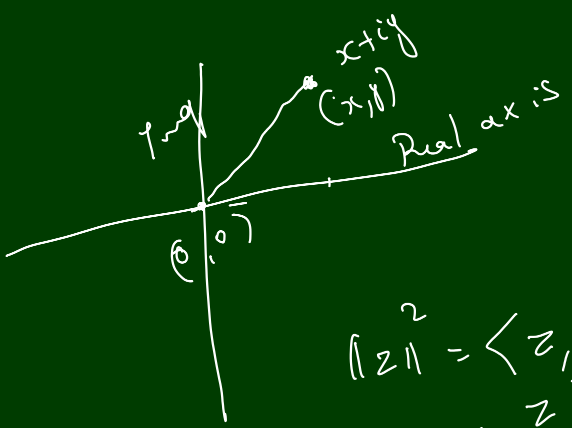
$-1 > 0$

$\Rightarrow \Leftarrow$

$\mathbb{C}$ as a normed space ✓

$(\mathbb{C}, +, \cdot)$

$\|x\| = |x|$



$\|z\| = |z|$

$|z| = \sqrt{x^2+y^2}$ ✓

$\langle z, w \rangle = z\bar{w}$

$\|z\|^2 = \langle z, z \rangle$

$= z\bar{z}$

$= x^2 + y^2$

$= |z|^2$

$\therefore \|z\| = |z|$

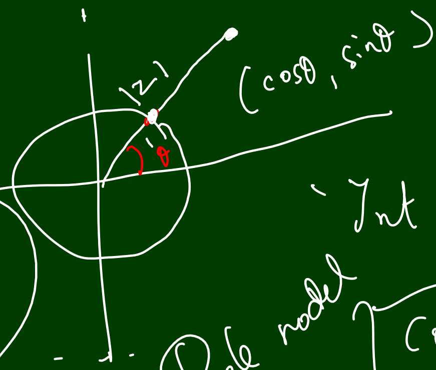
polar form $z = re^{i\theta}$ where $r = |z|$

$\theta \in \mathbb{R}$

θ is called argument of z

$e^{i\theta} := \cos\theta + i\sin\theta$ ✓

$(\cos\theta, \sin\theta)$



Measure Theory

Analysis

'Introduction to Complex Analysis'

Role model

'Complex Analysis'

✓ 1. Analysis ✓

✓ 2. Algebra ✓

✓ 3. Geometry

✓ 4. Topology ✓

✓

Applications

Complex Analysis, August 4, 2020..

Real Analysis

- LUB
- Seq
- Cont | unif. $f : A \to \mathbb{R}$
- Diff
- Rem Int.

Power series
uniform Convg

metric spaces

$\mathbb{C}^2$

Complex Analysis → Variables

$d(z, w) = |z - w|$

Metric spaces
limit pt
cluster point
dense
compact
connected
Pathwise connected

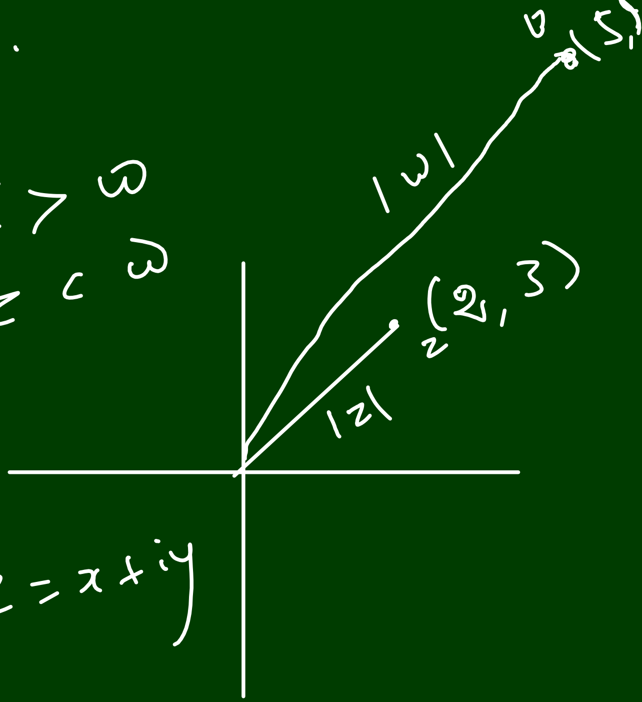
$(\mathbb{C}, +, \cdot)$ is a field.

$z = 2+3i$

$w = 5+4i$

$\cancel{z > w}$

$z < w$



$(\mathbb{C}, d)$ - is m.s.

$(\mathbb{C}, \|\ \|)$ is NLS

$z = x+iy$

$\|z\| = |z|$

$= \sqrt{x^2+y^2}$

$\langle z, w \rangle = z\bar{w}$

$\mathbb{C}$ over $\mathbb{C}$ V.S $\{1\}$ $z = 1 \cdot z$

$\mathbb{C}$ over $\mathbb{R}$. V.S

$\cdot : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$

$z = x+iy$ $\{1, i\}$

$x+iy = 0+i0$

$x = 0, y = 0$

Complex Analysis

1. Analysis
2. Algebra → Pr → Linear
3. Geometry
4. Topology

Role model for Higher Courses.

$\mathbb{C}$ is not an ordered field

Why?

Suppose there is order $>$ s.t $(\mathbb{C}, >)$ is ordered field.

Exactly one holds

$i > 0$, $\cancel{i = 0}$, $i < 0$

$i = (0,1)$

$0 = (0,0)$

$z = x + iy$

$w = a + ib$

$z = w$ iff $x = a$ & $y = b$

Suppose $i > 0$ ✗

$i \cdot i > i \cdot 0$ ✓

$i^2 > 0$

$-1 > 0$

$1 > 0$, $1 + (-1) > 0$

$0 > 0$

$\Rightarrow F$

$i < 0$

$-i > 0$

$(-i)^2 > 0$

$i^2 > 0$

$-1 > 0$.

$Z = x + iy$

$x = \operatorname{Re} Z$

$y = \operatorname{Im} Z$

✓ $f_i : \mathbb{C} \to \mathbb{R}$ $\quad 1 \le i \le 2$

$$\left.\begin{aligned} f_1(Z) &= \operatorname{Re}(z) \\ f_2(z) &= \operatorname{Im}(z) \end{aligned}\right\}$$

1) $|z|^2 = x^2 + y^2$

$\bar{z} = x - iy$

$z\bar{z} = x^2 + y^2$

$= |z|$

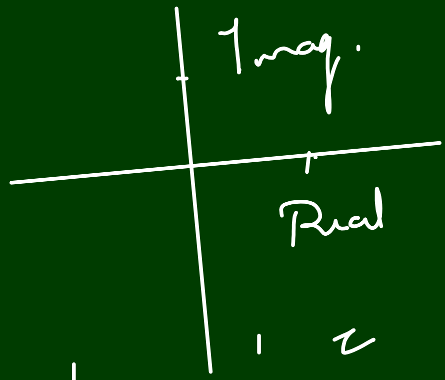
What is the Geometry?

2) ✓ $|z + w| \le |z| + |w|$ When the equality occurs.

3) $||z| - |w|| \le |z - w|$ ✓

$Z = (x, y) \in \mathbb{R}^2$

$Z = x + iy$



$z \neq 0$

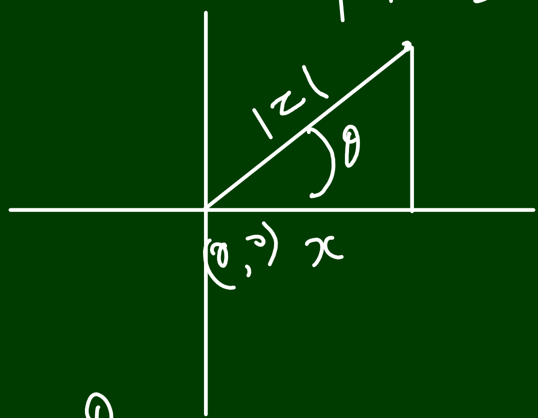
$z \in \mathbb{C}^*$

$Z = x + iy$

$x = |z| \cos\theta$

$y = |z| \sin\theta$

$|z| = r$



$Z = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

Polar form of com.

$Z = r \operatorname{cis} \theta$

$Z = |z| \operatorname{cis} \theta$

θ is called argument of Z.

What is argument of z for any $z \in \mathbb{C}^*$

$A(z) = \{ \theta \in \mathbb{R} : z = |z| e^{i\theta} \}$

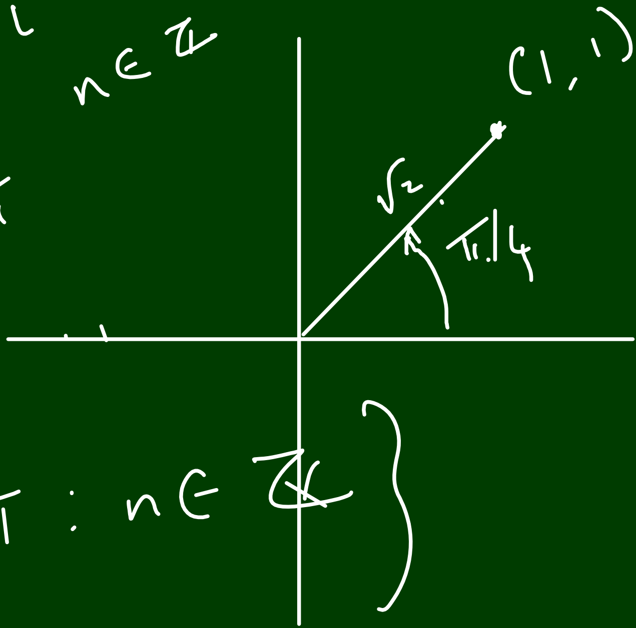
$Arg(z)$

$1+i$

$n \in \mathbb{Z}$

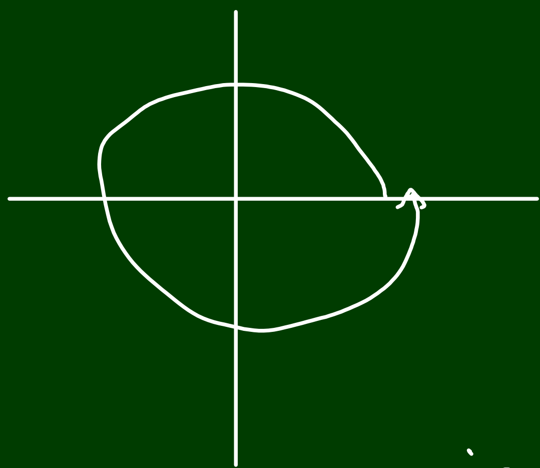
$1+i = \sqrt{2}\, e^{i\pi/4}$

$\arg(1+i) = \pi/4 + 2n\pi$ ✓



$A(z) = \left\{ \pi/4 + 2n\pi : n \in \mathbb{Z} \right\}$

$\arg : \mathbb{C}^* \to [0, 2\pi]$ ✓



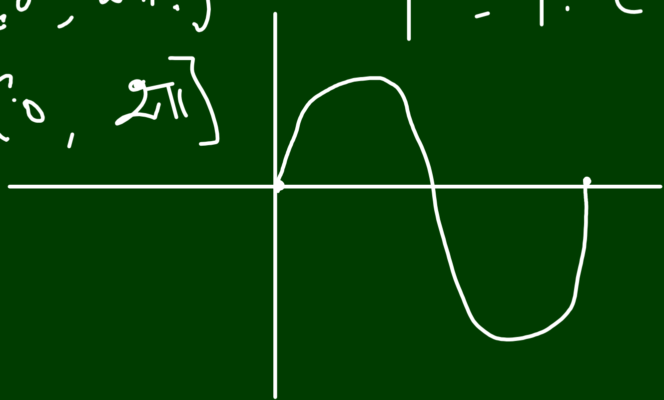
$\arg(1) = \begin{cases} 0 \\ 2\pi \end{cases}$

$1 = 1 \cdot e^{i0}$

$1 = 1 \cdot e^{i2\pi}$

$\arg : \mathbb{C}^* \to [0, 2\pi)$

$(0, 2\pi]$



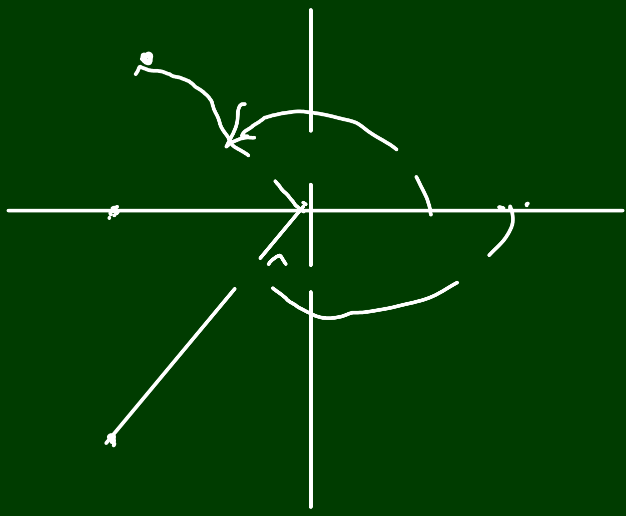
$(-\pi, \pi]$

$\arg : \mathbb{C}^* \to (-\pi, \pi]$

$\arg(-1) = \pi$

$\arg(-\pi) = \pi$

$\arg(1) = 0$



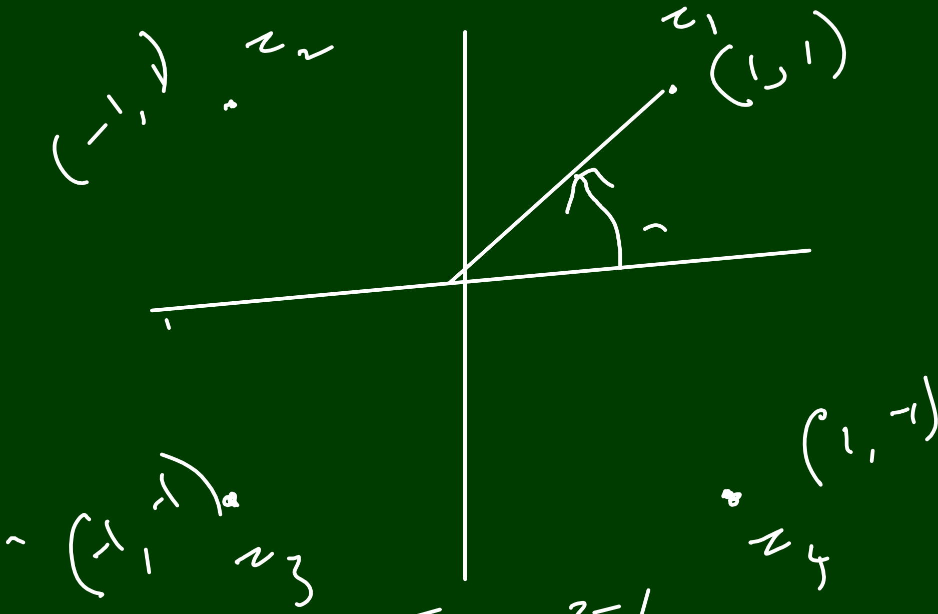
$\arg : \mathbb{C}^* \to (-\pi, \pi]$
$-\pi$

$z_1 = 1+i$

$z_2 = -1+i$

$z_3 = -1-i$

$z_4 = 1-i$



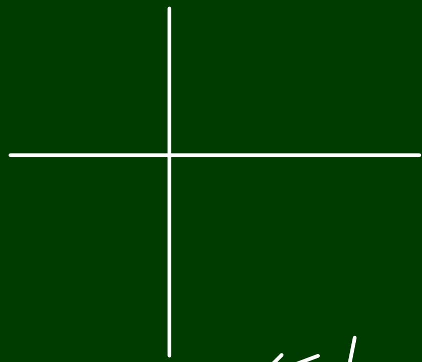
$\pi/4 + \frac{\pi}{2} = 3\pi/4$

z_i	$\arg z_i$
z_1	$\frac{\pi}{4}$
z_2	$\frac{3\pi}{4}$
z_3	$-3\pi/4$
z_4	$-\pi/4$

✓

$\arg_0 : \mathbb{C}^* \to [0, 2\pi)$

z_1	$\pi/4$
z_2	$3\pi/4$
z_3	$5\pi/4$
z_4	$7\pi/4$



$\pi + \frac{\pi}{4} = 5\pi/4$

$2\pi - \pi/4 = 7\pi/4$

Complex Analysis, Aug 10, 2021.

$(F, +, \cdot)$ is a field

F is said to be an ordered field if $\exists$ a subset $P \subseteq F$ s.t

(i) P is closed under $+$

(ii) P is closed under $\cdot$

(iii) $F = P \uplus (-P) \uplus \{0\}$ (law of trichotomy)

Observation

1. $-P := \{-x : x \in P\}$

2. $(\mathbb{R}, +, \cdot)$ is a field

What is P?

$P = \{x \in \mathbb{R} : x > 0\}$

$\therefore \boxed{P = \{x \in F : x > 0\}}$

In any Ordered field $1 \in P$ ✓

(p-Adic world of math)

$1 \in F$

$\therefore 1 \in P \uplus (-P) \uplus \{0\}$

Case (i) $1 \neq 0$ (one of the axiom in any field)

Why required? If $1 = 0$ then $F = \{0\}$

Let $a \in F$

$a \cdot 0 = 0$ (?)

$a \cdot 0 = a \cdot 1 = 0$

$\therefore a = 0$

$\therefore$ ✓

$\mathbb{Z}_2$

If $-1 \in P$ we want a contradiction

Let $-1 \in P$ $(-1) \cdot (-1) \in P$ $\quad -1, 1 \in P$.

$\therefore 1 \in P$ $\quad \therefore 0 \in P$

Take $F = \mathbb{C}$. There is order in $\mathbb{C}$ which makes

W.S.T $\mathbb{C}$ an ordered field.

~~Suppose~~ $\mathbb{C}$ is ordered field

satisf. (i), (ii), (iii) of defn

$\therefore$ By def $\exists\ P \subseteq \mathbb{C}$

$i \in \mathbb{C}$

Also $i \neq 0$

$\therefore\ i \in P$ or $i \in -P$

case (i) $i \in P$

$i^2 \in P$

$-1 \in P$

$(-1)^2 \in P$

$1 \in P$

$-1, 1 \in P$

$\therefore\ 0 \in P$

$-1 > 0$

$(-1)(-1) > (-1) \cdot 0$

$1 > 0$

case (ii) $i \in -P$

$\therefore\ -i \in P$

$(-i^2) \in P$

'$>$'

$x > 0$ Then $x \in P$

Def $x > y$ iff $x - y \in P$

$x > y$, $y > z$

i.e. $x > z$

$x - y > 0$ & $y - z > 0$

$\therefore\ x - y + y - z > 0$

$x - z > 0$

$\therefore\ x > z$

$i \neq 0$

$\therefore\ i > 0$

or $i < 0$

$i > 0$

$i \cdot i > i \cdot 0$

$\therefore\ -1 > 0$

$(-1) \cdot (-1) > (-1) \cdot 0$

$1 > 0$

Let $z, w \in \mathbb{C}$

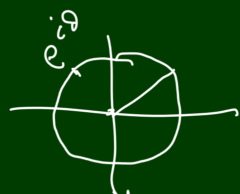
statement: If $|z| > |w|$ then $z^2 + w^2 = (w - z)(\_\_\_\_\_\_)$

If $z > w$

There is a total order on $\mathbb{C}$ {a partial order in which any 2 elem are comparable}

Revise from topology course — Dictionary order.

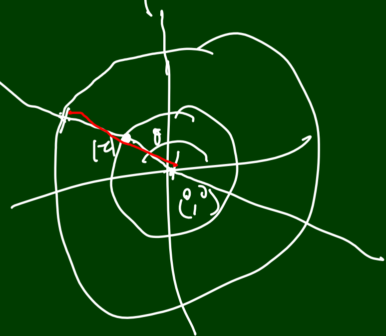
Argument of a Complex number



$$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

$$z = re^{i\theta} = |z| e^{i\theta}$$

$$z = e^{i\theta} = \cos\theta + i\sin\theta \quad (\cos\theta, \sin\theta)$$



Thm. Given $z \in \mathbb{C}^*$, $\exists$ unique $\theta \in [0, 2\pi)$ st $z = |z| e^{i\theta}$.

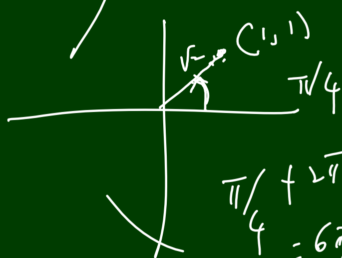
$\therefore$ Def $f : \mathbb{C}^* \to [0, 2\pi)$

$f(z) = \theta$ $\qquad \theta : \mathbb{C}^* \to [0, 2\pi)$

$\therefore$ argument ~~is~~ fn $\quad \theta(z) = \theta$

$\arg : \mathbb{C}^* \to [0, 2\pi)$

1+i



Let $z \in \mathbb{C}^*$

$$A(z) = \{ t \in \mathbb{R} : z = |z| e^{it} \}$$

$\frac{\pi}{4} + 2\pi = \frac{6\pi}{4}$

Any $t \in A(z)$ is called argument of z $\qquad \frac{\pi}{4} + 2n\pi$

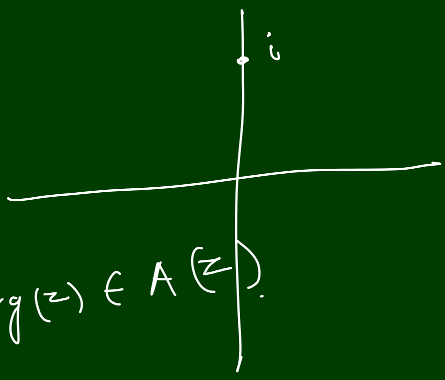
$\arg z$ is called Principal argument of z

$A(i)$

$\arg(i) = \pi/2$ ✓

$A(i) = \pi/2$ ✗

$\pi/2 \in A(i)$



$\arg(z) \in A(z)$

Arg ✓

arg ✓

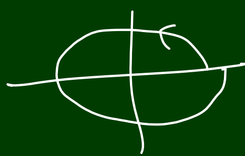
$\left\{ \frac{\pi}{2} + 2n\pi : n \in \mathbb{Z} \right\} \subseteq A(i)$ ✓

$\supseteq A(i)$?

Thm Given $z \in \mathbb{C}^*$, $\exists$ unique $\theta \in [0, 2\pi)$,

s.t. $z = |z| e^{i\theta}$

& $A(z) = \{ \theta + 2n\pi : n \in \mathbb{Z} \}$



(argument of z in $\mathbb{C}^*$ is not unique)

($\therefore$ This thm says we have fn for $z \in \mathbb{C}^*$ $z \mapsto \theta(z)$

$\theta(z)$ is denoted by $\arg_0(z)$)

$\therefore \arg_0 : \mathbb{C}^* \mapsto \mathbb{R}$, $z \in \mathbb{C}^*$

$\therefore \theta \in [0, 2\pi) \subset [0, 2\pi] \subseteq \mathbb{R}$

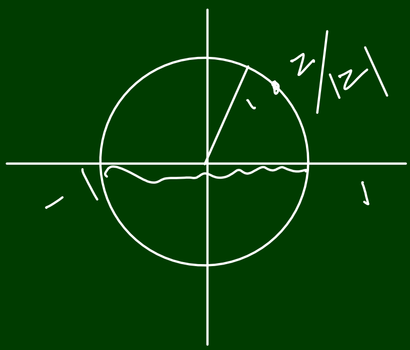
$z \in \mathbb{C}$

Find θ s.t $z = |z| e^{i\theta}$

$$\frac{z}{|z|} = e^{i\theta}$$

<u>Case(i)</u> z be s.t $\operatorname{Im} z \geq 0$

$$\frac{z}{|z|} = u + iv$$



$u^2 + v^2 = 1$

$-1 \leq u \leq 1$

$0 \leq v \leq 1$

$u + iv = \cos\theta + i\sin\theta$

$u = \cos\theta, \quad v = \sin\theta$

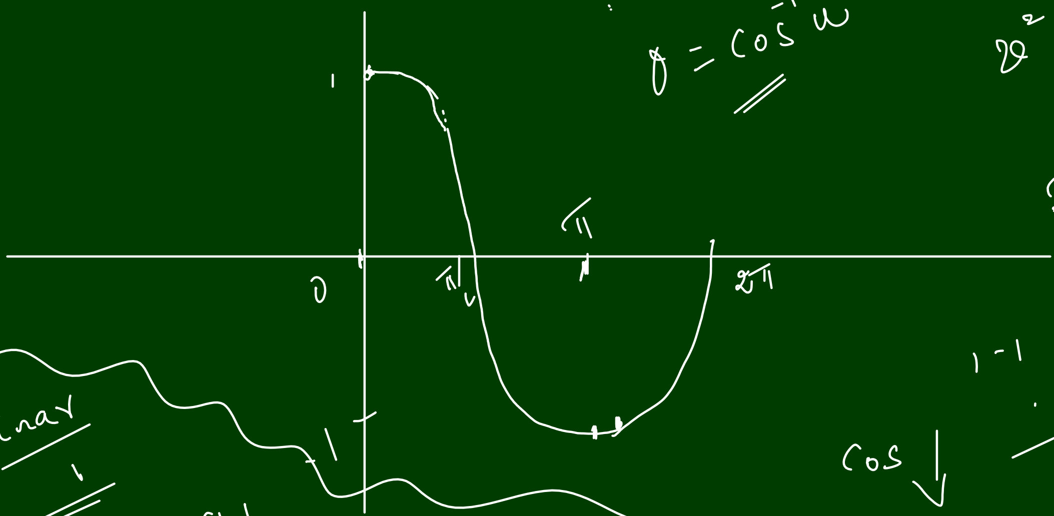
$\theta = \cos^{-1} u$

$u = \cos\theta$

$v^2 = 1 - \cos^2\theta$

$= \sin^2\theta$

$v = \sin\theta$



1-1

cos ↓

Seminar

Non trivial

$\cos : [0, \pi] \to [-1, 1]$ Cont, Red

cos is <u>bijective</u>,

$\cos^{-1}$ exists.

If z lies in closed upper half plane $\{w \in \mathbb{C} : \operatorname{Im} w \geq 0\}$

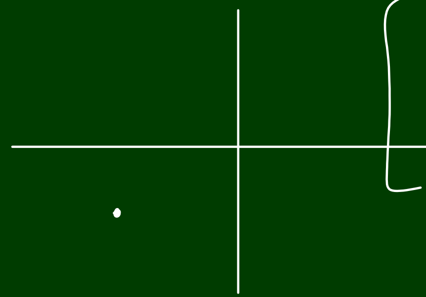
$\therefore \exists\, \theta \in [0, \pi]$ s.t $u = \cos\theta$, $v = \sin\theta$

$$\therefore \frac{z}{|z|} = u + iv = e^{i\theta}$$

$\therefore z = |z| e^{i\theta}$

$\operatorname{Im} z < 0$

Then $\operatorname{Im}(-z) = -\operatorname{Im} z > 0$



$\left[e^{it} = 1 \right.$ Then $\exists\, n \in \mathbb{Z}$ s.t $t = 2n\pi \left. \right]$

By Previous case

$\exists\, t \in (0, \pi)$

$$-z = |-z|\, e^{it}$$
$$-z = |z|\, e^{it}$$
$$z = |z|\, e^{it} \cdot (-1)$$
$$= |z|\, e^{it} e^{i\pi}$$
$$= |z|\, e^{i(t+\pi)}$$
$$= |z|\, e^{i\theta}$$

$\theta = t + \pi \in (\pi, 2\pi)$

∴ we get the result

suppose $\exists\, t, s \in [0, 2\pi)$ s t

$$z = |z|\, e^{it}$$
$$z = |z|\, e^{is}$$

$\left(e^{z} = 1 \text{ iff } \_\_\_\_ \right)$

$$e^{it} = e^{is}$$
$$e^{i(t-s)} = 1$$

∴

$\cancel{t - s \in \{2n\pi : n \in \mathbb{Z}\}}$ i.e $\exists\, n \in \mathbb{Z}$ $t - s = 2n\pi$ (?)

$$|t - s| < 2\pi$$
$$2|n|\pi < 2\pi$$

∴ $|n| < 1$ $\Rightarrow n = 0$

∴ $t - s = 0$

∴ $t = s$

∴ $\exists\, \theta \in [0, 2\pi)$ $z = |z|\, e^{i\theta}$

Suppose $\exists\, s \in \mathbb{R}$ s.t $z = |z|\, e^{is}$

$$e^{i\theta} = e^{is}$$
$$e^{i(\theta - s)} = 1$$

$\theta - s = 2n\pi$ when $n \in \mathbb{Z}$

∴ $s = \theta + 2n\pi$

Trignomtry can be (Reloaded using exponential fn)

$$e^z := 1 + z + \frac{z^2}{2!} + \dots + \dots$$

Complex Analysis, 11-8-2021

Argument of a Complex number:

$\arg_0 : \mathbb{C}^* \to \mathbb{R}$ is fn (Why? given $z \in \mathbb{C}^*$, $\exists !\ \theta \in [0, 2\pi)$ s.t $z = |z| e^{i\theta}$

Also $\left[A(z) := \{ t \in \mathbb{R} : z = |z| e^{it} \}\right]$

given $t \in A(z)$, $\exists\ k \in \mathbb{Z}$ s.t $t = \theta + 2k\pi$

$\therefore A(z) = \{ \theta + 2n\pi : n \in \mathbb{Z} \}$)

Proof (revise)

Let $z \in \mathbb{C}^*$

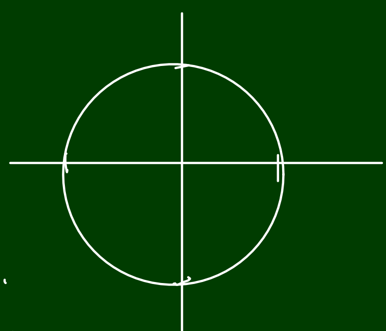
we want $\theta \in [0, 2\pi)$ s.t

$z = |z| e^{i\theta}$

$\frac{z}{|z|} = e^{i\theta}$

$w = \frac{z}{|z|} = u + iv$

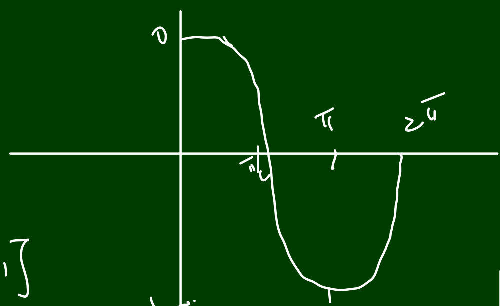
$|w| = 1 \implies u^2 + v^2 = 1$



Case(i) z lies in closed upper half plane $H = \{ w \in \mathbb{C}^* : \operatorname{Im} w \geq 0 \}$

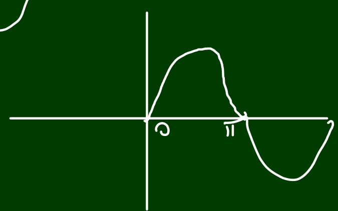
$-1 \leq u \leq 1$ $\qquad u = \operatorname{Re} e^{i\theta} = \cos\theta$

$0 \leq v \leq 1$



$\cos \downarrow$ in $[0, \pi]$

(Seminar thm) $\cos$ is a bijec from $\cos : [0, \pi] \to [-1, 1]$



$\therefore \cos^{-1}$ exists

$\therefore \exists\ \theta \in [0, \pi]$ s.t $u = \cos\theta$

$u^2 + v^2 = 1$

$v^2 = 1 - \cos^2\theta$

$= \sin^2\theta$

$v = \sin\theta$

$\therefore z \in H$ Then $\exists\ \theta \in [0, \pi]$ s.t $z = |z| e^{i\theta}$

Case(ii) If $\operatorname{Im} z < 0$

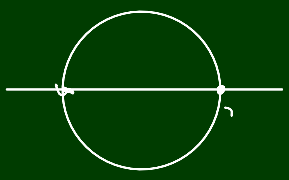
$\operatorname{Im}(-z) = -\operatorname{Im} z > 0$

$\mathbb{C} \to \mathbb{R}$
$z \mapsto \operatorname{Re}(z)$
$z \mapsto \operatorname{Im}(z)$ } linear map

$\therefore -z \in H$

$\therefore \exists\, t \in [0, \pi]$ s.t $-z = |z| e^{it}$

$-z = |z| e^{it}$



Also $\because$ $\operatorname{Im}(-z) > 0$

$\therefore z$ cannot lie in Real axis

$\therefore t \in (0, \pi)$

$\therefore z = |z| e^{it} (-1)$

$= |z| e^{it} e^{i\pi}$

$= |z| e^{i(t+\pi)}$

$\theta = t + \pi$ $\quad \theta \in (\pi, 2\pi)$

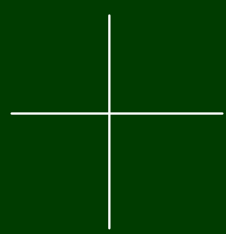
$\exists\, \theta \in [0, 2\pi)$ S.t $z = |z| e^{i\theta}$

Ex: { $e^{i\theta} = 1$ iff $\theta = 2n\pi$ for some $n \in \mathbb{Z}$ }

$\therefore \arg_0 : \mathbb{C}^* \to \mathbb{R}$. (is a fn).

Given $z \in \mathbb{C}^*$, $\exists!\, \theta \in [-\pi, \pi)$

St $z = |z| e^{i\theta}$



Exercise

Complete the proof

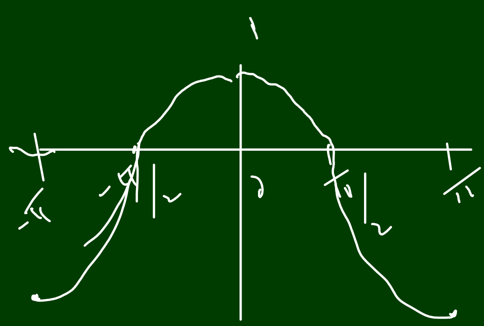
$w = \dfrac{z}{|z|} = e^{i\theta}$

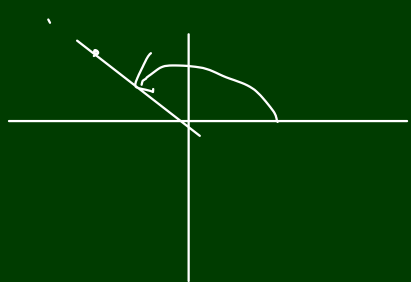
$w = u + iv$

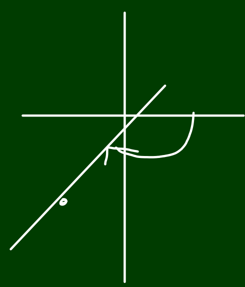
$u^2 + v^2 = 1$

$u = \cos\theta$

$v = \sin\theta$



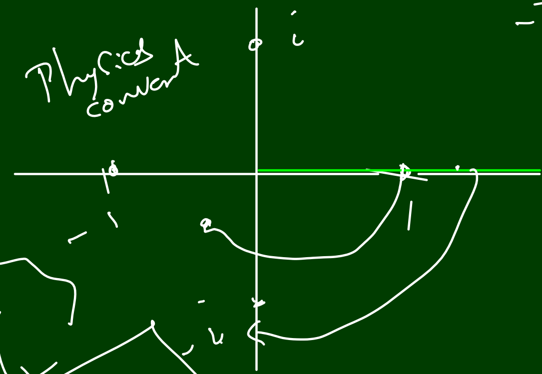




$[-\pi, \pi)$

Physics convention

$-\pi + \pi = 0$

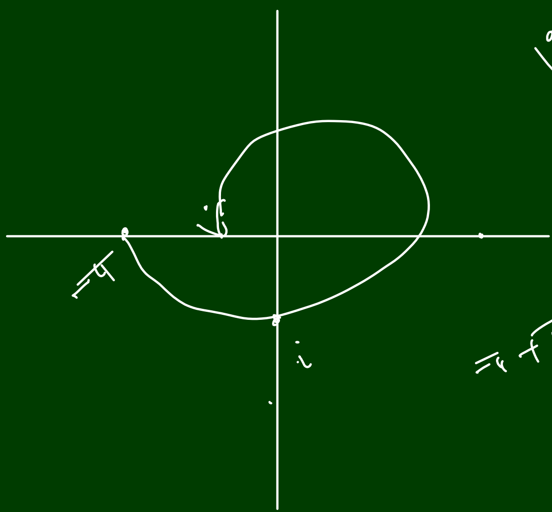


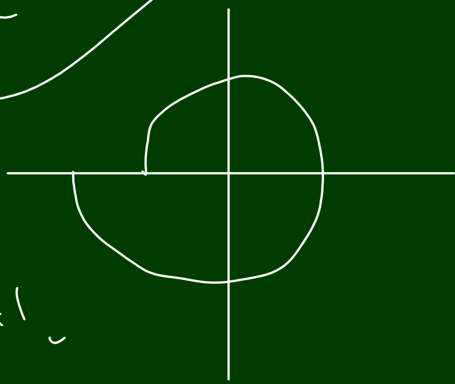
$\arg_{-\pi}(1) = 0$

$\arg_{-\pi}(i) = \pi/2$

$\arg_{-\pi}(-1) = -\pi$

$\arg_{-\pi}(-i) = -\pi/2$



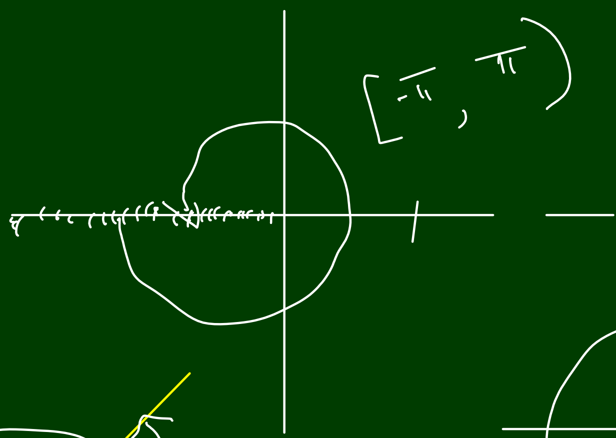


$-\pi + \pi/2 = -\pi/2$

$\theta \in [-\pi, \pi)$

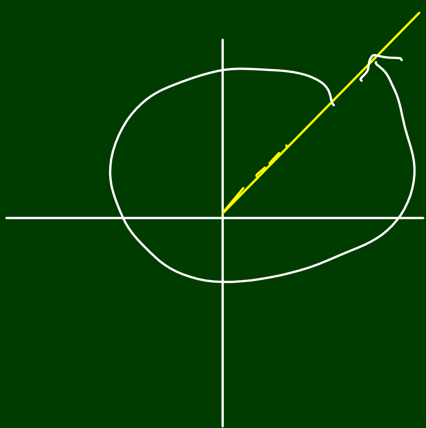
$\in [0, 2\pi)$

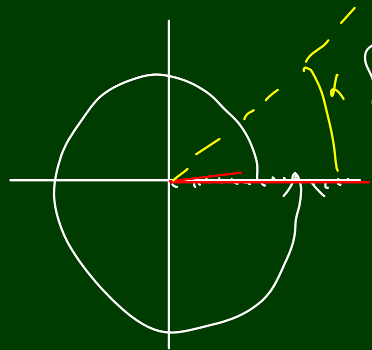
$\in [\alpha, \alpha + 2\pi)$



$[-\pi, \pi)$

sin, cos are func of period 2π





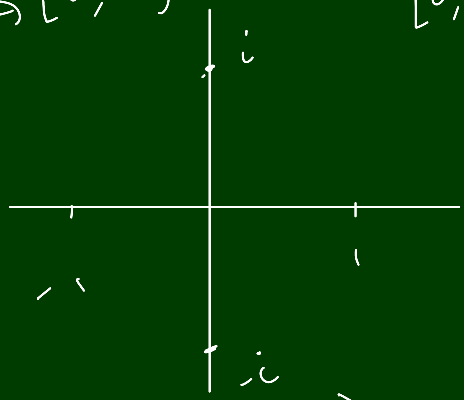
$\alpha \in [0, 2\pi)$

Referential

$\boxed{\arg_\alpha.}$

$\arg_0$

$\arg_0 : \mathbb{C}^* \to [0, 2\pi)$



$[0, 2\pi)$

$$\left.\begin{aligned} &\arg_0(1) = 0 \\ &\arg_0(i) = \frac{\pi}{2} \\ &\arg_0(-1) = \pi \\ &\arg_0(-i) = \frac{3\pi}{2} \\ &\vdots \end{aligned}\right\}$$

$[2\pi, 4\pi)$

$\arg_{2\pi}(i) = \frac{5\pi}{2}$

$\arg_{-\pi} : \mathbb{C}^* \to [-\pi, \pi)$

There are some many choices for arg fn

i.e given $\alpha \in \mathbb{R}$ we get a fn $\arg_\alpha : \mathbb{C}^\alpha \to [\alpha, \alpha + 2\pi)$

$\arg_0 : \mathbb{C}^* \to [0, 2\pi)$ is a fn

Is it a conts fn ?

$Re : \mathbb{C} \to \mathbb{R}$

$Im : \mathbb{C} \to \mathbb{R}$

Are they conts!

Let $a \in \mathbb{C}$

$(\mathbb{C}, |\ |)$

$$|f(z) - f(a)| = |Re\, z - Re\, a|$$
$$= |Re(z-a)|$$
$$\leq |z-a|$$

$w = x + iy$

$Re\, w = x$

$|w|^2 = x^2 + y^2$

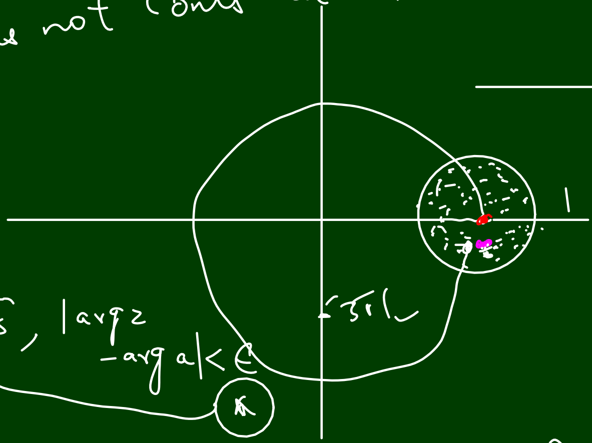
$x^2 \leq x^2 + y^2$

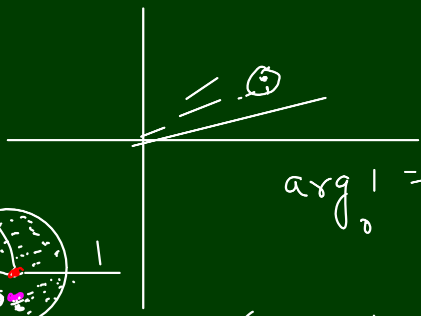
$|x|^2 \leq (x^2 + y^2)^{1/2}$

$|x| \leq |w|$

$|Re\, w| \leq |w|$

This $\arg_0$ is not conts at 1





$\arg_0 1 = 0$

$(1, 0)$

$(1, y)$

For $\epsilon = \pi$

$\exists\ \delta > 0$ & $\forall y$ $|z - a| < \delta$, $|\arg z - \arg a| < \epsilon$ (*)

$a = 1$

$z = 1 + iy$

Let $y = -\delta/2$ $\quad |z - a| = |iy| = |y| = \delta/2 < \delta$

$$|\arg z - \arg a| = |\arg z| = \arg z \geq \frac{3\pi}{2} > \pi$$

~~Thus~~ There is no choice of argument in $A(z)$ s.t which make it conts

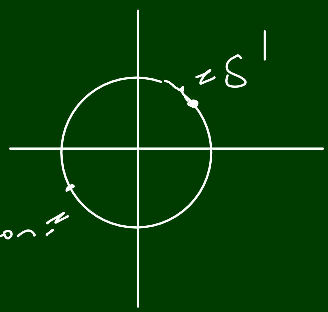
i.e There is no conts fn $\theta : \mathbb{C}^{*} \to \mathbb{R}$ s.t $\forall z \in \mathbb{C}^{*}$, $z = |z| e^{i\theta}$

Complex Analysis, Aug 12, 2021

Thm There is no conts fn $\theta : \mathbb{C}^* \to \mathbb{R}$ s.t $\forall z \in \mathbb{C}^*$, $z = |z| e^{i\theta(z)}$

Proof Suppose $\exists$ conts fn $\theta : \mathbb{C}^* \to \mathbb{R}$ s.t, $\forall z \in \mathbb{C}^*$, $z = |z| e^{i\theta(z)}$

$$S^1 := \{ z \in \mathbb{C} : |z| = 1 \}$$



We use Connectedness of S^1 to arrive at a contradiction.

We have to find a conts $f : S^1 \to \{\pm 1\}$ which is non-constant.

Claim θ is 1-1 on S^1

Let $z_1, z_2 \in S^1$

s.t $\theta(z_1) = \theta(z_2)$

$$i\theta(z_1) = i\theta(z_2)$$

$$e^{i\theta(z_1)} = e^{i\theta(z_2)}$$

also $|z_1| = |z_2| = 1$

$$\therefore |z_1| e^{i\theta(z_1)} = |z_2| e^{i\theta(z_2)}$$

$$\therefore z_1 = z_2$$

$\therefore$ Let $z \in S^1$. Then $-z \in S^1$

Also $z \neq -z$

$$\therefore \theta(z) \neq \theta(-z) \qquad \therefore x := \theta(z) - \theta(-z) \neq 0$$

Define $f : S^1 \to \{\pm 1\}$ by

$$f(z) = \frac{\theta(z) - \theta(-z)}{|\theta(z) - \theta(-z)|}$$

$\frac{x}{|x|} \in \{\pm 1\}$

Let $w \in S^1$

Then $f(w) \in \{\pm 1\}$

$f(w) = \gamma$ where $\gamma \in \{\pm 1\}$

$f(-w) = -f(w) = -\gamma$

$\therefore f$ is onto

$\Rightarrow \Leftarrow$

Aliter use IVT

Ex

$[0, 2\pi] \to S^1 \to \mathbb{R}$

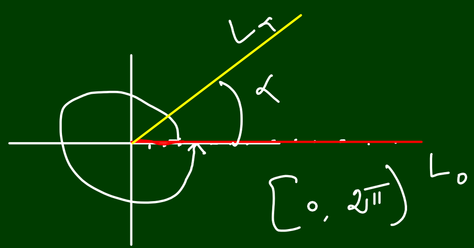
e^{it}

$g : [0, 2\pi] \to \mathbb{R}$

$g(t) = \theta(e^{it}) - \theta(-e^{it})$

For $\alpha \in \mathbb{R}$ $L_\alpha := \{t e^{i\alpha} : t \geq 0\}$

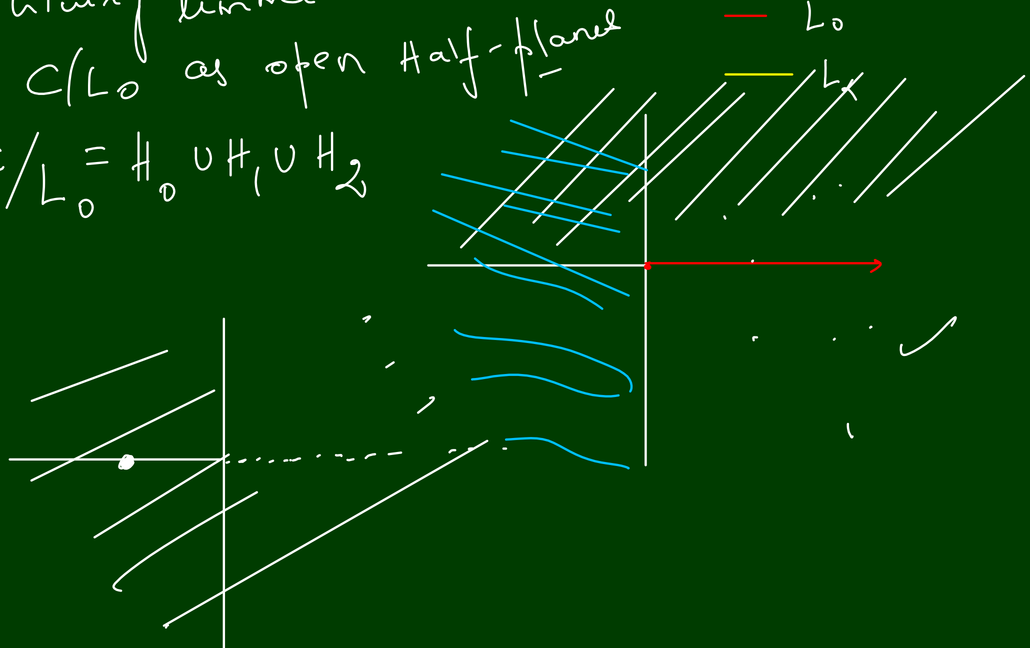
Thm $\arg_0 : \mathbb{C}^* \setminus L_0 \to (0, 2\pi)$ is a fn



Proof Using Gluing lemma

We divide $\mathbb{C} / L_0$ as open Half-planes

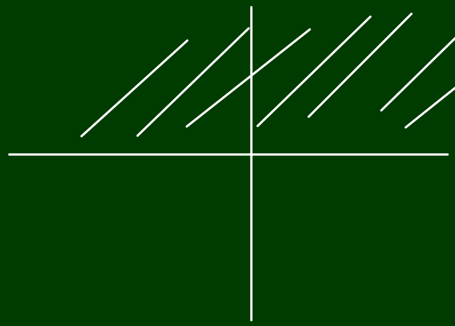
$\mathbb{C} / L_0 = H_0 \cup H_1 \cup H_2$



$H_0 = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ $H_3 = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$

$H_1 = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$

Let $z \in H_0$



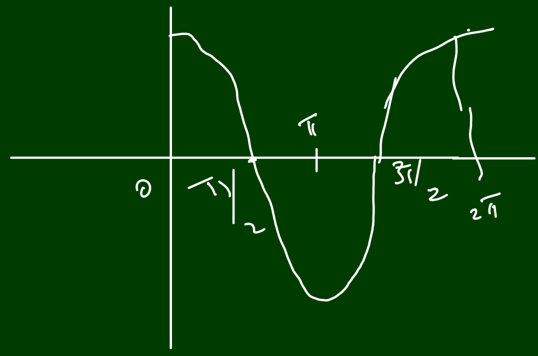
$\frac{z}{|z|} = u + iv$

$u^2 + v^2 = 1$

$-1 \le u \le 1$

$0 \le v \le 1$

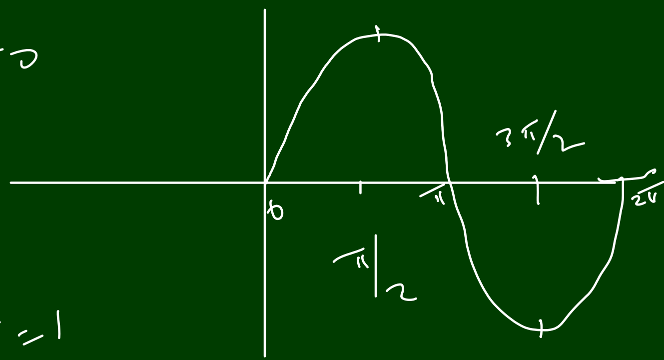
$\cos : [0, \pi] \to [-1, 1]$ is a bijection



$\cos\theta = u$ $\quad u = \cos\theta$ ✓

$= \operatorname{Re} \frac{z}{|z|}$ ✓ $\quad v = \sin\theta$ ✓

$\therefore \theta_1 = \cos^{-1} \operatorname{Re}\left(\frac{z}{|z|}\right), \ z \in H_0$



Let $z \in H_1$

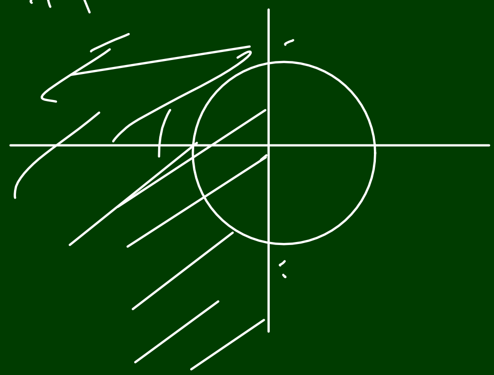
$v = \sin\theta$

$\cancel{\sin^{-1}}\left(\operatorname{Im} \frac{z}{|z|}\right) = \theta$

$u^2 + v^2 = 1$

$-1 \le u \le 0$

$-1 \le v \le 1$ $\quad H_1$



$\sin : \left(\pi/2, \ 3\pi/2\right) \to (-1, 1)$

is a bijection

Do this & correct

$$\theta_2 = \pi - \sin^{-1}\left(\operatorname{Im} \frac{z}{|z|}\right), \ z \in H_1$$

$$\theta_3 = \ ?$$

Let

Complex Analysis, Aug 12, 2021

Thm There is no conts fn $\theta : \mathbb{C}^* \to \mathbb{R}$ s.t $\forall z \in \mathbb{C}^*$, $z = |z| e^{i\theta(z)}$

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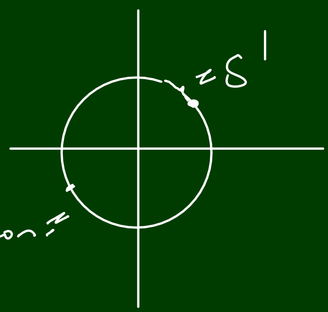

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$\therefore$ Let $z \in S^1$. Then $-z \in S^1$

Also $z \neq -z$

$$\therefore \underline{\theta(z) \neq \theta(-z)} \quad \therefore x := \theta(z) - \theta(-z) \neq 0$$

Define $f : S^1 \to \{\pm 1\}$ by

$$f(z) = \frac{\theta(z) - \theta(-z)}{|\theta(z) - \theta(-z)|}$$

$\frac{x}{|x|} \in \{\pm 1\}$ ✓

Let $w \in S^1$

Then $f(w) \in \{\pm 1\}$

$f(w) = r$ where $r \in \{\pm 1\}$

$f(-w) = -f(w) = -r$

$\therefore f$ is onto

$\Rightarrow \Leftarrow$

Aliter use IVT

$[0, 2\pi] \to S^1 \to \mathbb{R}$

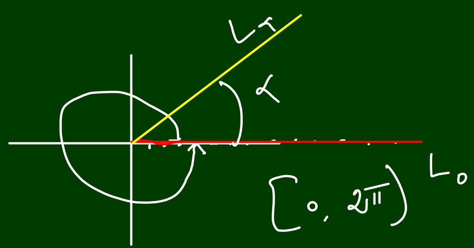
e^{it}

Ex

$g : [0, 2\pi] \to \mathbb{R}$

$g(t) = \theta(e^{it}) - \theta(-e^{it})$

For $\alpha \in \mathbb{R}$ $L_\alpha := \{t e^{i\alpha} : t \geq 0\}$



Thm $\arg_0 : \mathbb{C} \setminus L_0 \to (0, 2\pi)$ is a fn

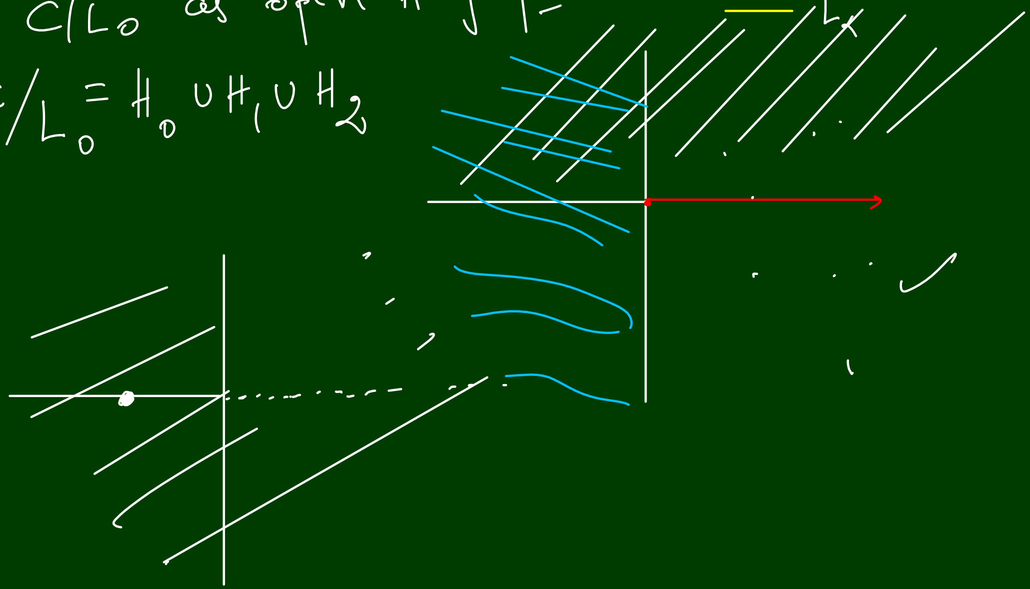
Proof Using Gluing lemma

We divide $\mathbb{C} \setminus L_0$ as open Half-planes

— L_0

— L_α

$\mathbb{C} / L_0 = H_0 \cup H_1 \cup H_2$



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$H_1 = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$

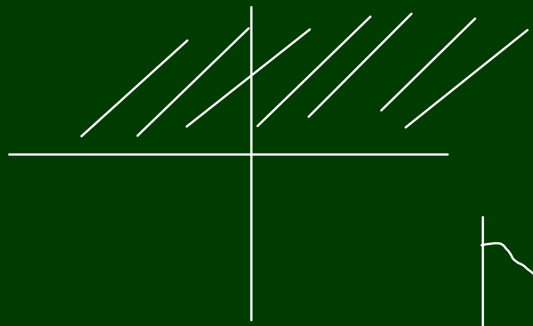
Let $z \in H_0$

$$\frac{z}{|z|} = u + iv$$

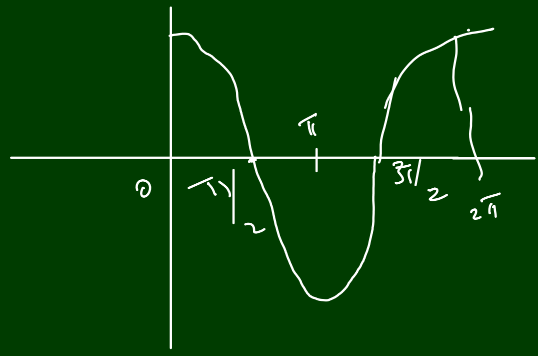
$u^2 + v^2 = 1$

$-1 \le u \le 1$

$0 \le v \le 1$



$(0, 2\pi)$



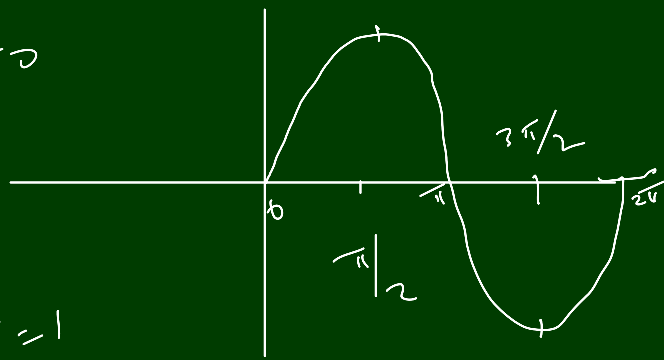
$\cos : [0,\pi] \to [-1, 1]$ is a bijection

$\therefore u = \cos\theta$ ✓

$v = \sin\theta$ ✓

$\cos\theta = u = \mathrm{Re}\,\frac{z}{|z|}$ ✓

$$\therefore \theta_1 = \cos^{-1} \mathrm{Re}\left(\frac{z}{|z|}\right), \quad z \in H_0$$



— $z \in H_1$

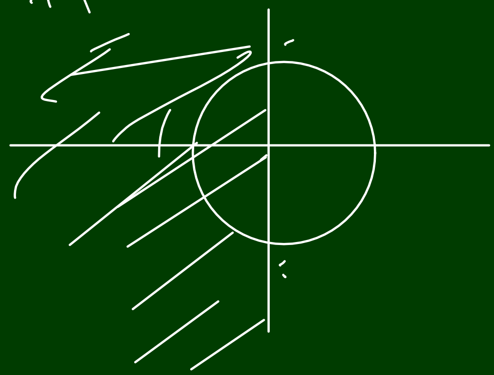
$v = \sin\theta$

$\cancel{\sin^{-1}}\left(\mathrm{Im}\,\frac{z}{|z|}\right) = \theta$?

$u^2 + v^2 = 1$

$-1 \le u \le 0$

$-1 \le v \le 1$ H_1



$\sin : \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \to (-1, 1)$ is a bijection

Do this & correct

$$\theta_2 = \pi - \sin^{-1}\left(\mathrm{Im}\,\frac{z}{|z|}\right), \quad z \in H_1$$

$\theta_3 = ?$

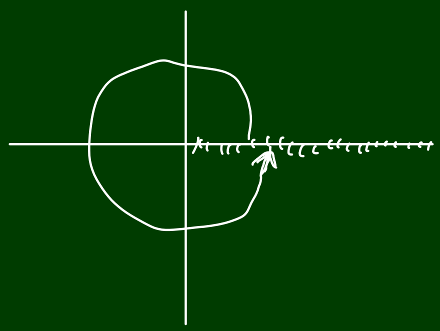
—

Complex Analysis, Aug 13, 2021

Thm Let given $\alpha \in \mathbb{R}$, $L_\alpha := \{t e^{i\alpha} : t \geq 0\}$

$\therefore L_0 = \{t : t \geq 0\}$

$\arg_0 : \mathbb{C}^* \setminus L_0 \to (0, 2\pi)$ is a contd fn



gluing lemma:

X - m.s.

$\forall \alpha$ $f_\alpha : U_\alpha \to \mathbb{R}$ where U_α open, $X = \bigcup_{\alpha \in I} U_\alpha$

or $X = \bigcup_{i=1}^{n} A_i$, A_i is cl.

$f_i : A_i \to \mathbb{R}$

Then $f = f_i(x)$ if $x \in A_i$

$f = f_\alpha(x)$ if $x \in U_\alpha$

$x \in U_\alpha \cap U_\beta$

$f_\alpha(x) = f_\beta(x)$

Proof

W.r.t.

$$\mathbb{C}^* \setminus L_0 = H_0 \cup H_1 \cup H_2$$

where $H_0 = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ - upper

$H_1 = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ - Left half

$H_2 = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$

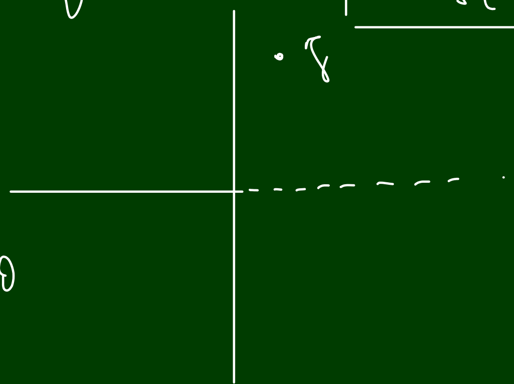
$z \in \mathbb{C}^*$, $\exists!\, \theta \in [0, 2\pi)$ s.t. $z = |z| e^{i\theta}$

Case (i)

Let $z \in H_0$

$\exists\, \theta \in (0, 2\pi)$ s.t. $z = |z| e^{i\theta}$

$$\frac{z}{|z|} = e^{i\theta}$$



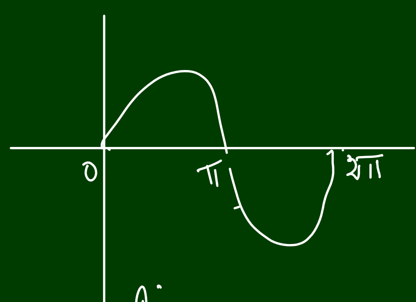
$\cos\theta = \operatorname{Re}\left(\frac{z}{|z|}\right)$, $\sin\theta = \operatorname{Im}\left(\frac{z}{|z|}\right)$

$\because z \in H_0$, $\operatorname{Im} z > 0$

$\therefore \sin\theta > 0$

$\therefore \theta \in (0, \pi)$

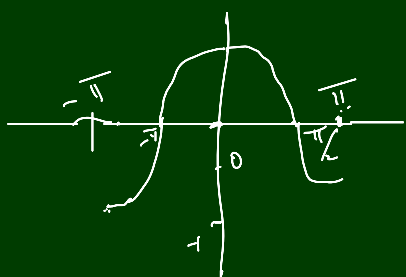
Draw sin graph for $[0, 2\pi]$



$\cos : [0, \pi] \to [-1, 1]$ is bijective contd fn

$\therefore \cos$ has a contd inverse

Draw graph for finding right inverse.



$\therefore \quad \cos^{-1} [-1, 1] \to [0, \pi]$ is conts

$\cos^{-1} (-1, 1) \to (0, \pi)$ is conts

$\because \cos\theta = Re\left(\frac{z}{|z|}\right)$

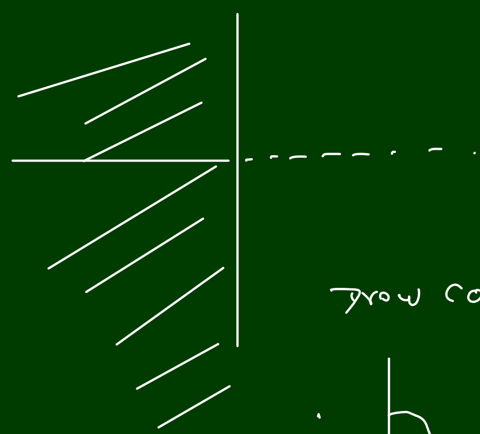
$$\theta = \cos^{-1}\left(Re\left(\frac{z}{|z|}\right)\right)$$

We define $\theta_1 : H_0 \to (0, \pi)$

$$\theta_1(z) = \cos^{-1}\left(Re\left(\frac{z}{|z|}\right)\right)$$

Then we have proved that θ_1 is conts on H_0

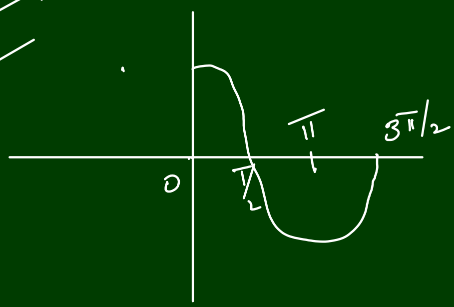
Case(ii) $\quad z \in H_1$



$\exists\ \theta \in (0, 2\pi)$ s.t

$$\frac{z}{|z|} = e^{i\theta}$$

Draw cosine fn in $(0, 2\pi)$

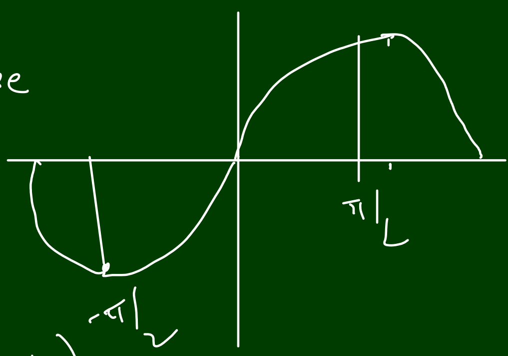


$\because z \in H_1, \quad Re\, z < 0$

$\therefore \quad \theta \in (\pi/2, 3\pi/2)$

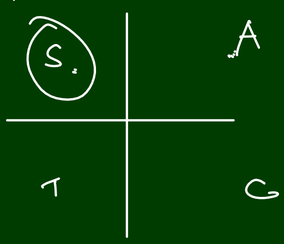
$$\pi - \theta \in \left(\frac{-\pi}{2}, \pi/2\right)$$

$\sin : (-\pi/2, \pi/2) \to (-1, 1)$ has conts inverse



$\sin\pi\cos\theta - \cos\pi\sin\theta$

$\therefore \quad \sin(\pi - \theta) = \sin\theta = Im\frac{z}{|z|}$



$$\therefore \pi - \theta = \sin^{-1}\left(Im\left(\frac{z}{|z|}\right)\right)$$

$$\theta = \pi - \sin^{-1}\left(Im\left(\frac{z}{|z|}\right)\right)$$

$\theta \leq 3\pi/2 \qquad \theta \geq \pi/2$

$\pi - \theta \geq \pi - 3\pi/2 \qquad -\theta \leq -\pi/2$

$\pi - \theta \geq -\pi/2 \qquad \pi - \theta \leq \pi/2$

Define $\theta_2 : H_1 \to (\pi/2, 3\pi/2)$ by

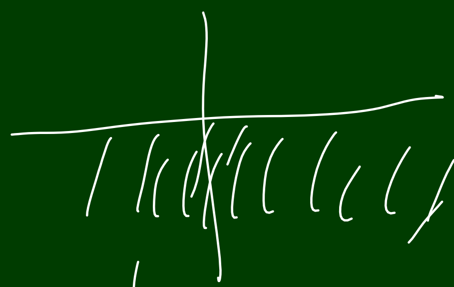
$$\theta_2(z) = \pi - \sin^{-1}\left(\operatorname{Im}\left(\frac{z}{|z|}\right)\right)$$

Then θ_2 is conts on H_1

Case (iii) $\quad z \in H_2$

$\exists\ \theta \in (0, 2\pi)$ s.t. $\dfrac{z}{|z|} = e^{i\theta}$ ✓

$\cos\theta = \operatorname{Re}\dfrac{z}{|z|}$, $\quad \sin\theta = \operatorname{Im}\dfrac{z}{|z|}$

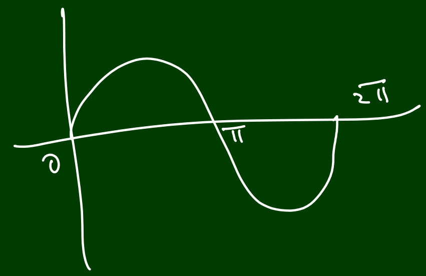


$\because z \in H_2$, $\operatorname{Im} z < 0$

$\therefore \quad \because \sin\theta < 0$

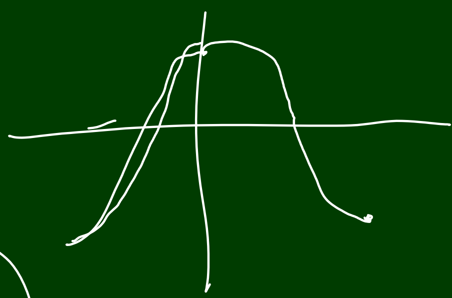
By sine graph $\theta \in (\pi, 2\pi)$

$2\pi - \theta \in (-\pi, 0)$.



$\because \cos^{-1} : (-1, 1) \to (-\pi, 0)$ is conts fn

Inverse fn



Also $\cos(2\pi - \theta) = \cos\theta$

$= \operatorname{Re}\dfrac{z}{|z|}$

$$\therefore 2\pi - \theta = \cos^{-1}\left(\operatorname{Re}\frac{z}{|z|}\right)$$

$$\therefore \theta = 2\pi - \cos^{-1}\left(\operatorname{Re}\frac{z}{|z|}\right)$$

Define $\theta_3 : H_3 \to (\pi, 2\pi)$ by

$$\theta_3(z) = 2\pi - \cos^{-1}\left(\operatorname{Re}\left(\frac{z}{|z|}\right)\right)$$

Define $\theta : \mathbb{C} \setminus L_0 \to (0, 2\pi)$ by

$$\theta(z) = \begin{cases} \theta_1(z) & \text{if } z \in H_0 \\ \theta_2(z) & \text{if } z \in H_1 \\ \theta_3(z) & \text{if } z \in H_2 \end{cases}$$

Note: H_0, H_1, H_2 are open sets.

Let $z \in H_0 \cap H_1$

Then $\theta_1(z) = \cos^{-1}\left(\mathrm{Re}\, \frac{z}{|z|}\right)$

$$\theta_2(z) = \pi - \sin^{-1}\left(\mathrm{Im}\, \frac{z}{|z|}\right)$$

$\therefore$

$$\mathrm{Im}\, \frac{z}{|z|} = \sin(\pi - \theta_2) = \sin\theta_2$$

$$\mathrm{Re}\, \frac{z}{|z|} = \cos\theta_2$$

$\therefore \boxed{\theta_2 = \theta_1}$ on $H_0 \cap H_1$

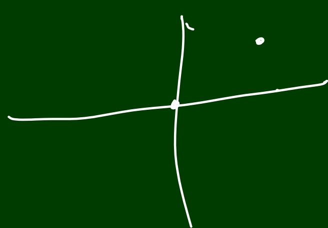
Also, $\theta_2 = \theta_3$ on $H_1 \cap H_2$

By Gluing lemma $\theta : \mathbb{C} \setminus L_0 \to (0, 2\pi)$ is a conti fn

1. Ex $\arg_\alpha : \mathbb{C} \setminus L_\alpha \to (\alpha, \alpha + 2\pi)$ is conti fn

2.) S.T $\forall z \in \mathbb{C}^*$, $\exists$ continous choice of A on $B(z, |z|)$

(Important problem)



Complex Analysis, Aug 6, 2020

Review

Thm $\forall z \in \mathbb{C}^*$, $\exists!\ \theta \in [0, 2\pi)$ s.t $z = |z| e^{i\theta}$

$z \to \theta(z)$

is called the argument

$\therefore$ $\arg_0 : \mathbb{C}^* \to [0, 2\pi)$

$\arg_0(z) = \theta$

Also $A(z) = \{\theta + 2n\pi : n \in \mathbb{Z}\}$

$A(z) := \{\theta \in \mathbb{R} : z = |z| e^{i\theta}\}$

Let $s \in A(z)$

$z = |z| e^{is}$, $z = |z| e^{i\theta}$

$e^{is} = e^{i\theta}$

$s - \theta \in \{2n\pi : n \in \mathbb{Z}\}$

$\therefore \exists\ n \in \mathbb{Z}$ s.t

$s - \theta = 2n\pi$

$\therefore s = \theta + 2n\pi$

$\mathbb{C}$

$z \xrightarrow{f_1} \operatorname{Re} z$

$z \xrightarrow{f_2} \operatorname{Im} z$

$z \xrightarrow{f_3} |z|$

Let $z_0 \in \mathbb{C}$

Given $\epsilon > 0$

Find $\delta > 0$

s.t $|z - z_0| < \delta$

$\Rightarrow |f(z) - f(z_0)| < \epsilon$

$$|f(z) - f(z_0)| = |\operatorname{Re} z - \operatorname{Re} z_0|$$

$z = x + iy \qquad \operatorname{Re} z = x$

$z_0 = x_0 + iy_0 \qquad \operatorname{Re} z_0 = x_0$

$|w|^2 =$

$w = a + ib$

$a^2 = |a|^2$

$\leq |a|^2 + |b|^2$

$a \leq |w|$

✓ $|x - x_0| \leq |z - z_0|$

$$|\operatorname{Re} z - \operatorname{Re} z_0| = |\operatorname{Re}(z - z_0)|$$

$\leq |z - z_0|$

$< \delta$

$\leq \epsilon$ ✓

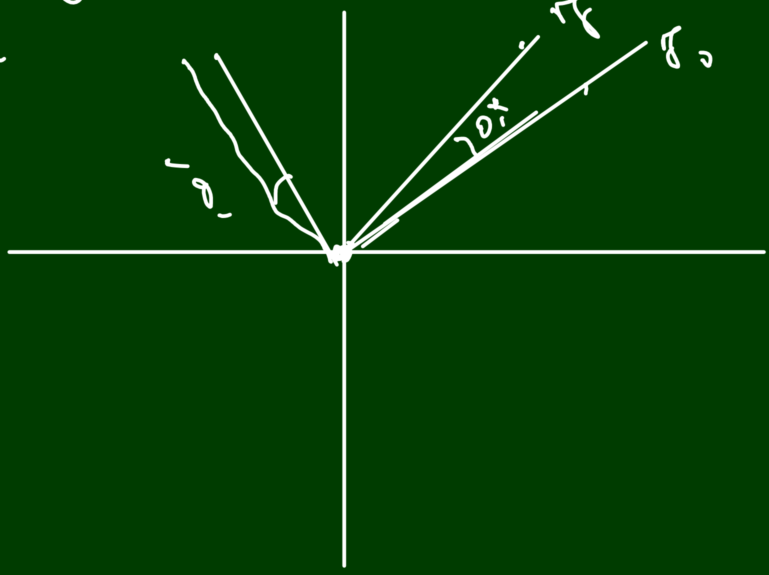
$\operatorname{Re} z \leq |z|$

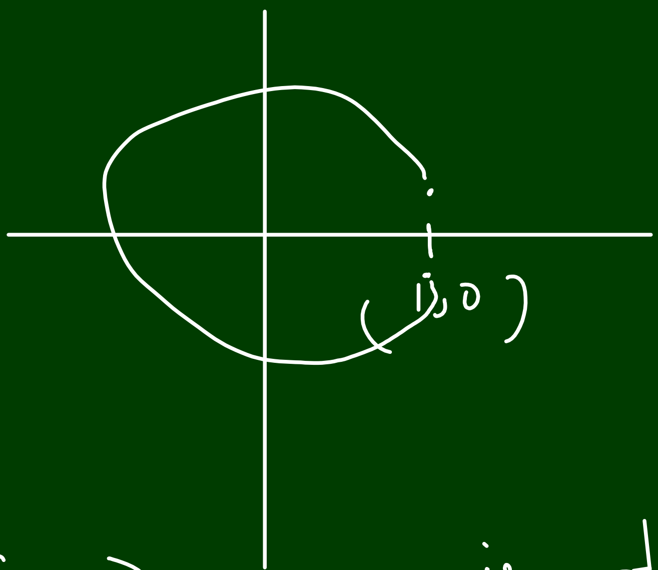
$\operatorname{Im} z \leq |z|$

$\arg_0 : \mathbb{C}^* \to [0, 2\pi)$

is fn ✓

Is it a conts fn





At $z_0 = (1,0)$, $\arg_0$ is not conts

Suppose conts at z_0

For $\epsilon = 1$, $\exists \delta > 0$ s.t. $\forall z \in \mathbb{C}^*$, $|z - z_0| < \delta$

$\left(|\arg z - \arg z_0| < 1\right)$ — (1)

$$z = x - \frac{i}{N} \qquad z_0 = x$$

$$z - z_0 = \frac{-i}{N}$$

By A.P. choose $N \in \mathbb{N}$ s.t.

$$|z - z_0| = \frac{1}{N} < \delta$$

$$|\arg z - \arg z_0| = |\arg z| > \pi \geq 1$$

$\arg z_0$ is not conts at any point

$$\{z \in \mathbb{C} : z \in \mathbb{R}, z \geq 0\}$$

$$\{x + iy : x \geq 0\}$$

Complex Variables.

1. R. V. Churchill & R. Brown

Thm

There is no Cont fn $\theta : \mathbb{C}^* \to \mathbb{R}$ s.t

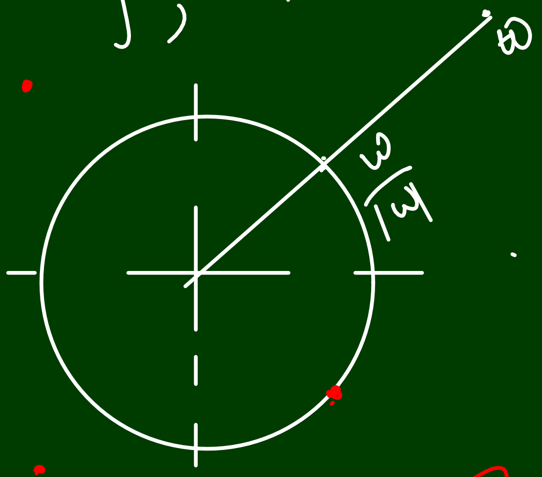
$\forall z \in \mathbb{C}^*$, $z = |z| e^{i\theta(z)}$

Proof:

// There is no cont fn $\theta : S^1 \to \mathbb{R}$ s.t

$\forall z \in S^1$, $z = |z| e^{i\theta(z)}$

(For if; there is cont $\theta : S^1 \to \mathbb{R}$



$z = \frac{\omega}{|\omega|}$

$|z| = 1$

$z = e^{i\theta(z)}$

$\frac{\omega}{|\omega|} = e^{i\theta\left(\frac{\omega}{|\omega|}\right)}$

$\omega = |\omega| e^{i\theta\left(\frac{\omega}{|\omega|}\right)}$

$= |\omega| e^{i\theta'(\omega)}$

$\theta'(\omega) = \theta\left(\frac{\omega}{|\omega|}\right)$ ✓

Main proof: Suppose $\exists$ cont fn $\theta : S^1 \to \mathbb{R}$

s.t $z = |z| e^{i\theta(z)}$, $\forall z \in S^1$

I look for a contradiction

Which do I use from m.s?

contradicts (compact, connected)

∴ Look for a cont fn

$f : S^1 \to \{\pm 1\}$ which is onto.

∴ $\theta : S^1 \to \mathbb{R}$

θ is 1-1

Supp $\theta(z_1) = \theta(z_2)$

∴ $i\theta(z_1) = i\theta(z_2)$

$z_1 = e^{i\theta(z_1)} = e^{i\theta(z_2)} = z_2$ ✓

∴ θ is 1-1

In particular $\theta(z) \neq \theta(-z)$

$\theta(z) - \theta(-z) \neq 0$

$f : S^1 \to \{-1, 1\}$

$$f(z) = \frac{\theta(z) - \theta(-z)}{|\theta(z) - \theta(-z)|}$$

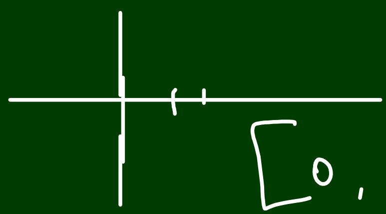
Cont f follows from cont θ, $|\,|$, $\frac{?}{z}$, $\frac{?}{z}$,

$$f(-z) = \frac{\theta(-z) - \theta(z)}{|\theta(-z) - \theta(z)|} = -\frac{(\theta(z) - \theta(-z))}{|\qquad|}$$

$\therefore$ if $f(z) = 1$, $f(-z) = -1$

$f(z) = -1$, $f(-z) = 1$

$\therefore f$ is onto



$[0, 2\pi)$

$\arg_0 : \mathbb{C} \setminus \{x+iy : x \geq 0\} \to (0, 2\pi)$

is conts fn

$z_1 = r_1 e^{i\theta_1}$

$z_2 = r_2 e^{i\theta_2}$

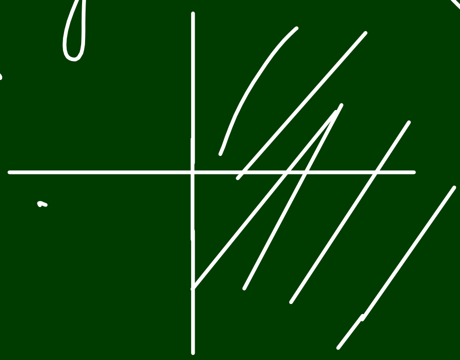
$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

Use Gluing Lemma

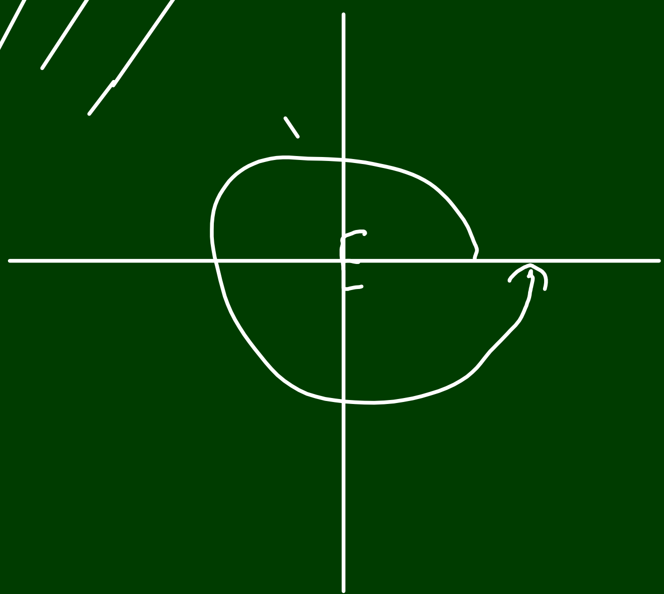
H_0 Right

H_1 - upr

H_2

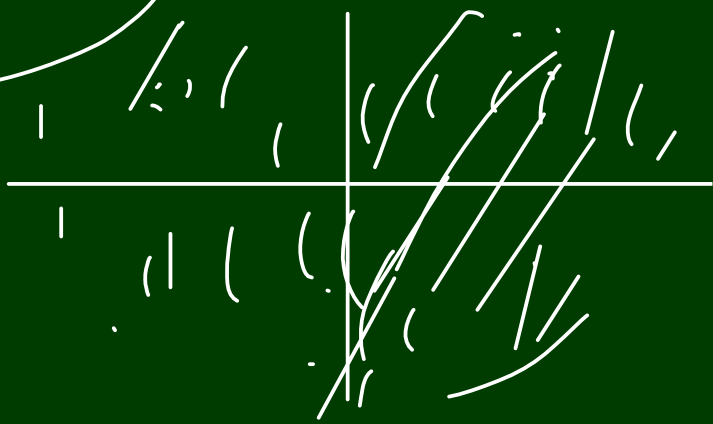


$\arg_{-\pi}$

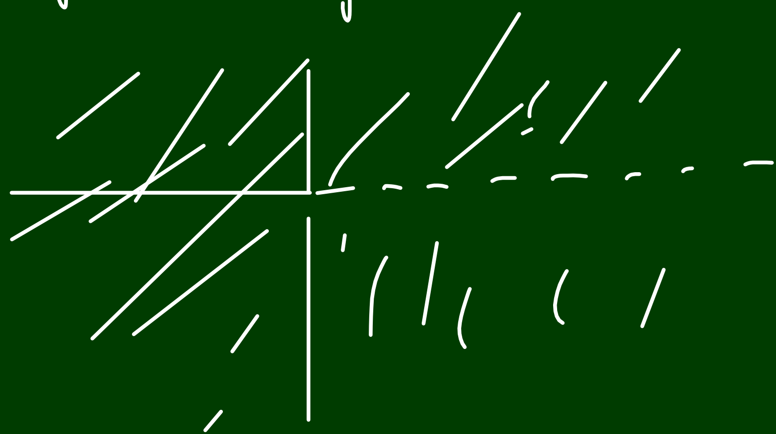


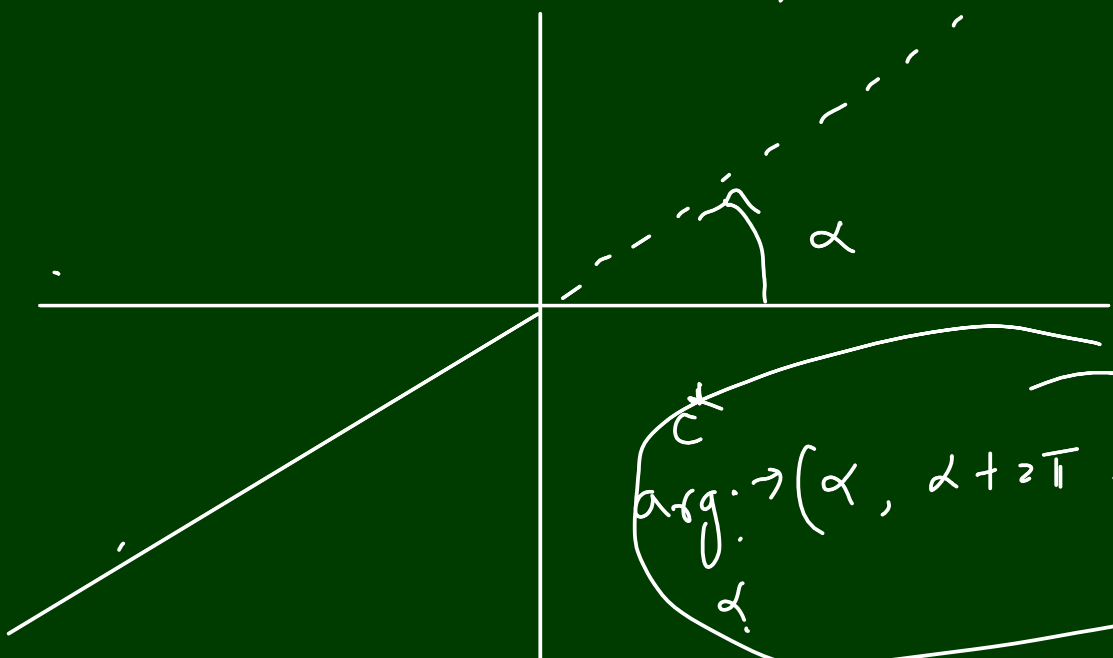
$\arg_{-\pi} : \mathbb{C}^* \setminus \{x+iy : x \le 0\} \to (-\pi, \pi)$

✓



$\arg_0 : \mathbb{C}^* \setminus \{x+iy : x \geq 0\} \to (0, 2\pi)$





$\mathbb{C}^*$

$\arg : \to (\alpha, \alpha + 2\pi)$

α?

Complex Analysis, Aug 17, 2021

Heading for defing logarithm of a complex no:

Lesson!

exp(x) $e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ —— (α)

The series on (α) converges $\forall x \in \mathbb{R}$ & hence exp fn is defined.

Derive all the well-known properties e^x using (α)

1. $e^x \cdot e^y = e^{x+y}$ [Exercise] [To be done from Cauchy-product of series]

$s_0 = 1$
$s_1 = 1+x$
$s_2 = 1+x+\frac{x^2}{2}$

2. If $x > 0$ then $e^x > 1+x > x$ where $s_n = \sum_{k=0}^{n} \frac{x^k}{k!}$

If $x > 0$, $e^x = \text{lub}\{s_n : n \in \mathbb{N} \cup \{0\}\}$

If $x > 0$ $\boxed{e^x > 1}$

$s_1 < s_2 \leq e^x$

$\therefore s_1 < e^x$

$1+x < e^x$ ✓

$\boxed{e^0 = 1}$

3. If $x \in \mathbb{R}$, $e^x > 0$

Trick use $e^0 = 1$

$e^{x-x} = e^x \cdot e^{-x} = e^0 = 1$

$\therefore e^x \cdot e^{-x} = 1$

$e^x = \frac{1}{e^{-x}}$

(g1) If $x = 0$, $e^x = 1$

(g2) If $x > 0$, by (1) $e^x > x > 0$

(g3) If $x < 0$, then $-x > 0$ $\therefore$ by (g2), $e^{-x} > 0$

& $e^x = \frac{1}{e^{-x}} > 0$

4) exp is strictly increasing fn

Pf: If $x > y$ then $e^x > e^y$

Then $x - y > 0$

$\therefore e^{x-y} > 1 + (x-y)$

$$e^x = e^{y+(x-y)} = e^{x-y} \cdot e^y > (1+x-y)\, e^y$$

$$> e^y \quad [1 + x - y > 1 \because x - y > 0]$$

5) $\exp : \mathbb{R} \to \mathbb{R}$ is conty (on $\mathbb{R}$)

Let $a \in \mathbb{R}$

$\sum x_n < \infty$, $\sum y_n < \infty$ Then $\sum (x_n + y_n) = \sum x_n + \sum y_n$

$$|\exp(x) - \exp(a)| = \left| \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{a^n}{n!} \right|$$

$$= \left| \sum_{n=0}^{\infty} \frac{(x^n - a^n)}{n!} \right|$$

$$= \left| \lim_{k} \sum_{n=0}^{k} \frac{(x-a)(x^{n-1} + a x^{n-2} + \cdots + a^j x^{n-j-1} + \cdots + a^{n-1})}{n!} \right|$$

$$\leq \lim_{k} \sum_{n=0}^{k} \frac{|x-a|}{n!} |x^{n-1} + \cdots + a^{n-1}|$$

$$\leq \sum_{n=0}^{\infty} \frac{|x-a|}{n!} n (1 + |a|)^{n-1}$$

$|x| = |x - a + a|$
$\leq |x-a| + |a|$
$< \delta + |a|$
$= 1 + |a|$

$|a^j x^{n-j+1}| = |a|^j |x|^{n-j-1}$
$< |a|^j (1+|a|)^{n-j-1}$
$< (1+|a|)^j (1+|a|)^{n-j-1}$

$$= (1+|a|)^{n-1}$$

$$\therefore |\exp(x) - \exp(a)| \le |x-a| \boxed{\sum_{n=1}^{\infty} \frac{(1+|a|)^{n-1}}{(n-1)!}}$$

$$= |x-a| e^{(1+|a|)}$$

$$< \delta e^{(1+|a|)}$$

$$\delta = \frac{\epsilon}{e^{1+|a|}}$$

$$\le \epsilon$$

$$\delta = \min\left\{1, \frac{\epsilon}{e^{1+|a|}}\right\}$$

5. $\exp : \mathbb{R} \to (0, \infty)$ is onto

In addi $\forall y > 0 \ \exists ! x \in \mathbb{R}, \ e^x = y$

onto foll monotonicity

$x_1 \ne x_2$

WLG $x_1 > x_2$

$e^{x_1} > e^{x_2}$

Given $t > 0$, $\exists c \in \mathbb{R}$, $e^c = t$

Use intermediate value thm

Let $t > 0$ choose $a = -1/t$

choose $b = t$

$e^a \quad t \quad e^b$

$\exp : [-1/t, t] \to \mathbb{R}$

$e^{1/t} \cdot e^{-1/t} = 1$

$e^{-1/t} < t < e^t$

$e^{-1/t} = \frac{1}{e^{1/t}} < \frac{1}{1} = t$

By IVT, $\exists c \in (-1/t, 1/t)$ s.t. $e^c = t$

$\exp : \mathbb{R} \to (0, \infty)$ is str$\uparrow$ and onto, conts

The inverse fn is denot as $\ln : (0, \infty) \to \mathbb{R}$ $(\log_e)$

(if $x > 0$, $e^{\ln x} = x$)

I - Interval.

$f : I \to \mathbb{R}$ conts strictly mon

Then $f^{-1} : f(I) \to \mathbb{R}$ is conts

$\therefore \ln : (0, \infty) \to \mathbb{R}$ is conts & $\uparrow$

Lesson 2 Let $z \in \mathbb{C}$

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

S.T. $\exp : \mathbb{C} \to \mathbb{C}^*$ is onto.

$\forall z \in \mathbb{C}^*$, $\exists\, w \in \mathbb{C}$ s.t. $e^w = z$

Let $w = x + iy$

$$e^x \cdot e^{iy} = z$$

$\because z \in \mathbb{C}^*$, $\exists!\, \theta \in [0, 2\pi)$ s.t. $z = |z| e^{i\theta}$

$$e^x \cdot e^{iy} = |z| e^{i\theta}$$

$$e^x = |z|$$

$$x = \ln|z|$$

$$y = \theta$$

choose $w = \ln|z| + i\theta$

Then $e^w = z$

Complex Analysis, Aug 18, 2021

Given $z \in \mathbb{C}^*$

How to define of logarithm of z?

(NOTE $\exp : \mathbb{C} \to \mathbb{C}^*$ is onto)

See the analogue of real case

Capture that $\forall\, x > 0$, $\exp(\ln x) = x$

Definition

Given $z \in \mathbb{C}^*$, we say w is a logarithm of z if
$$\exp(w) = z.$$

$$\operatorname{Log}(z) = \{ w \in \mathbb{C} : \exp(w) = z \}$$

Given $z \in \mathbb{C}^*$

$\operatorname{Log}(z) \neq \phi$ because $\exp : \mathbb{C} \to \mathbb{C}^*$ is onto.

$$e^{w} = z$$
$$e^{x+iy} = |z| e^{i\theta}$$
$$|z| = e^{x}, \quad e^{iy} = e^{i\theta}$$
$$x = \ln|z| \qquad y = \theta$$

$w = x + iy \in \operatorname{Log}(z)$

$z \in \mathbb{C} \setminus L_0$

$\boxed{\arg_\alpha}$

$\log_0(z) = \ln|z| + i \arg_0(z)$

$z \in \mathbb{C} \setminus L_\alpha$.

$\log_\alpha(z) = \ln|z| + i \arg_\alpha(z)$

Exercise

$\forall\, z \in \mathbb{C}^*$, $\exists$ conts choices of θ s.t. $\theta(z) \in A(z)$ in $B(z, |z|)$

$\log(z) := \ln|z| + i\theta(z)$

Defn Let U be open subset of $\mathbb{C}^*$
We say that $\exists$ a conts argument on U if $\exists$
a fn $\theta : U \to \mathbb{R}$ s.t. θ is conts on U and $\forall\, z \in U$ $z = |z| e^{i\theta(z)}$

We have already proved: If $U = \mathbb{C}^*$
There is no conts arg on U
If $U = \mathbb{C}\setminus L_0$
There is a conts arg on U
$U = \mathbb{C}\setminus L_\alpha$
''

Defn: Let $U \overset{\text{open}}{\subseteq} \mathbb{C}^*$. We say that $\exists$ a conts logarithm on U if $\exists$ conts fn $F: U \to \mathbb{C}$ s.t.
$$\forall z \in U,\ \exp(F(z)) = z$$

Thm: Let $U \overset{\text{open}}{\subseteq} \mathbb{C}^*$
U has conts argument on U iff U has conts logarithm on U

Proof: $\Rightarrow$ Let $\theta: U \to \mathbb{R}$ be the conts argument
Define $F: U \to \mathbb{C}$
$$F(z) = \ln|z| + i\,\theta(z)$$
Then $\exp(F(z)) = z$ (done already).
Also F is conts ✓ (Chg arg conts fn)

$\Leftarrow$ Let $F: U \to \mathbb{C}$ be the given conts logarithm
$$\forall z \in U,\ \exp(F(z)) = z$$
Define $\theta: U \to \mathbb{R}$ by $\theta = \operatorname{Im}(F(z))$

$e^u = |z|$
$u = \ln|z|$

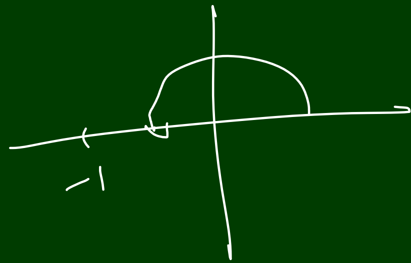
Verify $z = |z|e^{i\theta}$

$F(z) = u + iv$
$z = \exp(F(z)) = e^u \cdot e^{iv}$
$\therefore z = |z|e^{iv}$

True or false

$\forall\, U = \mathbb{C}^{\alpha}$, There is cont log on U false

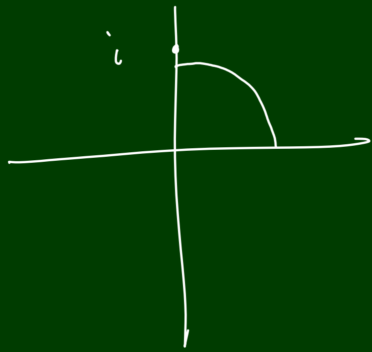
$$\operatorname{Log}(-1) = \{ i(2k+1)\pi : k \in \mathbb{Z} \}$$



$$w = \ln|z| + i\arg_0(-1)$$

$$w = i\pi$$

$$\operatorname{Log}(i) = \{ i(4k+1)\tfrac{\pi}{2} : k \in \mathbb{Z} \}$$



$$w = i\pi/2$$

$$\operatorname{Log}(\exp(z)) = \{ z + i2k\pi : k \in \mathbb{Z} \}$$

$w \in \operatorname{Log}(\exp(z)) \Rightarrow \exp(w) = \exp(\operatorname{Re} z + i \operatorname{Im} z) = e^{\operatorname{Re} z} \cdot e^{i \operatorname{Im} z}$

$w = x + iy$ $\qquad e^{x} \cdot e^{iy}$

$x = \operatorname{Re} z$

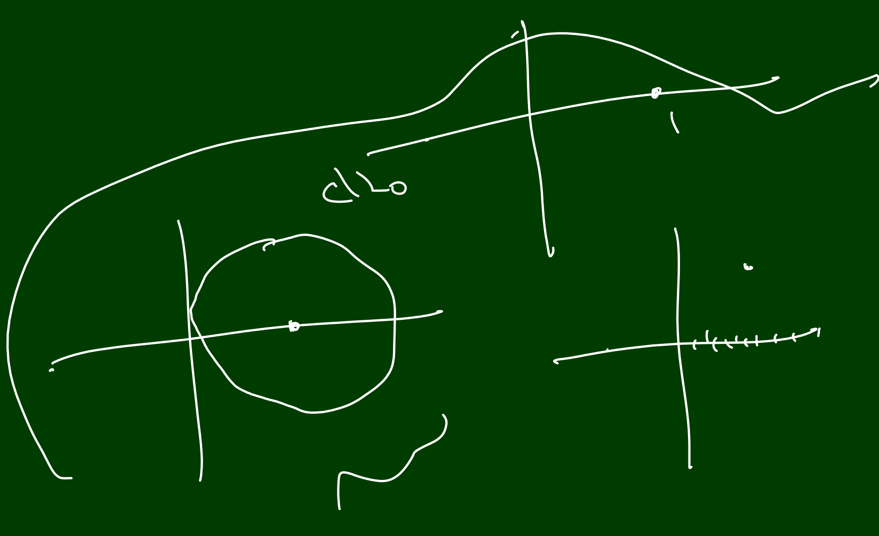
$y = \operatorname{Im} z + 2k\pi$

$x + iy = \underbrace{\operatorname{Re} z + i \operatorname{Im} z}_{=z} + i2k\pi = z + i2k\pi$

$$\operatorname{Log}(1) \overset{?}{=} \{ i2k\pi : k \in \mathbb{Z} \}$$

$\cancel{\arg_0}$

$$\arg_{-\pi}(1) = 0$$



Ex: S.T $\forall\, z \in \mathbb{C}^{\alpha}$, $\exists$ cont log on $B(z, |z|)$.

Fix $z_0 \in U$ $\quad U = B(z_0, |z_0|)$

Then $\exists$ conts logarithm on U

Real number

$a > 0$ $\quad x \in \mathbb{R}$ $\quad$ How to define a^x?

Let $a > 0$

$r \in \mathbb{Q}$ $\quad n \in \mathbb{N}$ $\quad a^n$

a^r $\quad a^{-n}$

If $n \in \mathbb{Z}$

a^n

Complex Analysis, Aug 19, 2021

Fix $a > 0$, $x \in \mathbb{R}$ what is a^x?

$y = a^x$

$\frac{dy}{dx} = a^x \log a$

$y = a^x$

$\log y = \log(a^x)$

$= x \log a$

$\log y$ =

$= \log a$

$\frac{1}{y}\frac{dy}{dx} = \log a$

$\frac{dy}{dx} = y \log a$

$= a^x \log a$

$a^x := \exp(x \ln a)$

P.T. $a^{x+y} = a^x \cdot a^y$

Let $\lambda \in \mathbb{C}^*$, let $x > 0$

$x^\lambda := \exp(\lambda \ln x)$

Law of exponents are satisfied

i^i

$25^{4+i} = ?$

$\lambda, z \in \mathbb{C}^*$

z^λ

$z^\lambda := \{ \exp(\lambda \cdot w) : w \in \operatorname{Log}(z) \}$

self study

1. Elementary Complex Nos
2. Sequences & series of Complex Nos.
3. Continuity
4. $\boxed{\text{Exponential map}}$ Self study
5. Differentiability (C.R. equations)

6. Integration
7. Ca
8
9
10. Rou

1) Argument & logarithm
Power fns:
2) power series

(a_n)

$\limsup(a_n)$, $\liminf(a_n)$

Let (a_n) be bdd sequence in $\mathbb{R}$

$t_1 := \text{lub}\{a_1, a_2, \dots\}$
$t_2 := \text{lub}\{a_2, a_3, \dots\}$
$t_3 := \text{lub}\{a_3, a_4, \dots\}$
$t_k = \text{lub}\{a_k, a_{k+1}, \dots\}$

$A_k := \{a_n : n \geq k\}$

$t_k := \text{lub}\, A_k$ $\qquad a_k \in A_k$

$A_{k+1} \subseteq A_k$

$t_{k+1} = \text{lub}\, A_{k+1} \leq \text{lub} A_k = t_k$

$\therefore t_{k+1} \leq t_k$

$(t_k) \downarrow$ $\qquad a_k \in A_k$

$\forall k \in \mathbb{N},\quad m \leq a_k \leq t_k$

$\therefore (t_k)$ converges to t where $t := \text{glb}\{t_k : k \in \mathbb{N}\}$

$t := \limsup a_n$

$s_k := \text{glb}\, A_k$ $\qquad A_{k+1} \subseteq A_k$

$\therefore \text{glb}\, A_{k+1} \geq \text{glb}\, A_k$

$s_{k+1} \geq s_k$

$\therefore (s_k) \uparrow$

$s_k = \text{glb}\, A_k$ & $a_k \in A_k$

$\forall k \in \mathbb{N}\quad s_k \leq a_k \leq M$

$\therefore (s_k)$ is bdd above

$s_k \to s$ $\qquad \therefore s = \text{lub}\{s_k : k \in \mathbb{N}\}$ & s is called $\liminf a_n$

Let (a_n) be the seq $(1, 2, 3, 1, 2, 3, \dots)$

$t_1 = 3$
$t_2 = 3$

$\forall k \in \mathbb{N},\ t_k = 3$ $\qquad t = \limsup a_n = 3$

$\therefore t_k \to t$

$s = 1$ $\qquad \liminf a_n = 1$

Ex: 1. $a_n = (-1)^n$ $\qquad$ 2) $a_n = \frac{1}{n}$ $\qquad$ 3) $a_n = 1 - \frac{1}{n}$

Compare s t p.T $s \le t$
liminf limsup

$s_k \to s$
$t_k \to t$

$s_k \le a_k \le t_k$

$\therefore \forall k \in \mathbb{N}, \; s_k \le t_k$

$t_k - s_k \geq 0$

$\downarrow t - s$

$\therefore t - s \geq 0$

$s \le t$ ✓

$a_n = (100, 1, 2, 3, 4, 1, 2, 3, \ldots \ldots)$

$a_n \to a$, $a = \lim a_n$ $\left(\forall \epsilon > 0 \left(\exists N \in \mathbb{N} \left(\forall n \geq N \left(a_n \in (a - \epsilon, a + \epsilon) \right) \right) \right) \right)$

Thm (lim sup is t)

(i) $\forall \epsilon > 0 \left(\exists N \in \mathbb{N} \left(\forall n \geq N \left(a_n < t + \epsilon \right) \right) \right)$

(ii) $\forall \epsilon > 0$ there are infinitely many n s.t. $a_n > t - \epsilon$

i.e $\{ n \in \mathbb{N} : a_n > t - \epsilon \}$ is an infinite set

— x —

Complex Analysis, Aug 23, 2021.

Given a bdd seq (a_n) of real numbers

$t_k = \text{lub } A_k$ where $A_k = \{a_n : n \geq k\}$

Then $(t_k) \downarrow$ & $t = \text{glb } t_k$

Then t is called $\limsup a_n$

$s_k = \text{glb } A_k$ Then $(s_k) \uparrow$ & $s = \text{lub } s_k$

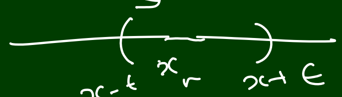
s is called $\liminf a_n$

Recall

(i) $s \leq t$

$x_n \to x$ $\forall n \geq N$

$\exists N$,



$x = \lim x_n$

n_2 n_3

$t = \limsup t_k$

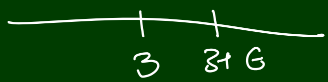
$(a_n) = (100, 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots)$

a_4 a_7 a_{10}

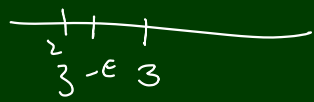
$t = 3$

Thm (lim sup)

(i) $\forall \epsilon > 0 \left(\exists N \in \mathbb{N} \left(\forall n \geq N \left(a_n < t + \epsilon \right) \right) \right)$



(ii) $\forall \epsilon > 0$, there are infinitely many $n \in \mathbb{N}$ s.t $a_n > t - \epsilon$

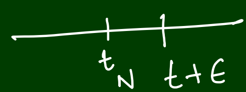


Proof Let $\epsilon > 0$.

B

$t = \text{glb}\{t_k : k \in \mathbb{N}\}$

$\therefore t + \epsilon$ is not a lb for B



$\exists N \in \mathbb{N}$ s.t $t_N < t + \epsilon$

$\because t_N = \text{lub} \cdot A_N$

$\therefore t + \epsilon$ is an u.b for A_N,

$\therefore \forall k \geq N, \quad a_k \leq t_N < t + \epsilon$

(2) Extract a subsequence of (a_n) s.t each term is greater than $3 - \epsilon$

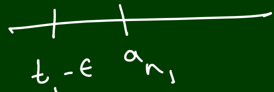
$\forall j \in \mathbb{N} \quad t \leq t_j$.

$\forall j \in \mathbb{N} \quad t - \epsilon \leq t_j - \epsilon$

Choose $j = 1$

$t_1 - \epsilon$ is not u.b for A_1

$\therefore \exists n_1 \in \mathbb{N}$ s.t $a_{n_1} > t_1 - \epsilon$



For $j = n_1 + 1$

$\therefore \exists n_2 \in \mathbb{N}$ s.t $a_{n_2} > t_{n_1+1} - \epsilon \geq t - \epsilon$

$a_{n_2} \in A_{n_1+1} = \{a_{n_1+1}, a_{n_1+2}, \dots\}$

$\therefore n_2 \geq n_1 + 1 > n_1$

Proceeding like this

For $j = n_k + 1$

$\exists n_{k+1} \in \mathbb{N}$ s.t $a_{n_{k+1}} > t_{n_k+1} - \epsilon$

$\therefore n_{k+1} \geq n_k + 1 > n_k$

$\therefore \forall k \in \mathbb{N}$ $a_{n_{k+1}} > t_{n_k+1} - \epsilon \geq t - \epsilon$

$\therefore \forall k \in \mathbb{N}$ $a_{n_{k+1}} > t - \epsilon$

$\therefore \{n_k : k \in \mathbb{N}\} \subseteq \{n \in \mathbb{N} : a_n > t - \epsilon\}$

Infinite set

$(100, 1, 2, 3, 1, 2, 3, \dots)$

Thm (lim inf)

(i) $\forall \epsilon > 0 (\exists N \in \mathbb{N} (\forall n \geq N (s - \epsilon < a_n)))$

(ii) $\forall \epsilon > 0$, $\{n \in \mathbb{N} : a_n < s + \epsilon\}$ is infinite

Exercise

Thm $a_n \to a$ iff $\limsup a_n = \liminf a_n = a$

(i.e $t = s = a$)

$\Leftarrow$ Let $\epsilon > 0$

By Thm 1, $\exists N_1 \in \mathbb{N}$ s.t $\forall n \geq N_1$ $a_n < t + \epsilon$

By Th 2, $\exists N_2 \in \mathbb{N}$ s.t $\forall n \geq N_2$ $a_n > s - \epsilon$

$\forall n \geq N$

$N = \max\{N_1, N_2\}$

$\forall n \geq N$, $\quad s - \epsilon < a_n < t + \epsilon$

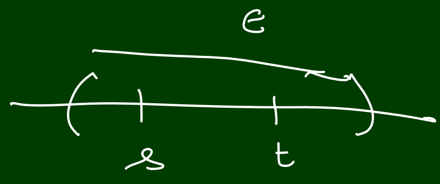
Given $s = t = a$ $\quad \therefore a - \epsilon < a_n < a + \epsilon$

. QED

$\Rightarrow$ Let $a_n \to a$

Claim (i) $s = t$

(ii) $s = t = a$



W.L.T $s \leq t$

Let $\epsilon > 0$.

To P.T $|s - t| < \epsilon$

$\because a_n \to a$, $\exists N \in \mathbb{N}$ s.t $\forall n \geq N$, $\quad a - \epsilon/2 < a_n < a + \epsilon/2$

$$a - \epsilon/2 < s_N \leq s \leq t \leq t_N < a + \epsilon/2$$

$\therefore |s - t| < \epsilon$

$\therefore s = t$

P.T $\quad t = a$ $\qquad a - \epsilon/2 < t < a + \epsilon/2$

$-\epsilon/2 < t - a < \epsilon/2$

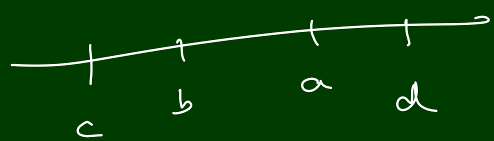
$\therefore |t - a| < \epsilon/2 < \epsilon$

Exercise

Let a, b be s.t $\quad a, b \in (c, d)$

where $\quad d - c = \epsilon$

S.T $|a - b| < \epsilon$



$\limsup\left(\frac{1}{n}\right) = 0$

$\frac{(-1)^n}{n}$

Power series

Sequences of complex no's

(Series)

Video lectures

$f :$

$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$

Find domain D s.o $\forall\, z \in D$

$f(z)$ is complete

i.e $\sum_{n=0}^{\infty} c_n (z-a)^n$,

(a_n) is seq

Given seq (a_n) it associated series $\sum a_n$

$\sum (-1)^n$

$s_n = \sum_{k=1}^{n} a_k$

$s_1 = -1$

$s_2 = 0$

$s_3 = -1$

$\sum a_n$ converges iff (s_n) converges

& if $s_n \to s$

Then $\boxed{\sum a_n = s}$

Complex Analysis, August 24, 2021

Given seq (a_n) in $\mathbb{C}$ the associated series $\sum_{n=1}^{\infty} a_n$

Def $\sum_n a_n$ converges if partial $S_n := \sum_{k=1}^{n} a_n$ is a convergent seq.

If $S_n \to S$ Then $S = \sum_n a_n$.

If the series does not converge then it is divergent

$x = 0.99999\ldots \longrightarrow$?

$y = 1$ Compare x & y

$x = 0.65$

$x = \frac{6}{10} + \frac{5}{10^2}$

0 .5 .6 .7 1

$\sum \frac{9}{10^n} = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots$

$\cancel{+ \frac{9}{10^n}}$

$\cancel{(1 + \frac{1}{10} + \cdots + \frac{1}{10^{n-1}})}$

$\frac{1}{1-r}$

$\frac{1}{1 - \frac{1}{10}}$

$\sum \frac{9}{10^n} = \frac{9}{10} \sum \frac{1}{10^{n-1}}$

$= \frac{9}{10} \cdot \frac{1}{\frac{9}{10}}$

$= 1$

Checking Answers.

$x = 0.99999\ldots$ — ①

$10x = 9.999\ldots$ — ②

(?)

② − ① ; $9x = 9$.

$\boxed{x = 1}$

Thm 1

If a series $\sum a_n$ converges then $a_n \to 0$

$S_n = a_1 + \cdots + a_n$

$S_{n-1} = a_1 + \cdots + a_{n-1}$

$a_n = S_n - S_{n-1}$

$S_n \to S$

$S_{n-1} \to S$

$\therefore a_n \to 0$ //

Contrapositive of Thm

(Thm) If $a_n \not\to 0$ then $\sum a_n$ does not converge

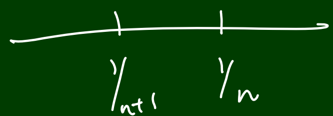
Is converse true of Thm

If $a_n \to 0$ then $\sum a_n$ converges

Ans No $\quad \sum \frac{1}{n}$ does not converge. (Why?)

$\sum \frac{1}{n^2}$ converges $\qquad \boxed{\sum \frac{1}{n^2} = \frac{\pi^2}{6}}$ → Proof Fourier series,

$\sum \frac{1}{n^2}$ converges.



$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

W.S.T $\quad \sum \frac{1}{n(n+1)}$ converges.

$$a_n = \frac{1}{n(n+1)}$$

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

$$s_1 = 1 - \frac{1}{2}$$

$$s_2 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3} = 1 - \frac{1}{3}$$

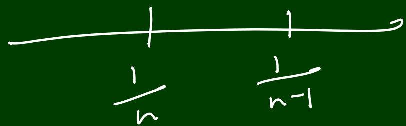
$\vdots$

$$s_n = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$s_n = 1 - \frac{1}{n+1}$$

$$\boxed{s_n \to 1} \qquad \boxed{\sum \frac{1}{n(n+1)} = 1}$$

$\sum_{2} \frac{1}{n(n-1)}$



$a_n = \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$

$n \geq 2$

$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots \left(\frac{1}{n-1} - \frac{1}{n}\right)$

$= 1 - \frac{1}{n}$

$\lim s_n = 1$

$n^2 > n^2 - n$

$\frac{1}{n^2} \leq \frac{1}{n^2 - n}$

$a_n \leq b_n$

$s_n \leq t_n \qquad \forall n \geq 2$

$s_n = a_1 + \cdots + a_n$

$\leq b_1 + \cdots + b_n$

$= t_n \leq t$

$\therefore s_n \leq t$

$\therefore (s_n)$ is convergent

If we series of non-negative terms

$s_1 = a_1$

$s_2 = a_1 + a_2$

$= s_1 + a_2$

$s_2 \geq s_1$

$\therefore s_{n+1} = s_n + a_n$

$\boxed{s_{n+1} \geq s_n}$

$\therefore (s_n)$ is $\uparrow$

If (a_n) is a non-neg seq then $\sum a_n$ Converges iff (s_n) is bdd above

[$(\Rightarrow)$ (s_n) converges $\therefore (s_n)$ is bdd $\therefore$ bdd above

$(\Leftarrow)$ $(s_n) \uparrow$ & bdd above then (s_n) converges to s

$\therefore s = \text{lub}\{s_n : n \in \mathbb{N}\}$]

$\forall n \in \mathbb{N}, \ s_n \leq s$

$\sum \frac{1}{n}$ diverges

$s_n = 1 + \cdot \quad + \frac{1}{n}$

E.S.T (s_n) is not bdd above

$s_{2^0} = 1$

$s_{2^1} = 1 + \frac{1}{2} >$

$3 < 4$

$\frac{1}{3} > \frac{1}{4}$

$s_{2^2} = \left(1 + \frac{1}{2}\right) + \left(\underbrace{\frac{1}{3} + \frac{1}{4}}_{> 2\left(\frac{1}{4}\right)}\right)$

$s_{2^2} = s_{2^1} + \left(\frac{1}{3} + \frac{1}{4}\right)$

$s_{2^2} > 1 + \frac{1}{2} + \frac{1}{2}$

$> 1 + 2\left(\frac{1}{2}\right)$

$s_{2^3} = s_{2^2} + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

$= s_{2^2} + \left(\underbrace{\frac{1}{2^2+1} + \frac{1}{2^2+2} + \frac{1}{2^2+3} + \frac{1}{2^2+2^2}}_{> \frac{2^2}{2^3}}\right)$

$s_{2^3} > 1 + 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2^3}\right)$

$= 1 + 2\left(\frac{1}{2}\right) + \frac{1}{2}$

$= 1 + 3\left(\frac{1}{2}\right)$

$2^2 + 1 < 2^3$

$\frac{1}{2^2+1} > \frac{1}{2^3}$

$S_{2^{k+1}} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\underbrace{\frac{1}{2^k+1} + \cdots + \frac{1}{2^k+2^k}}_{> \frac{2^k}{2^{k+1}} = \frac{1}{2}}\right)$

$S_{2^{k+1}} > 1 + (k+1)\frac{1}{2}$

$\forall k \in \mathbb{N}$

$\boxed{s_{2^k} > 1 + \frac{k}{2}}$

(s_n) is not bdd above

Let $\alpha \in \mathbb{R}$

$> \alpha$

$s_{2^k} > 1 + \frac{k}{2}$

'Geometric Series + Comparison test'

Mother of all series test? are the only important tests [1. Integral test is also test]

- Defn Given a seq (a_n) in $\mathbb{C}$
A series $\sum_n a_n$ is said to be absolutely convergent

if $\sum_n |a_n| < \infty$

1. absolutely converges $\Rightarrow$ converges.
2. Converges $\not\Rightarrow$ absolutely converges

$$\sum \frac{(-1)^n}{n}.$$

Let $\sum a_n$ be absolutely converge.

Claim $\sum a_n$ converges.

$$s_n = \sum_{k=1}^{n} a_k \qquad t_n = \sum_{k=1}^{n} |a_k|$$

$n > m$

$$s_n - s_m = \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k$$

$$= \sum_{k=m+1}^{n} a_k$$

$$|s_n - s_m| \le \sum_{k=m+1}^{n} |a_k| = t_n - t_m \le |t_n - t_m|$$

Complete the pf

Geometric Series

If $|z| < 1$ then $\sum z^n$ converges in $\mathbb{C}$

$$\text{Consider } \sum_{n=0}^{\infty} |z|^n$$

$$s_n = 1 + |z| + \dots + |z|^n$$

Let $|z| = r$

$$s_n = \underbrace{1 + r + \cdots + r^n}$$

$$r s_n = r + \cdots + r^n + r^{n+1}$$

$$s_n(1-r) = 1 - r^{n+1}$$

$$s_n = \frac{1 - r^{n+1}}{1-r}$$

$$= \frac{1}{1-r} - \frac{1}{1-r} \cdot r^{n+1}$$

$$\boxed{s_n \to \frac{1}{1-r}} \quad [\because r<1, \; r^n \to 0]$$

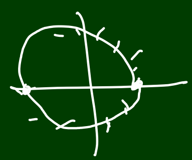
$\therefore \sum |z|^n$ converges $\frac{1}{1-|z|}$

& hence $\sum z^n$ converges

$a_n \to 0$ iff $|a_n| \to 0$

$\therefore$ If $|z|>1$, $\sum_{n=0}^{\infty} z^n$ diverges

$\forall n \in \mathbb{N}$ $|z^n| = |z|^n > 1$



$\therefore$ nth term does not go to zero $\therefore$ diverges

If $|z|=1$, diverges

Comparison test

If $\sum a_n, \sum b_n$ are given series of non-negative terms

$\forall n \in \mathbb{N}$, $a_n \le b_n$

(i) If $\sum b_n$ converges then $\sum a_n$ converges

(ii) If $\sum a_n$ diverges then $\sum b_n$ diverges.

$s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$ $\qquad t_n \to t$

$t = \text{lub}\{t_n : n \in \mathbb{N}\}$

$s_n \le t_n \le t$

$\therefore (s_n)$ is bdd

$\therefore \sum a_n$ Convergent

(ii) $s_n \le t_n$

Supp (t_n) is bdd above the (s_n) is bdd above

$\therefore \sum a_n$ converges

$\Rightarrow \Leftarrow$

$\therefore (t_n)$ is not bdd above

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

S.T the series in the R.H.S converges absolu for all $z \in \mathbb{C}$

(Compare with G. Ser & c

$$\left|\frac{z^n}{n!}\right| = \frac{|z|^n}{n!}$$

$N+1 > N$

$n \ge N$

$$= \frac{|z|}{1} \cdot \frac{|z|}{2} \cdot \ \frac{|z|}{n}$$

$$= \frac{|z|}{1} \cdot \frac{|z|}{2} \cdots \frac{|z|}{N} \cdot \frac{|z|}{N+1} \cdots \frac{|z|}{n}$$

By A.P $\forall N \in \mathbb{N}$ $\quad \frac{|z|}{N} < \frac{1}{2}$

$$\le \frac{|z|^N}{N!} \cdot \frac{|z|}{N} \cdots \frac{|z|}{N}$$

$$\le \frac{|z|^N}{N!} \left(\frac{|z|}{N}\right)^{n-N}$$

$$\le \frac{|z|^N}{N!} \left(\frac{1}{2}\right)^{n-N}$$

$$\sum_{n \ge N} \frac{|z|^n}{n!} \le \frac{|z|^N}{N!} \sum_{n \ge N} \left(\frac{1}{2}\right)^{n-N}$$

$$\le \frac{2|z|^N}{N!}.$$

$$\sum_{n=1}^{N} a_n + \boxed{\sum_{n=N+1}^{\infty} a_n} = \sum_{n=1}^{\infty} a_n$$

$\sum_{n=1}^{\infty} \frac{|z|^n}{n!}$ Converges

Complex Analysis, Aug 25, 2021

Direct Comparison: Let $(a_n)(b_n)$ be non negative sequences s.t

$\forall n \in \mathbb{N} \quad a_n \le b_n$

(1) If $\sum b_n$ converges then $\sum a_n$ converges

(2) If $\sum a_n$ diverges then $\sum b_n$ diverges.

$\frac{1}{n^2} \le \frac{1}{n(n-1)}$

$\frac{1}{n(n+1)}$

Exercise

1. Limit comparison test

Let $(a_n),(b_n) > 0$ &

$$\lim \frac{a_n}{b_n} = l$$

If $l > 0$ Then $\sum a_n$ & $\sum b_n$ both converge or both diverge

2. Let $(a_n),(b_n) > 0$ & $\forall n$ $\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}$.

(i) If $\sum b_n$ converges then $\sum a_n$ converges

(ii) If $b_n \to 0$ then $a_n \to 0$

Geometric + Comparison

$\frac{1}{n(n+1)} \times \frac{n^2}{1}$

$\lim_n \frac{1}{1+\frac{1}{n}} = 1$

Ratio test

Let $\forall n$ $a_n > 0$

Let $\limsup \frac{a_{n+1}}{a_n} = r$

If $r < 1$ then $\sum a_n$ converges

If $r > 1$ then $\sum a_n$ diverges.

If $r = 1$ the test fails

$\sum 1/n$ $\{1/n^2\}$

$\lim \frac{1}{n+1} \times \frac{n}{1} = 1$

$\lim \frac{1}{(n+1)^2} \cdot \frac{n^2}{1} = 1$

Proof

$$\frac{a_{n+1}}{a_n} \to r$$

Let $r < 1$

choose $s \in (r, 1)$

$\therefore \limsup \frac{a_{n+1}}{a_n} < s$

$\therefore \exists N \in \mathbb{N}, \forall n \ge N,$ $\frac{a_{n+1}}{a_n} < s$

$\forall\, n \geq N \quad a_{n+1} < s\, a_n$

(i.e.) $\quad a_{N+1} < s\, a_N$

$a_{N+2} < s\, a_{N+1}$
$< s^2 a_N.$

$\therefore \forall\, k \in \mathbb{N} \quad a_{N+k} < s^k a_N$

By Comp $\quad \sum_{k=1}^{\infty} a_{N+k}$ converges $\left(\because\ a_N \sum_k s^k < \infty\right)$

$\sum_{n=N+1}^{\infty} a_n$ converges

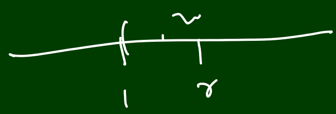
$\therefore \quad \sum_{n=1}^{\infty} a_n$ converges $(?)$.

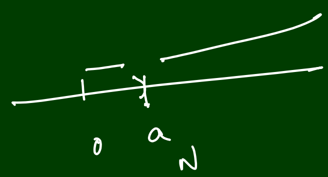
If $\sum a_n < \infty$

Then $\sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n$

Let $r > 1$

$\limsup \frac{a_{n+1}}{a_n} = r > 1$





$\exists\, N \in \mathbb{N}, \quad \forall\, n \geq N, \quad \frac{a_{n+1}}{a_n} > 1$

$\forall\, n \geq N, \quad a_{n+1} > a_n$

$\therefore \forall\, n \geq N, \quad a_{n+1} > a_N$

$\therefore \quad a_n \not\to 0$

Ratio-test

If $(a_n) \geq 0$ & $\limsup a_n^{1/n} = r$

If $r < 1$ $\quad \sum a_n$ Converges

If $r > 1$ $\quad \sum a_n$ diverges

If $r = 1$ $\quad$ the test fails

$\because \limsup a_n^{1/n} < s$

$\therefore \exists N \in \mathbb{N}$ s.t $\forall n \geq N \quad a_n^{1/n} < s$

$\forall n \geq N, \quad a_n < s^n$

$\because \sum_{n \geq N} s^n < \infty, \quad \sum_{n \geq N} a_n < \infty$

& hence $\sum_{n=1}^{\infty} a_n < \infty$

$\lim a_n^{1/n} > 1$

$\exists N \in \mathbb{N}, \ \forall n \geq N, \quad a_n^{1/n} > 1$

$a_n > 1$

$\forall n \geq N$

$\therefore a_n \not\to 0$

$\therefore \sum a_n < \infty$

If $r = 1$, $\quad \sum \frac{1}{n}$

$\left(\frac{1}{n}\right)^{\frac{1}{n}} = \frac{1}{n^{1/n}}$

$\lim a_n^{1/n} = 1$

$\limsup a_n^{1/n} = 1$.

$n^{1/n} \to 1$

If $a \geq 0$

$a^{1/n} \to 1$

case $0 < a < 1$

$\frac{1}{a} > 1$

$\therefore \left(\frac{1}{a}\right)^{1/n} \to 1$

$\therefore \frac{1}{a^{1/n}} \to 1$

$\therefore a^{1/n} \to 1$

$\sum \frac{1}{n^2} \qquad \limsup \frac{1}{n^{2/n}} = 1$.

Let $\sum_{n \in \mathbb{N}} a_n$ be a convergent series

What is the meaning of $\sum_{n=N+1}^{\infty} a_n$

It is also denoted $\sum_{n \geq N} a_n$ & called as tail of the series $\sum a_n$

$\forall k \in \mathbb{N} \quad b_k := a_{N+k}$

$s_n = \sum_{k=1}^{n} a_k \quad \therefore \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_{N+k} = \sum_{n=N+1}^{\infty} a_n$

$t_k = b_1 + \cdots + b_k$

$= a_{N+1} + \cdots + a_{N+k}$

$= s_{N+k} - s_N$

W.K.T
$s_n \to s$
(s_{N+k}) is subseq of (s_n)
$\therefore s_{N+k} \to s \quad \therefore t_k \to s - s_N$

To P.T. $\sum a_n = \sum_{n=1}^{N} a_n + \sum_{n\geq N} a_n$

Let $\sum a_n = s$

$\therefore$ P.T $\sum_{n \geq N} a_n = s - s_N$

$\therefore$ To P.T $\sum_{k=1}^{\infty} b_k = s - s_N$

Thm (Tail) Let $\sum a_n$ be convergent

Then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t $\sum_{n \geq N} a_n < \epsilon$

Proof (Take all the not def in before)

Let $\epsilon > 0$
$\because s_n \to s$, $\exists N \in \mathbb{N}$, $\forall n \geq N$, $s_n \in (s-\epsilon, s+\epsilon)$

In par $s_N \in (s-\epsilon, s+\epsilon)$

$\therefore s - \epsilon < s_N$

$s - s_N < \epsilon$

i.e $\sum_{n\geq N} a_n < \epsilon$

Exer S.T $\ell^1 \subseteq \ell^2$

Let $(a_n) \in \ell^1$

$\therefore \sum |a_n| < \infty$

S.T $\sum |a_n|^2 < \infty$

Let $a_n \geq 0$. If $\sum a_n < \infty$ then $\sum a_n^2 < \infty$.

Harmonic Series

$$\sum \frac{1}{n^p} < \infty \text{ if } p > 1$$

$$\sum \frac{1}{n^p} \text{ diverges if } 0 < p \leq 1$$

$$\frac{2^k}{2^{p(k+1)}} < \sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n^p} < \frac{2^k}{2^{pk}}.$$

$n < 2^{k+1}$

$n^p < 2^{p(k+1)}$

$\frac{1}{n^p} > \frac{1}{2^{p(k+1)}}$

$n > 2^k$

$n^p > 2^{pk}$

$\frac{1}{n^p} < \frac{1}{2^{pk}}$

If $p > 1$

$$\sum_{k=0}^{\infty} \sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n^p} \leq \sum_k \left(\frac{2}{2^p}\right)^k$$

$p > 1$

$2^p > 2$

$\frac{2}{2^p} < 1$

If $p \leq 1$ "

$$\geq \frac{1}{2^p} \sum_k \left(\frac{2}{2^p}\right)^k$$

$2^p \leq 2$

$\frac{2}{2^p} \geq 1$

diverges.

Seminar

1. Abel's Test (Seminar).
2. Cauchy product ( )

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

$\sum a_n = A$

$\sum b_n = B$

$\sum c_n = AB$

Find all $z \in \mathbb{C}$ s.t. ~~Series~~ converges

$\sum_n z^n$, $D = \{z \in \mathbb{C} : |z| < 1\}$

$\sum_n \frac{z^n}{n!}$, $D = \mathbb{C}$

$\sum_n n^n z^n$, $D = \{0\}$ If $z \neq 0$

Complex Analysis, Aug 26, 2021

$$\sum n^n z^n$$

If $z \neq 0$ then $\sum n^n z^n$ diverges.

T.S.T $n^n z^n \not\to 0$

$\forall n \geq N$

$\therefore |n^n z^n| > 1$

Find N s.t $|N^N z^N| > 1$

Write text book proof

1. $\frac{1}{n^2} \leq \frac{1}{n(n-1)}$

$n^2 > n^2 - n$

2. $n^2 + n^2 \geq n^2 + n$

$\frac{1}{2n^2} \leq \frac{1}{n(n+1)}$

$\frac{1}{2} \sum \frac{1}{n^2} \leq \sum \frac{1}{n(n+1)}$

$N|z| > 1$ $\quad \frac{1}{N} < |z|$

By prop $\exists N \in \mathbb{N}$ s.t $\frac{1}{N} < |z|$

$\therefore N|z| > 1$

$\therefore N^N |z|^N > 1$

$\therefore \forall n \geq N, \quad n^n |z|^n \geq N^N |z|^N > 1$

$\therefore |(n|z|)^n| > 1$

$\therefore$ nth term does not go to zero

$\forall n \geq N$
$a_n > 1$
Then $a_n \not\to 0$
Suppose $a_n \to 0$

0 $\quad$ 1/2 $\quad$ 1

$\exists N_1 \in \mathbb{N}$
$\forall n \geq N_1, \quad a_n < 1$ — (2)
$N' = \max\{N, N_1\}$
$\therefore$ by (2) $a_{N'} < 1$
by (1) $a_{N'} > 1$
$\Rightarrow \Leftarrow$

1) $\sum_{n=0}^{\infty} z^n$, $D = B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$

2) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, $D = \mathbb{C}$

so $\forall z \in D$, the series converges

3) $\sum_{n=0} n^n z^n$, $D = \{0\}$

Given $a \in \mathbb{C}$, a power series is a series of the form
$$\sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{when } (a_n) \text{ in } \mathbb{C}$$

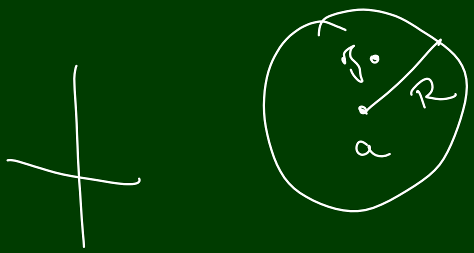
Thm Given a power series $\sum_{n=0}^{\infty} a_n (z-a)^n$

$\exists\, R$ s.t $0 \le R \le \infty$ s.t

$\forall\, z \in \mathbb{C}$ with $|z-a| < R$, the given series converges,

$\forall\, z \in \mathbb{C}$ with $|z-a| > R$, the given series diverges.

This R is called radius convergence of the power series.



$f : B(a,R) \to \mathbb{C}$,
$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$
$$f_n(z) = \sum_{k=0}^{n} a_k (z-a)^k$$

$\forall\, z \in B(a,R)$, $f_n(z) \to f(z)$

Proof We will find R s.t $\sum a_n (z-a)^n$ converges absolute for all z s.t $|z-a| < R$.

For the sake of simplicity (of course with coaching class exam in mind)

Assume $a = 0$

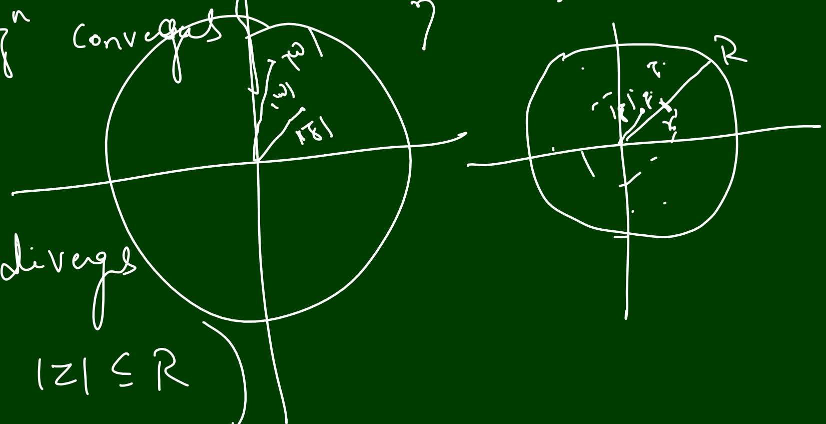
$\therefore$ Let the given series be $\sum_{n=0}^{\infty} a_n z^n$

Find R s.t if $|z| < R$, the series $\sum a_n z^n$ converges absolute

If $|r| < r$ ..

✓ $R := \text{lub}\{ |z| : \sum a_n z^n \text{ converges} \}$?

$R \ge 0$ ✓ ✓



If $|z| > R$ then $\sum a_n z^n$ diverges

(because If $\sum a_n z^n$ converges then $|z| \le R$)

T.P.T If $|z|<R$ then $\sum a_n z^n$ converges

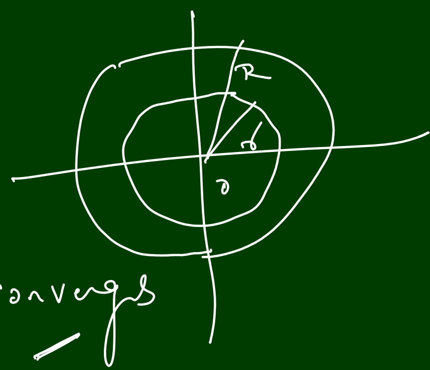
Let $0 < r < R$

Let $|z| \le r$

$r < R$

$\therefore \exists\, z_0 \in \mathbb{C}$ s.t $|z_0| > r$ and

$\sum a_n z_0^n$ converges



We have to p.T. $\sum a_n z^n$ converges absolutely.

{ We use comparison test. Find the series series to compare }

$\therefore a_n z_0^n \to 0$ $\therefore (a_n z_0^n)$ is a bdd seq

$\therefore \exists\, M > 0$ s.t $\forall n \in \mathbb{N}$, $|a_n z_0^n| \le M$

$|z_0| > r$

$\therefore \dfrac{r}{|z_0|} < 1$

$$
\begin{aligned}
|a_n z^n| &= |a_n|\,|z|^n \\
&\le |a_n|\, r^n \\
&= |a_n|\, r^n \cdot \frac{|z_0|^n}{|z_0|^n} \\
&= |a_n| \left(\frac{r}{|z_0|}\right)^n |z_0|^n \\
&= |a_n|\,|z_0|^n \cdot \left(\frac{r}{|z_0|}\right)^n \\
&= |a_n z_0^n| \left(\frac{r}{|z_0|}\right)^n \\
&\le M \left(\frac{r}{|z_0|}\right)^n
\end{aligned}
$$

$\therefore \sum |a_n z^n|$ converges (by comparison test)

$\therefore \sum a_n z^n$ converges absolutely $\therefore \sum a_n z^n$ converges

$\therefore \ \forall \eta \in \mathbb{C}$, s.t $|\eta| \leq r < R$, $\sum a_n \eta^n$ converges absolutely & hence converges.

Let $\eta \in \mathbb{C}$, s.t $|\eta| < R$

choose r s.t $|\eta| < r < R$

Exercise: Copy cat the same for $w = \eta - a$

Let $\sum a_n (\eta - a)^n$ be a power series with radius of Convergence R

Let $0 < r < R$

$f_n, f : B[a,r] \to \mathbb{C}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

$$f_n(z) = \sum_{k=0}^{n} a_k (z-a)^k$$

We have proved that, $\forall z \in B[a,r]$, $f_n(z) \to f(z)$

$f_n \to f$ pointwise on $B[a,r]$

Is the convergence uniform.

$\forall n \geq N$

$\forall \eta \in B[a,r]$

$$|f(z) - f_n(z)| \leq \left| \sum_{n \geq N} a_n (\eta - a)^n \right|$$

(Weierstrass M-test)

$$\leq \sum_{n \geq N} |a_n| |\eta - a|^n < \epsilon$$

$$\leq \sum_{n \geq N} M \left(\frac{r}{|\eta_0|}\right)^n$$

Complex Analysis, Aug 17, 2020.

Power Series $\sum a_n(z-a)^n$, (a_n) in $\mathbb{C}$, $a \in \mathbb{C}$. $\gamma \in A \subset \mathbb{C}$

Thm Given a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ where (a_n) in $\mathbb{C}$ & $a \in \mathbb{C}$

$\exists R \in [0, \infty]$ s.t

$B(a,R)$

(i) $\forall z \in \mathbb{C}, |z-a| < R$, $\sum a_n(z-a)^n$ converges

(ii) $\forall z \in \mathbb{C}, |z-a| > R$, $\sum a_n(z-a)^n$ diverges

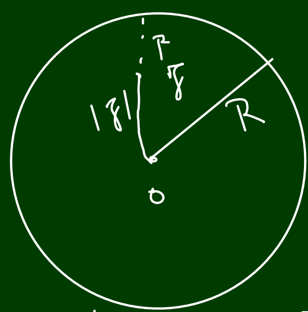
This R is called the radius of convergence.

Proof

For the sake of simplicity consider $\sum a_n z^n$

How to capture this R?

* LUB *



Define $S := \{ |\gamma| : \sum a_n \gamma^n$ converges in $\mathbb{C} \}$

& $R := \text{lub } S$ {Note R exists as we are in extended $[0, \infty]$}

Case(i) Suppose $|\gamma| > R$

$|\gamma| \notin S$ [For if, $|\gamma| \in S$, $|\gamma| \leq R$]

$\therefore \sum a_n \gamma^n$ diverges (by defn of S)

Case(ii) Let $|\gamma| < R$

claim: $\sum a_n \gamma^n$ converges

We S.T the $\sum a_n \gamma^n$ absolutely convergent & hence convergent

Choose $r \in \mathbb{R}$ s.t $0 < |\gamma| \leq r < R$

$\therefore r$ is not u.b for S

$\therefore \exists \gamma_0 \in \mathbb{C}$ s.t $|\gamma_0| > r$ & $\sum a_n \gamma_0^n$ converges.

$\therefore (a_n \gamma_0^n)$ is bdd

$\therefore \exists M > 0$ $\forall n \in \mathbb{N}, |a_n \gamma_0^n| \leq M$

$\frac{r}{|\gamma_0|} < 1$ $\therefore \sum \left(\frac{r}{|\gamma_0|}\right)^n < \infty$

Comparison test

If $a_n \geq 0$ & $b_n \geq 0$ $\forall n \in \mathbb{N}$

& $a_n \leq b_n$

(i) If $\sum b_n$ converges then $\sum a_n$ converges:

(ii) If $\sum a_n$ diverges then $\sum b_n$ diverges

For all z s.t $|z| \le r$

$$|a_n z^n| = |a_n|\,|z|^n$$
$$\le |a_n|\, r^n$$
$$= |a_n|\,|z_0|^n \cdot \frac{r^n}{|z_0|^n}$$
$$= |a_n z_0^n| \cdot \left(\frac{r}{|z_0|}\right)^n$$
$$\le M \cdot \left(\frac{r}{|z_0|}\right)^n$$

$\therefore$ The geometric series $\sum \left(\frac{r}{|z_0|}\right)^n < \infty$

$\therefore$ By Comparison test, $\sum |a_n z^n| < \infty$

Every absolutely convergent Series is convergent

$\sum a_n z^n$ converges in $\mathbb{C}$

x — —

Observe that we have proved something more.

$$f : B(0,R) \to \mathbb{C}$$
$$f(z) = \sum a_n z^n$$
$$f_n : B(0,R) \to \mathbb{C}$$
$$f_n(z) = \sum_{k=0}^{n} a_k z^k$$

$$\forall \epsilon > 0 \left(\exists n_0 \in \mathbb{N} \left(\forall n \ge n_0 \left(\forall z \in B(0,R) \left(|f_n(z) - f(z)| < \epsilon \right) \right) \right) \right).$$

i.e The power series uniformly converges on $B(0,R)$

$n \ge n_0$ $\forall z \in B[0,r]$, where $r < R$

$$|f_n(z) - f(z)| = |f_n(z) - f_m(z) + f_m(z) - f(z)|$$

$$\boxed{\begin{matrix} m \ge n_0 \\ m \ge n_1(z) \end{matrix}}$$

$$\le |f_n(z) - f_m(z)| + |f_m(z) - f(z)|$$

$$< \epsilon/2 + \epsilon/2$$

$$|f_n(z) - f_m(z)| = \left|\sum_{k=m+1}^{n} a_k z^k\right|$$

$$\leq \sum_{k=m+1}^{n} |a_k z^k|$$

$$\leq M \sum_{k=m+1}^{n} \left(\frac{r}{|z_0|}\right)^k$$

$$\leq M(t_n - t_m)$$

$\because (t_n - t_m)$ is Cauchy (f_n) is uniformly Cauchy. Weistress M-test

— x —

Examples

1. $\sum z^n$
2. $\sum \frac{z^n}{n^2}$
3. $\sum \frac{z^n}{n}$

Geometric series
Comparison test

Ans 1. $R = 1$, on $|z| = 1$, $\sum z^n$ diverge.

2. $R = 1$, on $|z| = 1$, $\sum \frac{z^n}{n^2}$ converges. $a_n = \frac{|z|^n}{n^2}$

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} \frac{1}{|z|} \qquad a_{n+1} = \frac{|z|^{n+1}}{(n+1)^2}$$

$R = 1$

$|z| < 1$ $\quad$ $|z| > 1$

$|z| = 1$,

$|z| < 1$

$c = (|z|, |z|, \ldots)$

$$\sum \frac{|z|^n}{n^2} = \langle c, (\tfrac{1}{n^2}) \rangle$$

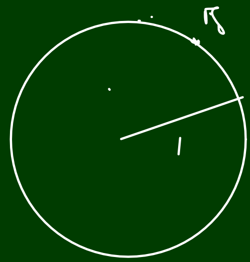
$$\leq \|c\|_2 \, \|(\tfrac{1}{n^2})\|_2$$

$< \infty$

3. $\sum \frac{z^n}{n}$

$\boxed{R = 1}$

On $|z| = 1$



$z = 1$, $\sum \frac{z^n}{n}$ diverges

$z = (-1)$, $\sum \frac{z^n}{n}$ converges

Good Exercise: $z \neq 1$ on $|z| = 1$, $\sum \frac{z^n}{n}$ converges.

Ratio test / Root test

Consider the power series $\sum a_n (z-a)^n$

Δ Assume one of the following limit exits

(i) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$

(ii) $\lim_{n\to\infty} |a_n|^{1/n} = \rho.$

Then $R = \frac{1}{\rho}$

Case (i) $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exits $=\rho.$

$$\frac{|a_{n+1}(z-a)^{n+1}|}{|a_n(z-a)^n|} = \frac{|a_{n+1}|}{|a_n|}|z-a|$$

$$\to \rho.|z-a|$$

Complex Analysis, Aug 18, 2020

$\sum a_n, \quad a_n \geq 0$

Comparison test

If $(a_n) \geq 0$ & $(b_n) \geq 0$ s.t

$\forall n \in \mathbb{N}, \quad a_n \leq b_n$

(i) if $\sum b_n$ converges then $\sum a_n$ converges

(ii) If $\sum a_n$ diverge then $\sum b_n$ diverges.

(iii) $\exists N \in \mathbb{N}$, s.t $\forall n \geq N$, $|a_n| \leq b_n$ & $\sum b_n$ is convergent then $\sum a_n$ is absolutely convergent

Proof $\quad s_n = \sum_{k=0}^{n} a_n, \quad t_n = \sum_{k=0}^{n} b_n$

Given $t_n \to t$ where $t = \text{lub}\{t_n : n \geq 0\}$

we have $\forall n \in \mathbb{N}$ $s_n \leq t_n \leq t$

$\therefore \forall n \in \mathbb{N}, \ s_n \leq t$

$\therefore \sum a_n$ Converges i.e $s_n \to s$

Then $s = \text{lub}\{s_n : n \geq 0\}$

$\therefore s \leq t$

$\therefore \sum a_n \leq \sum b_n$

(ii) $s_1 = a_1$ $\qquad a_n \geq 0$ $\qquad 1, -1, 1, -1,$

$s_2 = a_1 + a_2$ $\qquad \therefore s_n \geq 0$ $\qquad 1, 2, 3, 1, 2, 3, \ldots$

$s_3 = a_1 + a_2 + a_3$ $\qquad a_n \to \infty$

$s_n \uparrow$ & unbounded.

$\forall n \in \mathbb{N}$ $t_n \geq s_n$

Given $M \in \mathbb{R}$, $\exists N \in \mathbb{N}$

$\therefore \forall k \geq N, \ a_k > M$

$\therefore$ Given $M \in \mathbb{R}$ $\exists N \in \mathbb{N}$

$\forall k \geq N, \quad s_k > M$

$\forall k \geq N, \ t_k \geq s_k > M$

$\therefore t_n \to \infty$ In particular (t_n) is unbounded

$c_0 = |a_N|$ $\qquad d_0 = b_N$

$c_1 = |a_{N+1}|$

$\forall\, k \in \mathbb{N}$ $\quad c_k = |a_{N+k}|$ $\qquad d_k = b_{N+k}$

$\forall\, k \in \mathbb{N}$ $\quad c_k \leq d_k$

$\sum_{k=0}^{\infty} c_k$ is converg

$\sum_{n=N}^{\infty} |a_n|$ is converg. (by (i)).

$\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|$.

$\therefore$ $\sum_{n=0}^{\infty} a_n$ is absolutely conver

$\therefore$ $\sum_{n=0}^{\infty} a_n$ is convergent. (Proof in next step)

Thm

In $\mathbb{C}$, every absolutely Convergent Series in $\mathbb{C}$ is Convergent

Proof Let $\sum_{n=1}^{\infty} a_n$ be a series s.t $\sum_{n=1}^{\infty} |a_n| < \infty$

$s_n = \sum_{k=1}^{n} a_k$, $\quad t_n = \sum_{k=1}^{n} |a_k|$.

To S.T (s_n) converges. Ex.

E.S.T (s_n) is Cauchy [In $\mathbb{C}$, every Cauchy Seq in $\mathbb{C}$ converges]

$n > m$

$$|s_n - s_m| = \left|\sum_{k=m+1}^{n} a_k\right|$$

$$\leq \sum_{k=m+1}^{n} |a_k|$$

$$= t_n - t_m$$

$\therefore$ (t_n) is cauchy the result.

Root test

Let $c_n > 0$ $\forall n \in \mathbb{N}$ & Assume $\frac{c_{n+1}}{c_n} \to r$

Then

(i) (c_n) converges if $0 \le r < 1$

(ii) (c_n) diverges if $r > 1$

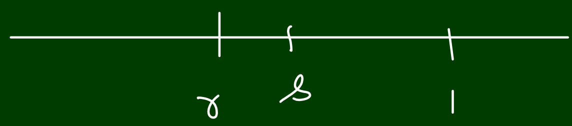
If $r = 1$, the result is inconclusive (Look at $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$)

Proof

(i) $\frac{c_{n+1}}{c_n} \to r$

& $0 \le r < 1$

To P.T. $\sum c_n$ converges.



Choose s s.t. $r < s < 1$

$\frac{c_{n+1}}{c_n} \to r$

∴ For $\epsilon = s - r$, $\exists N \in \mathbb{N}$, $\forall n \geq N$, $\frac{c_{n+1}}{c_n} < s$

$c_{n+1} < s c_n$

$c_{N+1} < s c_N$

$c_{N+2} < s c_{N+1} < s^2 c_N$

$\vdots$

$c_{N+K} < s^K c_N$

(by (iii) $\sum c_n$ converges)

(ii) $r > 1$,

$\forall n \geq N$ $\frac{c_{n+1}}{c_n} > 1$ $\forall k \in \mathbb{N}$ $c_{N+K} > c_N$

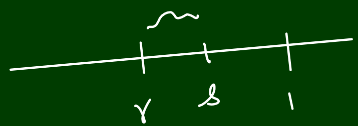
Root test

Let (c_n) be seq of posit reals

Assume $c_n^{1/n} \to r$

Then (i) (c_n) converges if $0 \leq r < 1$

(ii) (c_n) diverges if $r > 1$



$\exists N \in \mathbb{N}, \forall n \geq N,$ $\quad c_n^{1/n} < s$

$c_n < s^N$

$c_{N+k} < s^{N+k}$

Coming to C.A.

$$\sum a_n z^n$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$$

Then $R = \frac{1}{\rho}$.

'Cauchy-Hadamard formula'

lim sup, lim inf } of a sequence

Review Series using Real Analysis, S. Ajit k, Ku.

Bye.

Complex Analysis, 19th Aug, 2020.

lim sup. lim inf

Let (a_n) be bounded seq in $\mathbb{R}$

$t_1 := \text{lub}\{a_n : n \geq 1\}$

$t_2 := \text{lub}\{a_n : n \geq 2\}$

$A_k := \{a_n : n \geq k\}$

$$\boxed{t_n := \text{lub} A_n = \text{lub}\{a_n, a_{n+1}, \ldots\}}$$

Observe

1) $t_1 \geq t_2 \geq t_3 \ldots$

$\checkmark$ (t_n) is $\downarrow$

2) (t_n) is bdd below

Let $n \in \mathbb{N}$ $\alpha \leq a_n \leq \beta$

$a_n \in A_n$]

$\alpha \leq a_n \leq t_n$

$\therefore$ (t_n) converges say to t

$t = \limsup_n a_n$

By symmetry.

$s_n := \text{glb} A_n$

$A_1 \supset A_2$

$A_2 \subseteq A_1$

$\text{glb} A_2 \geq \text{glb} A_1$

$$\boxed{\begin{gathered} A \subset B \\ \alpha := \text{glb} A \geq \text{glb} B =: \beta \end{gathered}}$$

Observe

(i) (s_n) is $\uparrow$

(ii) (s_n) is bdd above.

$[\alpha \leq a_n \leq \beta$ $\quad s_n \leq a_n \leq \beta$]

$\therefore s_n \to s$ (A monot ↑ ind which is bdd above converges to the lub)

$s = \text{lub}\{s_n : n \in \mathbb{N}\}$

$s = \liminf_n a_n$

1) $a_n = (-1)^n$ $\quad -1, 1, -1, 1, -1, 1, \ldots$

$\forall n \in \mathbb{N}$, $A_n = \{-1, 1\}$

$\limsup_n a_n = \underline{1}$ $\quad \liminf_n a_n = \underline{-1}$

2) $(a_n) = (1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots)$

$s = 1, \ t = 3$

3) $(a_n) = (100, 1, 2, 3, 100, 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots \ldots)$

$t_1 = 100$ $\quad \limsup a_n = 3$, $\quad \liminf a_n = 1$

$t_2 = 100$

$t_5 = 100$

$t_6 = 3$

$t_7 = 3$

$t_8 = 3$

$t_9 = 3$

Observe

$s \leq t$ $\quad a_n \geq M$

$\Leftarrow$ $\quad s_n \to s$ $\quad a_n \to a$

$t_n \to t$ $\quad$ Then $a \geq M$

$s_n = \text{glb } A_n$

$t_n = \text{lub } A_n$

$s_n \leq t_n$

$t_n - s_n \geq 0$

$t_n - s_n \to t - s$

$t - s \geq 0$

$t \geq s$

$\liminf a_n \leq \limsup a_n$

Theorem Let (a_n) be a bdd seq

$a_n \to a \Rightarrow \liminf a_n = \limsup a_n = a$

Converse If $\liminf a_n = \limsup a_n = a \Rightarrow a_n \to a$

$(-1)^n = (-1, 1, -1, 1, \ldots)$

$\limsup a_n = 1$

Lemma-1

With usual notations.

(i) $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, a_n < t+\epsilon$

(ii) $\forall \epsilon > 0, \{n \in \mathbb{N} : a_n > t-\epsilon\}$ is infinite

Proof

(i) Let $\epsilon > 0$.

$t+\epsilon$

$t = glb\{t_n : n \in \mathbb{N}\}$

$\therefore t+\epsilon$ is not a lb for $\{t_n : n \in \mathbb{N}\}$

$\therefore \exists N \in \mathbb{N}$ s.t. $t_N < t+\epsilon$

$lub\, A_N = t_N$

$lub\{a_n : n \geq N\} = t_N$

$\therefore \forall n \geq N, \; a_n \leq t_N < t+\epsilon$

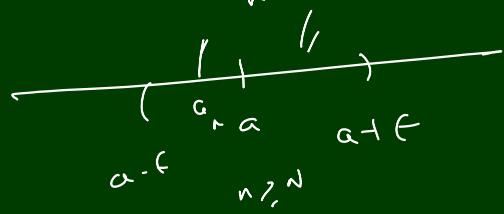
$\therefore \forall n \geq N, \; a_n < t+\epsilon$

(ii)

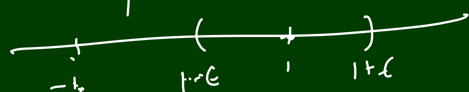
$a_n \to a$

$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N$

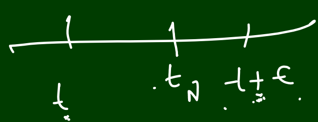
$a_n \in (a-\epsilon, a+\epsilon)$



$\limsup a_n = a$



$-1, 1, -1, 1, \ldots$



Complex Analysis, Aug 20, 2020

Lemma 1. (Fact)

With usual not,

(i) $\forall \epsilon > 0 \left(\exists N \in \mathbb{N} \left(\forall n \geq N \left(a_n < t + \epsilon \right) \right) \right)$

(ii) $\forall \epsilon > 0$ $\{ n \in \mathbb{N} : a_n > t - \epsilon \}$ is infinite

Let $\epsilon > 0$ $\quad t = \inf \{ t_n : n \in \mathbb{N} \}$

$\therefore t + \epsilon$ is not a l.b for $\{ t_n : n \in \mathbb{N} \}$

$\therefore \exists N \in \mathbb{N}$ s.t $t_N < t + \epsilon$

$\because t_N = \text{lub } A_N = \{ a_N, a_{N+1}, \dots \}$

$\forall K \geq N$, $\quad a_K \leq t_N < t + \epsilon$

(ii) $\quad t - \epsilon < t \leq t_K, \ \forall K \in \mathbb{N}$

Let $K = 1$ $\quad t - \epsilon < t_1 = \text{lub } A_1$

$t - \epsilon$ is not an u.b for A_1 $\qquad t-\epsilon \quad a_{n_1}$

$\therefore \exists n_1 \in \mathbb{N}$ s.t $a_{n_1} > t - \epsilon$

$K = n_1 + 1$ $\quad t - \epsilon < t_{n_1+1} = \text{lub } A_{n_1+1}$

$\exists n_2 \in \mathbb{N}$ s.t $a_{n_2} > t - \epsilon$

$n_2 \geq n_1 + 1 > n_1$

$K = n_2 + 1$

Proce $\quad \exists n_{K+1} \in \mathbb{N}$ s.t $n_{K+1} > n_K$ & $a_{n_K} > t - \epsilon$

$\therefore \{ n_K : K \in \mathbb{N} \} \subseteq \{ n \in \mathbb{N} : a_n > t - \epsilon \}$

infinite

$\therefore$ The result is proved.

Lemma 2 (For s)

With usual notations,

(i) $\forall \epsilon > 0 \,(\exists N \in \mathbb{N} \,(\forall n \geq N \,(a_n > s - \epsilon)$

(ii) $\forall \epsilon > 0$, $\{n \in \mathbb{N} : a_n < s + \epsilon\}$ is infinite.

Proof: Exercise.

Thm: Let (a_n) be a bdd seq of real nos. & $a \in \mathbb{R}$

$a_n \to a$ iff $\liminf_n a_n = \limsup_n a_n = a$

Proof:

~~Let $a_n \to a$~~

~~claim $t = s = a$~~

~~$\forall \epsilon > 0$, we prove that $0 < t - s < \epsilon$,~~

~~$s = \text{lub}\{s_n : n \in \mathbb{N}\}$~~

~~$t = \text{glb}\{t_n : n \in \mathbb{N}\}$~~

~~$\forall n \in \mathbb{N}$ $\quad t \leq t_n$~~

~~$s_n \leq s$~~

~~|||y~~

~~$\therefore s_n \leq s \leq t \leq t_n$~~

~~By Lemma 1 (i) $\exists N_1 \in \mathbb{N}$ $\forall n \geq N_1$,~~

~~$a_n < t + \epsilon$~~

$s_n \leq a_n \leq t_n$

$a_n < t + \epsilon$

W.S.T $t = a$

$\Rightarrow$

Let $a_n \to a$ ✓

claim $\boxed{t = s = a}$

First we prove $t = s$ ✓

For that, EST, $\forall \epsilon > 0$, $t - s < \epsilon$ —— (*)

Let $\epsilon > 0$

$a_n \to a$, $\exists N \in \mathbb{N} \left(\forall n \geq N \left(a - \epsilon < a_n < a + \epsilon \right) \right)$.

$a - \epsilon$ is a l.b for A_N

$a - \epsilon \leq s_N \leq s$ —— (1)

Also $a + \epsilon$ is an u.b for A_N

$\therefore t \leq t_N \leq a + \epsilon$ —— (2)

$\therefore s \leq t$ —— (3)

Claim: γ (1)(2)(3)

Recall $t = \text{glb}\{t_n : n \in \mathbb{N}\}$
$s = \text{lub}\{s_n : n \in \mathbb{N}\}$

$a - \epsilon \quad t \quad a \quad a + \epsilon$

$\therefore |t - s| = t - s \leq 2\epsilon$

$\therefore s = t$

Claim $t = a$

$\Leftarrow$ Given $s = t = a$

Claim W.S.T $a_n \to a$

Let $\epsilon > 0$

By Lemma 1, $\exists N_1 \in \mathbb{N}$ s.t $\forall n \geq N_1$, $a_n < t + \epsilon = a + \epsilon$ —— (1)

By Lemma 2, $\exists N_2 \in \mathbb{N}$ s.t $\forall n \geq N_2$, $a - \epsilon = s - \epsilon < a_n$ —— (2)

$N := \max\{N_1, N_2\}$ —— (3)

$\forall k \geq N (\geq N_1)$ $a_k < a + \epsilon$ (by (3) & (1)) —— (4)

$\forall k \geq N (\geq N_2)$ $a - \epsilon < a_k$ —— (5)

By (4) & (5) $\therefore \forall k \geq N$, $a - \epsilon < a_k < a + \epsilon$

Recap

Thm Let (a_n) be bdd seq of real numbers & $a \in \mathbb{R}$

$a_n \to a$ iff $t = s = a$

$\Leftarrow$ very easy (by using Lemma 1 & $\overset{\text{Lem}}{2}$)

$\Rightarrow$ Attention required

Use Convergence first to get a stage

claim: $t = s = a$

First we show $t = s$ (we know that $s \le t$)

EST, $\forall \epsilon > 0$, $t - s < \epsilon$

Let $\epsilon > 0$

$\because a_n \to a$, $\exists N \in \mathbb{N} \; (\forall n \in \mathbb{N} \; (a - \epsilon < a_n < a + \epsilon))$

$\therefore a - \epsilon$ is l.b for A_N

$\therefore s_N$ is g.l.b for A_N, $a - \epsilon \le s_N \le s$

$(s = \text{lub}\{s_n : n \in \mathbb{N}\})$

Hly. $a + \epsilon$ is u.b for A_N

$\therefore t_N$ is lub for A_N $t \le t_N \le a + \epsilon$

$$a - \epsilon \le s_N \le s \le t \le t_N \le a + \epsilon$$

Ex: Prove that $\mathbb{R}$ is complete

Cauchy-Hadamard formula for radius of Convergence

Consider the power Series $\sum_n a_n (z-a)^n$.

if R is radius of convergence then

$$\boxed{\frac{1}{R} = \limsup |c_n|^{1/n}}$$

or

$$R = \liminf |c_n|^{-1/n}$$

Complex Analysis, Aug 21, 2020.

Cauchy-Hadamard formula

For the power series $\sum a_n(z-a)^n$, if the radius of convergence is R then

$$\frac{1}{R} = \limsup |a_n|^{1/n} \quad \text{or}$$

$$R = \liminf |a_n|^{-1/n}$$

Proof $\because R$ is radius of convergence

$|z-a| \le r < R$ then series $\sum a_n(z-a)^n$ converges absolutely

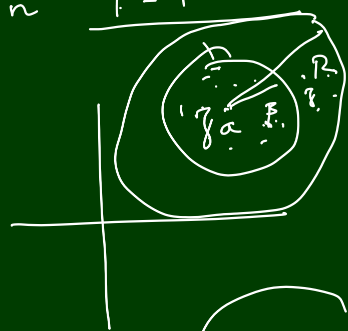
& if $|z-a| > R$ then series diver

Let $\boxed{\frac{1}{\beta} := \limsup |a_n|^{1/n}}$

Claim $R = \beta$ $\quad (R \le \beta \ \& \ \beta \le R)$

Let $|z-a| < \beta$ $\forall z \in B(a,\beta)$ $\quad R = \sup\{|z-a| : \sum a_n(z-a)^n \text{ converges}\}$

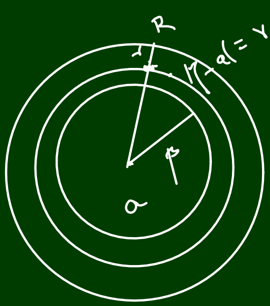
If we prove $\sum a_n(z-a)$ converges. Then $\beta \le R$



$\forall z \in \mathbb{C}$, $|z-a| > \beta$

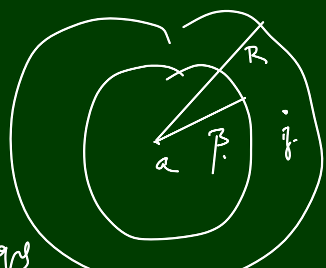
If we prove $\sum a_n(z-a)$ diverges

then $R \not> \beta$



If $R > \beta$ choose z s.t. $\beta < |z-a| < R$

$\Rightarrow \sum a_n(z-a)$ converges $\Rightarrow\Leftarrow$



First Let $|z-a| < \beta$

choose r s.t. $|z-a| < r < \beta$

$\forall \epsilon > 0, \exists N \in \mathbb{N}$
$\forall n \geq N, \; c_n < t + \epsilon$

$|z - a| < r < \beta, \quad \boxed{\frac{1}{\beta} = \limsup |a_n|^{1/n}}$

$\therefore \frac{1}{r} > \frac{1}{\beta}.$

$\frac{1}{\beta} = t$

$\frac{1}{r} = \frac{1}{\beta} + \epsilon$

$\epsilon = \frac{1}{r} - \frac{1}{\beta} > 0$

By lemma of limsup, $\exists N \in \mathbb{N}$

$\forall n \geq N, \quad |a_n|^{1/n} < \frac{1}{r}$

$|a_n| < \frac{1}{r^n}$

$\forall n \geq N, \quad |a_n| r^n < 1$

$\Rightarrow (|a_n| r^n)$ is bdd Say by M.

$\frac{|z-a|}{r} < 1$

$$|a_n(z-a)^n| = |a_n| \, |z-a|^n$$
$$= |a_n| \cdot r^n \, \frac{|z-a|^n}{r^n}$$
$$= |a_n| r^n \cdot \left(\frac{|z-a|}{r}\right)^n$$
$$\leq M \left(\frac{|z-a|}{r}\right)^n.$$

By Comparison test $\sum |a_n(z-a)^n| < \infty$

$\therefore \sum a_n(z-a)^n$ converges.

Let $|z-a| > \beta$

Claim $\sum a_n(z-a)^n$ diverges.

$\frac{1}{|z-a|} < \frac{1}{\beta}$

(ii) $\forall \epsilon > 0, \; \{n \in \mathbb{N} : c_n > t - \epsilon\}$ is infinite

There exists infinite many $n \in \mathbb{N}$ s.t

$$\frac{1}{|z-a|} < |a_n|^{1/n}$$

for infinite, $|a_n|^{1/n} |z-a| > 1$

for infinite many n. $|a_n| |z-a|^n > 1$

$\therefore$ nth term does not converge to 0

$\therefore \sum a_n (z-a)^n$ diverges

' Please go through the notes'

Radius convergence of the power series problem has to be done.

Complex Analysis, Aug 25, 2020

Recall power series $\sum a_n(z-a)^n$ - R - radius Convergence

$$R := \underset{r<R}{\text{lub}} \{ |z-a| : \sum a_n(z-a)^n \text{ converges} \}$$

i.e

(i) $\forall z \in B[a, r]$, $\sum a_n(z-a)^n$ converges (absolutely)

(ii) $\forall z$ s.t $|z-a| > R$, $\sum a_n(z-a)^n$ diverges.

Tests

a) Ratio test

If $\lim \frac{|a_{n+1}|}{|a_n|} = \rho$ then $R = \frac{1}{\rho}$

b) Root test

If $\lim |a_n|^{1/n} = \rho$ then $R = 1/\rho$

c) Cauchy - Hadamard formula

$$\frac{1}{R} = \lim_n \sup |a_n|^{1/n}$$

or $R = \lim_n \inf |a_n|^{-1/n}$

1) $a_n = 2^n$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2^n} = 2$$

$R = \frac{1}{2}$

$a_n^{1/n} = 2$

$R = 1/2$

2) $a_n = 2^n - 1$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} - 1}{2^n - 1} = \frac{2 - \frac{1}{2^n}}{1 - \frac{1}{2^n}}$$

$$\lim \frac{a_{n+1}}{a_n} = 2$$

$R = 1/2$

3) $a_n = a^n + b^n$ where $a > b > 0$

$a_n^{1/n} = (a^n + b^n)^{1/n} \longrightarrow a$ $\quad R = \frac{1}{a}$

4) $\sum a_n z^n$, $\sum a_n z^{n+1}$ have same radii of con R

Let R be of convergence of $\sum a_n z^n$

Let $|z| \leq r < R$

Claim $\sum a_n z^{n+1}$ converges absolutely

$$|a_n z^{n+1}| = |z|\,|a_n z^n| \leq r\,|a_n z^n|$$

$|t_n| = \left|\sum_{k=0}^{n} a_k z^k\right|$

$\therefore$ By comparison test $\sum |a_n z^{n+1}|$ converges.

$\therefore \sum a_n z^{n+1}$ converges

Let $|z| > R$ claim: $\sum a_n z^{n+1}$ diverges

claim $|a_n z^{n+1}| = |a_n z^n|\,|z|$

$$> R\,|a_n z^n|$$

$$> R \cdot \frac{1}{R} = 1$$

$|z_1| - |z_2|$

There are infinitely many 'n'

Ex

5) $\limsup (a_n + b_n) \leq \limsup (a_n) + \limsup (b_n)$

$a_n = (-1)^n$

$b_n = (-1)^{n+1}$

6) $\limsup (a_n b_n) \leq \limsup (a_n)(\limsup b_n)$

$a_n = \begin{cases} 1 \\ 2 \end{cases}$ $\quad b_n = \begin{cases} 2 \\ 1 \end{cases}$

$a_n = (1, 2, 1, 2, \ldots)$

$b_n = (2, 1, 2, 1, 2, 1, \ldots)$

$a_n b_n = (2, 2, 2, \ldots)$

$2 < 4$

~~Lemma~~ $a_n \geq 0$, $a \in \mathbb{R}$, $\limsup a_n = a$

1) If $b_n \to b$, $b_n \geq 0$ $\limsup a_n b_n = ab$ //

$\sum a_n z^n$, $\sum \frac{a_n}{n+1} z^n$ $\qquad (n+1)^{1/n} \to 1$

R

$$\limsup \frac{|a_n|^{1/n}}{(n+1)^{1/n}} = \lim |a_n|^{1/n} = R$$

$$\sum n a_n z^n \quad \limsup n^{1/n} |a_n|^{1/n} = \limsup |a_n|^{1/n} = R$$

— × —

$$\boxed{\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}}$$

$\exp : \mathbb{C} \to \mathbb{C}$

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{\cancel{n!}}{\cancel{n!}(n+1)} = \frac{1}{n+1}$$

$$\lim \frac{a_{n+1}}{a_n} = 0$$

$R = \infty$

Through from ratio test

We learn estimate via the following (another) proof the power series converge $\forall z \in \mathbb{C}$

By A.P $\exists N \in \mathbb{N}$, $\frac{|z|}{N} < \frac{1}{2}$ $\qquad N+1 > N$, $\frac{1}{N+1} < \frac{1}{N}$

$$\left| \frac{z^{N+k}}{(N+k)!} \right| = \underbrace{\frac{|z|}{1} \cdots \cdots \frac{|z|}{N}}_{} \cdot \frac{|z|}{N+1} \cdots \frac{|z|}{N+k}$$

$$\leq \frac{|z|^N}{N!} \frac{|z|}{N} \cdots \frac{|z|}{N}$$

$$\leq \frac{|z|^N}{N!} \left(\frac{1}{2}\right)^k$$

$$\le \frac{|z|^N}{N!} \frac{1}{2^K}$$

$$\sum_{n \geq N} \frac{|\xi|^n}{n!} \le \frac{|z|^N}{N!} \sum_{n \geq N} \frac{1}{2^{n-N}}$$

$$\le 2 \frac{|z|^N}{N!}$$

exp is a contin fn on $\mathbb{C}$

$\sum a_n - \sum b_n = \sum (a_n - b_n)$

$$|\exp z - \exp w| = \left| \sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{w^n}{n!} \right|$$

$$= \left| \sum_{n=0}^{\infty} \frac{(z^n - w^n)}{n!} \right|$$

$z^n - w^n = (z-w)(\xi^{n-1} + \xi^{n-2} w + \dots + \xi w^{n-2} + w^{n-1})$

$$\le \sum_{n=0}^{\infty} \frac{|z^n - w^n|}{n!} , \ (\ ?\ )$$

$$= \sum_{n=0}^{\infty} \frac{|z-w| \, |\xi^{n-1} + \xi^{n-2} w + \dots + w^{n-1}|}{n!}$$

choose $\delta < 1$

$|w - z| < \delta$

$|w| = |w - z + z|$

$\le |w - z| + |z|$

$< 1 + |z|$

$$\le \sum_{n=0}^{\infty} \frac{|z-w|}{n!} \left(|\xi|^{n-1} + |\xi|^{n-2} |w| + \dots + |w|^{n-1} \right)$$

$$\le \sum_{n=0}^{\infty} \frac{|z-w|}{(n-1)!} (1 + |\xi|)^{n-1}$$

$|z| < 1 + |z|$

$$|\xi|^{n-1} < (1 + |\xi|)^{n-1}$$

$$|\xi|^{n-1} |w| < (1 + |\xi|)^{n-2} (1 + |\xi|)$$

$\delta = \min\{1, \exp(1 + |\xi|)\}$

$$= |z-w| \sum_{n=1}^{\infty} \frac{(1 + |\xi|)^{n-1}}{(n-1)!}$$

$$= |z-w| \exp(1 + |\xi|)$$

$$< C_a |z - a|$$

Complex Analysis, Aug 25, 2020 Lecture 2.

$f : \mathbb{C} \to \mathbb{C}$

$f(z) = \exp(z)$ is conts fn on $\mathbb{C}$.

$f(z) = \sum a_n (z-a)^n$ is power series & R be radius

$f : B(a,R) \longrightarrow \mathbb{C}$

f is conts on $B(a,r)$

For simplicity assume $a = 0$.

$f : B(0,R) \to \mathbb{C}$

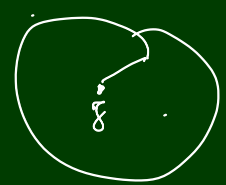
$f(z) = \sum_n a_n z^n$

f is conts on $B(0,R)$

Let. $z \in B(0,R)$ claim f is conts at z.

choose δ s.t $|z| + 2\delta < R$ &

Let $|h| < \delta$. —— (1)



$$|f(w) - f(z)| = \left|\sum_{n=0}^{\infty} a_n (w^n - z^n)\right|$$

$$(w^n - z^n) = (w - z)(w^{n-1} + w^{n-2} z + \cdots + z^{n-1})$$

$$\boxed{\frac{w^n - z^n}{w - z} = w^{n-1} + w^{n-2} z + \cdots + z^{n-1}}$$

$w = z + h$ $\quad |w| = |z+h| \le |z| + |h| < |z| + \delta$

choose $|h| < \delta^*$

where $\delta' = \min\{\delta, \frac{\epsilon\delta}{m}\}$

$$|(z+h)^n - z^n| \le |h|\, n \left(|z| + |h|\right)^{n-1}$$
$$\le |h|\, n (|z| + \delta)^{n-1} \text{ —— } (*)$$

$$\left((|z| + \delta) + \delta\right)^n \ge n (|z| + \delta)^{n-1} \delta \quad (\text{By Binom thm})$$

$$\therefore (|z| + 2\delta)^n \ge n (|z| + \delta)^{n-1} \delta$$

Using in $(*)$, $\quad |(z+h)^n - z^n| \le \dfrac{|h| (|z| + 2\delta)^n}{\delta}$

$$|f(q+h)-f(q)| \le \sum_{n=0}^{\infty} |a_n|\, |(q+h)^n - q^n|$$

$$\le \sum_{n=0}^{\infty} |a_n| |h| \frac{(|q|+2\delta)^n}{\delta}$$

$$= \frac{|h|}{\delta} \sum_{n=0}^{\infty} |a_n| (|q|+2\delta)^n$$

$$\le \frac{M|h|}{\delta}$$

$$\le \frac{M \epsilon \delta}{M\delta}$$

$$< \epsilon$$

$w_1 = q + 2\delta$

$|w_1| \le |q| + 2\delta < R$

$\sum a_n w_1^n$ converges

$\frac{|h|}{\delta} M < \epsilon$

$|h| < \epsilon$

$|h| < \delta_1$

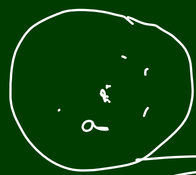
$\frac{M}{\delta} \cdot \delta_1 < \epsilon$

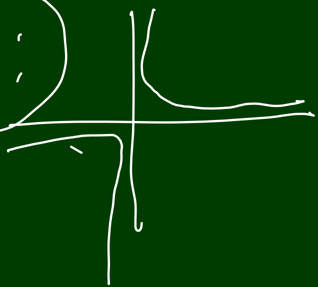
$\delta_1 < \frac{\epsilon\delta}{M}$

$$\boxed{\lim_{q \to a} f(q) = l}$$

$f : A \to \mathbb{R}$

a is cluster pt of A





$xy = 1$

$y = 1/x$

$\forall \epsilon > 0 \left(\exists \delta > 0 \left(0 < |q-a| < \delta \left(|f(q) - l| < \epsilon \right) \right) \right)$

$q \ne a,\ q \in B(a,\delta) \,/\, f(q) \in (l-\epsilon, l+\epsilon)$

$f : \mathbb{R}^a \to \mathbb{R}$

$f(x) = \frac{1}{x}$

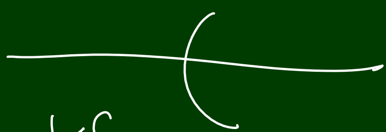
$\lim_{x \to 0} f(x)$

$$\lim_{x \to a} f(x) = \infty$$

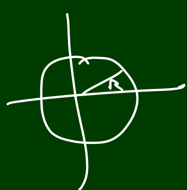
$\forall R \in \mathbb{R},\ \exists \delta > 0,\quad 0 < |x-a| < \delta$

$f(x) \in (R, \infty)$

$f(x) > R$



R



$w = q - a$

$|w| = |q-a| < R$

We have every power Series

is a conts fn on $B(a, r)$

Complex Analysis, Aug 26, 2020

Thm $\quad f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \left\{ \begin{array}{l} \sum a_n(z-a)^n \\ \text{with radius conv } R \end{array} \right.$

$0 < R \leq \infty]$

$$f_n(z) = \sum_{k=0}^{n} a_k(z-a)^k$$

$f_n \to f$ uniformly on $\bar{B}[a,r]$ for $r < R$

This thm is an application of Weistress M-test.

Thm (W-M-test)

Let X be any set $f_n : X \to \mathbb{C}$ be seq of fns such that $\forall n \in \mathbb{N}$, $\exists M_n \geq 0$ ($\forall x \in X$, $|f_n(x)| \leq M_n$). Also assume $\sum_n M_n < \infty$ Then the series $\sum f_n$ is uniformly conv on X.

Proof:

$$s_n := \sum_{k=0}^{n} f_n$$

i.e $\quad s_n(x) = \sum_{k=0}^{n} f_n(x)$

Given $\forall n \in \mathbb{N}$ $|f_n(x)| \leq M_n$

By Comparison test

$\sum_n |f_n(x)|$ converges

$\therefore \sum_n f_n(x)$ converges $\forall x \in X$

Define $f : X \to \mathbb{C}$

by $\quad f(x) := \sum_{n=0}^{\infty} f_n(x)$

i.e $\quad s_n \to f$ pointwise on X

Claim: $s_n \rightrightarrows f$ on X.

Let $\epsilon > 0$.

Let $n \geq N$

Let $x \in X$

$$|f(x) - f_n(x)| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right|$$

$$\leq \sum_{k=n+1}^{\infty} |f_k(x)|$$

$$\leq \sum_{k=n+1}^{\infty} M_k < \epsilon$$

Given $\epsilon > 0$ choose $N \in \mathbb{N}$ s.t $\sum_{n>N} M_n < \epsilon$ —(1)

For $n > N$

$$\therefore \left| \sum_{k=N+1}^{n} f_k(x) \right| \leq \sum_{k=N+1}^{n} |f_k(x)| \leq \sum_{k=N+1}^{n} M_k \leq \sum_{k>N} M_k < \epsilon$$

$$\therefore \left| \sum_{n>N} f_n(x) \right| \leq \epsilon$$

$n > N$

$$|f(x) - f_n(x)| = \left| \sum_{n+1}^{\infty} f_n(x) \right| \leq \epsilon \quad f_n \to f \text{ uniform}$$

$\forall x \in X$ $\therefore \sum f_n$ is uniformly con

Thm Let $\sum a_n (z-a)^n$ be given & $0 < R \leq \infty$.

The for any r s.t $0 \leq r < R$ the power se is uniformly convergent on $B[a, r]$

Proof: Let $z \in B[a, r]$

$|z - a| \leq r$

$\sum f_n$

$\forall z \in D$ $|f_n(z)| \leq M_n$

$\sum M_n < \infty$

$0 \leq r < R$

$\exists z_0 \in \mathbb{C}$

$r < |z_0 - a|$

& $\sum a_n (z_0 - a)^n$ is convergent

$$\frac{r}{|z_0 - a|} < 1$$

$$|a_n| |z - a|^n \leq |a_n| \cdot r^n$$

$$\leq |a_n| \cdot r^n \cdot \frac{|z_0 - a|^n}{|z_0 - a|^n}$$

$$\leq |a_n| |z_0 - a|^n \left(\frac{r}{|z_0 - a|} \right)^n$$

$$\leq M \cdot \left(\frac{r}{|z_0 - a|} \right)^n$$

$$M_n = M \left(\frac{r}{|z_0 - a|} \right)^n$$

$\sum M_n < \infty$

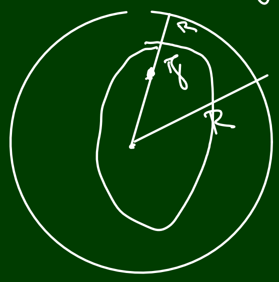
$\therefore$ By Weierstrass M-test the series converges.

Ex: S.T $\sum z^n$ does not converge uniformly on $\overline{B(0,1)}$

$f_n(x) = x^n$ on $(0,1)$

Cor: Any power series with $R>0$ is conts on its disk of convergence.

$f : B(0,R) \to \mathbb{R}$ by $f(z) = \sum a_n z^n$



Let $z \in B(0,r)$

Claim: f is continuous at z. Choose r s.t $0 < |z| < r < R$.

On $B[0,r]$ $\quad s_n = \sum_{k=0}^{n} a_k z^k$

$s_n \rightrightarrows f$

$\therefore$ s_n's are conts on $B[0,r]$

$\therefore$ f is conts on $B[0,r]$

In fact f is conts at z $\left[z \in B[0,r] \right]$

Do this Exercis

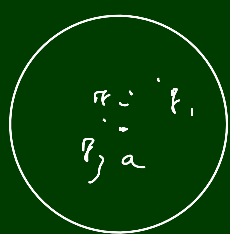
$f_n \rightrightarrows f$ on X & f_n's are conts then f is conts.

$B(X, \mathbb{R})$ is complete

Uniqueness theorem for power series.

$R>0$

$f(z) = \sum a_n (z-a)^n$
$g(z) = \sum b_n (z-a)^n$ $\quad \forall z \in B(a,R)$.



(1) $z_n \neq a$ $\forall n$ (2) $\lim z_n = a$ (3) $\forall n$, $f(z_n) = g(z_n)$

Then $\forall n \in \mathbb{N}$ $a_n = b_n$

Proof: For simp $a = 0$ $\quad f(z) = a_0 + a_1 z + \cdots \quad g(z) = b_0 + b_1 z + \cdots$

W.k.t f & g are conts at 0 $\quad z_n \to 0$

$f(z_n) \to f(0)$

$$\therefore \quad f(0) = \lim f(z_n)$$
$$\qquad\qquad\qquad \shortparallel$$
$$g(0) = \lim g(z_n)$$

$$\therefore \quad f(0) = g(0)$$
$$\shortparallel \qquad \shortparallel$$
$$a_0 = b_0$$

For indn assn, $\forall\, k$ s.t $0 \le k \le n$, $a_k = b_k$

$f(z)$

$$\phi(z) = a_{n+1} + a_{n+2}(z) + \cdots$$

$$\psi(z) = b_{n+1} + b_{n+2}(z) \cdot$$

$$f(z) = \sum_{k=0}^{n} a_k z^k + z^{n+1}\phi(z)$$

$$\phi(z) = \begin{cases} \dfrac{f(z) - \sum_{k=0}^{n} a_k z^k}{z^{n+1}} & z \neq 0,\ |z| < R \\ a_{n+1} & \text{at } z = 0 \end{cases}$$

$$\Big\{ g(z) = z^{n+1}\psi(z)$$

Do similr ψ.

ϕ, ψ conts on $B(0,R)$.

$\forall\, r \in \mathbb{N}$ $\phi(z_r) = \psi(z_r)$

By Conv $\phi(0) = \psi(0)$ $\quad$ i.e $a_{n+1} = b_{n+1}$

— x —

Complex Analysis, Aug 27, 2020.

Revise Uniqueness theorem proved yesterday:

State
$f(z) = \sum_n a_n (z-a)^n$ & $g(z) = \sum_n b_n (z-a)^n$, $z \in B(a,R)$

is s.t. Given
(i) $z_n \in B(a,r)$ s.t. $\forall n$ $z_n' \neq a$

(ii) $z_n \to a$

(iii) $\forall n$, $f(z_n) = g(z_n)$

Then $\forall n \in \mathbb{N}$, $a_n = b_n$

Proof: Based on Induction

First we prove $a_0 = b_0$

$f(a) = a_0$, $g(a) = b_0$

$z_n \to a$

$\because f, g$ are conts at a, $f(z_n) \to f(a)$, $g(z_n) \to g(a)$

By (iii) $f(a) = g(a)$

We assume $\forall K$, $0 \leq K \leq n$, $a_K = b_K$

We claim $a_{K+1} = b_{K+1}$.

$f(z) = a_0 + a_1(z-a) \cdot + a_n(z-a)^n a_{n+1}(z-a)^{n+1}$.

$f(z) - \sum_{k=0}^{n} a_k (z-a)^k = (z-a)^{n+1} \phi(z)$

$$\phi(z) = \begin{cases} \dfrac{f(z) - \sum_{k=0}^{n} a_k (z-a)^k}{(z-a)^{n+1}} & 0 < |z-a| < R \\ a_{n+1} & z = a \end{cases}$$

where $\phi(z) = a_{n+1} + a_{n+2}(z-a) + \cdots$

Then ϕ is conts a $\left[\lim_{z \to a} \phi(z) = \phi(a)\right]$

$$\psi(\eta) = \begin{cases} \dfrac{g(\eta) - \sum_{k=0}^{n} b_k (\eta - a)^k}{(\eta-a)^{n+1}}, & 0 < |\eta - a| < R \\ b_{n+1} & \eta = a \end{cases}$$

ψ is cont^s a.

$\eta_r \to a$

$\forall\, r \in \mathbb{N}$ $\quad \phi(\eta_r) = \psi(\eta_r)$

$\therefore$ $\because$ ϕ & ψ are cont

$$\phi(a) = \psi(a)$$

i.e. $\quad a_{n+1} = b_{n+1}$

Class Ex:

1. If 2 power series agree on the set $S = \{\eta_n : n \in \mathbb{N}\}$ with a as a cluster point in $B(a,R)$ then the power series are equal on $B(a,R)$.

2. If 2 power series agree on a non-empty open set $G \subset B(a,R)$ then the power series is equal on $B(a,R)$

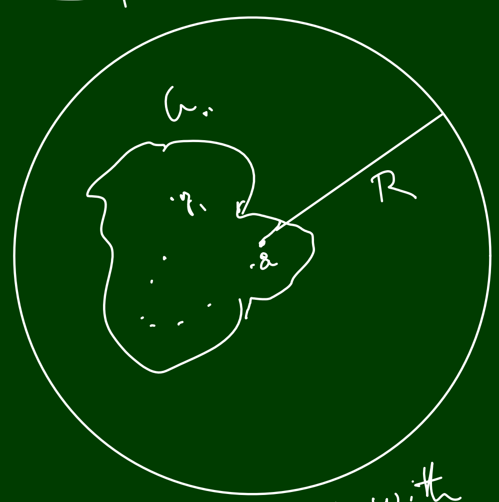
$a \in G$

G is open in $\mathbb{C}$

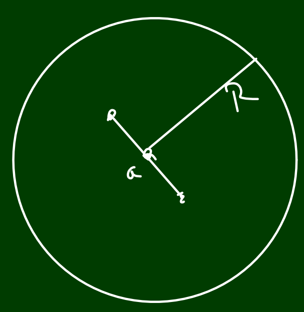
$\therefore$ a is a cluster point G

$\therefore \exists (\eta_n)$ in G s.t

$\forall n\ \eta_n \neq a$ & $\eta_n \to a$



3.



Replace (2) with L - line segment cont a

$$L = \{(1-t)a + ta : t \in [l_1, l_2]\}$$

a is cluster point of a open set (G) and L.

Power Series and Differentiation

Thm Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $z \in B(0,R)$ [R is radi conv of P.S $\sum_n a_n z^n$]

Then f is diff on $B(0,R)$

& $f'(z) = \sum_{n=0}^{\infty} a_n \, n \, z^{n-1}$, $z \in B(0,R)$

That is the power Series can be differentiated term by term in its disk of convergence.

Before Proof:

$$f : U \subseteq \mathbb{C} \to \mathbb{C}$$

Let $a \in U$

Then f is diff at a if $\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$ exists

if $\forall \epsilon > 0 \left(\exists \delta > 0 \left(\forall z \text{ s } 0 < |z - a| < \delta \right.\right.$

$\left.\left.\left(\left| \frac{f(z) - f(a)}{z - a} - f'(a) \right| < \epsilon \right)\right.\right)$

1) LUB
2) Sequences
3) Continuit
4) Diff

Revise Real Analysis Book

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